2 (b)

for Poisson distribution;

Given, mean value, $\lambda = 0.8$ Per year.

1. Find the possibility of occuring more than two hurricanes in a year.

here, expected number of events in the interval, $\lambda = 0.8$

Number of events, $k > 2$

from Poisson distribution formula,

$$P(x=k) = \frac{e^{-\lambda} \cdot \lambda^{k}}{k!}$$

$> 3 - 0, 1, 2$

$\geq 3 \cdot 0, 1, 2$

$x \geq 2$

$0 \quad 1 \quad 2$

$n-1$

Probability more than 2 hurrikanes

$$P(X>2) = 1 - P(X \leq 2)$$

$$= 1 - \{P(x=0) + P(x=1) + P(x=2)\}$$

$$= 1 - \left[\frac{e^{-0.8} \cdot (0.8)^{0}}{0!} + \frac{e^{-0.8} \cdot (0.8)^{1}}{1!} + \frac{e^{-0.8} (0.8)^{2}}{2!}\right]$$

$$= 1 - 0.9525 = 0.0475$$

Probability more than 2 hurricane ≈ 0.0475 (4.75%)

(2) probability of exactly one hurricane

$$P(X=1) = \frac{e^{-0.8} \cdot (0.8)^{1}}{1!}$$

$$= 0.8 \times e^{-0.8}$$

$$= 0.8 \times 0.4493$$

$$\approx 0.3594 \text{ or } 35.94\%$$

(3.a) Given,

x and y are jointly discrete random variables, and $p(x,y) = \frac{x+y}{30}$

where, $x = 0, 1, 2$ $y = 0, 1, 2, 3$.

Otherwise, $P(x,y) = 0$

we need to check if x and y are independent or not.

the condition for independence is,

$$P(x,y) = P_X(x) \cdot P_Y(y) \text{ for all } x, y$$

where. $P(x)$ is the marginal Probability for $X=x$

$P_Y(y)$ is the marginal Probability for $Y=y$.

marginal distribution for X.

$$P_X(x) = \sum_{y=0}^{3} p(x,y)$$

$$= \sum_{y=0}^{3} \frac{x+y}{30}$$

$$= \frac{x+0}{30} + \frac{x+1}{30} + \frac{x+2}{30} + \frac{x+3}{30}$$

$$= \frac{x+x+1+x+2+x+3}{30}$$

$$= \frac{4x+6}{30}$$

X এর টা বের করতে y গুলো হিসেব করতে হবে। যদি continuous random variable হয় তখন integration করতে হবে।

So, for $P_X(0) = \frac{6}{30} = 0.2$

$P_X(1) = \frac{4+6}{30} = \frac{1}{3} = 0.333$

$P_X(2) = \frac{8+6}{30} = 0.466$

Marginal distribution for Y

$$P_Y(y) = \sum_{x=0}^{2} p(x,y) = \sum_{x=0}^{2} \frac{x+y}{30}$$

$$= \frac{0+y}{30} + \frac{1+y}{30} + \frac{2+y}{30} = \frac{3+3y}{30}$$

So, for $P_Y(0) = \frac{3}{30} = 0.1$

$P_Y(1) = \frac{6}{30} = 0.2$

$P_Y(2) = \frac{9}{30} = 0.3$

$P_Y(3) = \frac{12}{30} = 0.4$

So, for pair (x, y)

when, $x=0$, $y=0$

$$P(x,y) = \frac{0+0}{30} = 0$$

Product of marginals

$$P_X(0) \cdot P_Y(0) = 0.2 \times 0.1 = 0.02$$

Here, $P(0,0) \neq P_X(0) \cdot P_Y(0)$

So, X and Y are not independent.

<u>b</u>

Suppose that x and y are jointly continuous random variables with, $f(x,y) = y - x$

for $0 < x < 1$ and $1 < y < 2$

otherwise $f(x,y) = 0$

Compute: $E(x)$, $var(x)$, $E(y)$, $Var(y)$, $cov(x,y)$, $corr(x,y)$

(x এর marginal density বের করতে y এর সাপেক্ষে integration)

For finding $E(x)$ and $E(y)$ we need marginal densities of $f(x,y)$

marginal density of $X=x$ is $f_X(x) = \int_{y=1}^{2} f(x,y)\, dy$

$$= \int_{y=1}^{2} (y-x)\, dy \qquad = \cancel{\left[\frac{y^2}{2} - x\right]_1^2}$$

$$= \int_{y=1}^{2} y\, dy - \int_{y=1}^{2} x\, dy \qquad = \cancel{\frac{4}{2} - x - \frac{1}{2} + x}$$

$$= \left[\frac{y^2}{2}\right]_1^2 - [xy]_1^2 \qquad =$$

$$= \left(\frac{4}{2} - \frac{1}{2}\right) - [2x - x]$$

$$= \frac{3}{2} - x$$

marginal density of $Y=y$ is $f_Y(x) = \int_{x=0}^{1} f(x,y)\, dx$

$$= \int_{x=0}^{1} y - x\, dx$$

$$= \int_{x=0}^{1} y\, dx - \int_{x=0}^{1} x\, dx$$

$$= [yx]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= y - \frac{1}{2}$$

Expectation $= \int$ যার expection বের করবো $\times$ তার merginal density

x হলে x এর সাপেক্ষে, y হলে y এর সাপেক্ষে.

$$E(x) = \int_{x=0}^{1} x \times f_X(x)\, dx$$

$$= \int_{x=0}^{1} x \times \left(\frac{3}{2} - x\right) dx$$

$$= \int_{x=0}^{1} \frac{3x}{2} - x^2\, dx$$

$$= \left[\frac{3 \cdot x^2}{2 \cdot 2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{3}{4} - \frac{1}{3}\right)$$

$$= \frac{9-4}{12}$$

$$= \frac{5}{12} = 0.4167$$

$$E(y) = \int_{y=1}^{2} y \times f_Y(y)\, dy$$

$$= \int_{y=1}^{2} y \times \left(y - \frac{1}{2}\right) dy$$

$$= \int_{y=1}^{2} \left(y^2 - \frac{y}{2}\right) dy$$

$$= \left[\frac{y^3}{3} - \frac{y^2}{4}\right]_1^2$$

$$= \left(\frac{2^3}{3} - \frac{4}{4} - \frac{1}{3} + \frac{1}{4}\right)$$

$$= \frac{19}{12}$$

$$= 1.5833$$

Variance $= E(x^2) - [E(x)]^2$

2nd moment for variance.

$$E(x^2) = \int_0^1 x^2 \times f_X(x)\, dx$$

$$= \int_0^1 x^2 \left(\frac{3}{2} - x\right) dx$$

$$= \int_0^1 \frac{3x^2}{2} - x^3\, dx$$

$$= \left[\frac{\cancel{3}\cdot x^3}{2\cdot \cancel{3}} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{4}$$

$$= \frac{1}{4}$$

$$E(y^2) = \int_1^2 y^2 \cdot f_Y(y)\, dy$$

$$= \int_1^2 y^2 \left(y - \frac{1}{2}\right) dy$$

$$= \int_1^2 y^3 - \frac{y^2}{2}\, dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{6}\right]_1^2$$

$$= \frac{16}{4} - \frac{8}{6} - \frac{1}{4} + \frac{1}{6}$$

$$= \frac{31}{12}$$

$$\text{var}(X) = E(x^2) - E(x)^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2$$

$$= \frac{11}{144}$$

$$\text{var}(Y) = E(y^2) - E(y)^2$$

$$= \frac{31}{12} - \left(\frac{19}{12}\right)^2$$

$$= \frac{11}{144}$$

**

Covariance $\text{Cov}(X,Y) = E[XY] - E[X]\cdot E[Y]$

$$E[XY] = \int_{x=0}^{1}\int_{y=1}^{2} xy\, f(x)\cdot f(y)\, dy\, dx$$

$$= \int_{x=0}^{1} \cancel{xy} \int_{y=1}^{2} xy = \int_{x=0}^{1} xy$$

Covariance : $Cov(x,y) = E(xy) - E(x) \cdot E(y)$

$$E(xy) = \int_{x=0}^{1} \int_{y=1}^{2} xy \, f(xy) \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=1}^{2} xy \, (y-x) \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=1}^{2} (xy^2 - x^2 y \, dy) \, dx$$

$$= \boxed{\int_{x=0}^{1} x \frac{y^3}{\cancel{3}} - x^2 \frac{y^2}{2}}$$

$$= \int_{x=0}^{1} \left[\int_{y=1}^{2} xy^2 dy - \int_{y=1}^{2} x^2 y \, dy \right] dx$$

$$= \int_{x=0}^{1} x \left[\frac{y^3}{3} \right]_1^2 - x^2 \left[\frac{y^2}{2} \right]_1^2 dx$$

$$= \int_{x=0}^{1} x \left(\frac{8}{3} - \frac{1}{3} \right) - x^2 \left(\frac{4}{2} - \frac{1}{2} \right) dx$$

$$= \int_{x=0}^{1} \frac{7x}{3} - \frac{3x^2}{2} \, dx$$

$$= \left[\frac{7x^2}{3 \cdot 2} - \frac{3 \cdot x^3}{2 \cdot 3} \right]_0^1$$

$$= \frac{7}{6} - \frac{3}{6}$$

$$= \frac{4}{6} = \frac{2}{3}$$

$$\therefore \mathrm{Cov}(x,y) = E(xy) - E(x)\cdot E(y)$$

$$= \frac{2}{3} - \left(\frac{5}{12}\cdot\frac{19}{12}\right)$$

$$= \frac{2}{3} - \cancel{\frac{55}{108}}\ \frac{95}{144}$$

$$= \cancel{\frac{72-55}{108}} = \cancel{\frac{17}{18}}\ \frac{1}{144}$$

x, y independent হলে.

$E(xy) = E(x)\cdot E(y)$ লেখা যেত,

তখন $\mathrm{Cov}(x,y) = 0$ আসত

Correlation

$$\mathrm{Cor}(x,y) = \rho_{xy}$$

$$= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{var}(X)\cdot \mathrm{var}(Y)}}$$

we have, $\mathrm{var}(X) = \mathrm{var}(Y) = \frac{11}{144}$

$$\rho_{xy} = \frac{1/144}{11/144} = \frac{1}{11} = 0.09$$

to inspect

5(a) b

Generate three Poisson variants with mean $\alpha = 0.2$ using acceptance rejection techniques. Given random numbers are 0.4347, 0.4196, 0.8353, 0.9952 and 0.8004

Ans: given, mean $\lambda = 0.2$

random numbers = 0.4347, 0.4196, 0.8353, 0.9952, 0.8004.

we need three poisson variants (0.2)

For poisson, $P(X=K) = \frac{e^{-\lambda} \cdot \lambda^K}{K!}$ where, $K = 0, 1, 2 \cdots$

$$P(X=0) = \frac{e^{-0.2} \times (0.2)^0}{0!} = e^{-0.2} \approx 0.8187$$

$$P(X=1) = \frac{e^{-0.2} \times (0.2)^1}{1!} = e^{-0.2} \times 0.2 \approx 0.1637$$

$$P(X=2) = \frac{e^{-0.2} \times (0.2)^2}{2!} = \frac{0.2^2 \times e^{-0.2}}{2} \approx 0.0164$$

Now, cumulative,

$F(0) = P(X \leq 0) = 0.8187$

$F(1) = P(X \leq 1) = 0.8187 + 0.1637 = 0.9824$

$F(2) = P(X \leq 2) = 0.9824 + 0.0164 = 0.9988$

So we have. [U is the uniform random number in [0.1)

X	Interval for U
0	$0 \le U \le 0.8187$
1	$0.8187 < U \le 0.9824$
2	$0.9824 < U \le 0.9988$

when . $U = 0.4357 \rightarrow$ first range $\rightarrow X = 0$

$U = 0.4146 \rightarrow$ first range $\rightarrow X = 0$

$U = 0.8353 \rightarrow$ 2nd " $\rightarrow X = 1$

(U denotes a Uniform random numbers between 0 and 1, used to generate Poisson random variates by comparison with Cumulative probabilities.)

So the Poisson variates are $\boxed{0, 0, 1}$

5(c) The life of a device used to inspect cracks in aircraft wings is given by X. The cdf of the devices lifetime, in years, is as follows,

$$\cancel{F(x) = 1-e}^{x/2} \quad \text{for } x \geq 0$$

$$F(x) = \frac{1}{2}\int_0^x e^{-t/2}\,dt$$

(i) Find the probability that the device will last for 2 years.

(ii) Find the probability that the device will last between 2 and 3 years.

Ans: CDF of the devices lifetime.

$$F(x) = \int_0^x 1/2\, e^{-t/2}\, dx$$

$$= \frac{1}{2}\left[e^{-t/2} \cdot \frac{1}{-\frac{1}{2}}\right]_0^x$$

$$= \frac{1}{2}\left[-e^{-x/2} \cdot 2\right]_0^x$$

$$= \frac{1}{2}\left[-2e^{-x/2} + 2 \cdot e^0\right]$$

$$= 1 - e^{-x/2}$$

PDF for the defice $f(x) = \frac{1}{2} e^{-x/2}$

the question is (i) $P(x=2)$

(ii) $P(2 \le x \le 3)$

(i) For a Continuous random variable the Probability at a single Point is always 0.

$P(x=2) = 0$

Beause for continuous distributions, Probabilities are computed over interval, not at exact Point.

(ii) $P(2 \le x \le 3)$

$= f(3) - f(2)$

$f(x) = 1 - e^{-x/2}$

$f(3) = 1 - e^{-3/2} = 1 - e^{-1.5} = 0.7769$

$f(2) = 1 - e^{-2/2} = 1 - e^{-1} = 0.6321$

$P(2 \le x \le 3) = f(3) - f(2)$

$= 0.7769 - 0.6321$

$= 0.145$

6 (c)

Used the mixed congruential method to generate a sequence of three two digit numbers with $x_0 = 37$, alpha $= 7 = \alpha$, $c = 29$ and $m = 100$

<u>Ans:</u> given, seed $x_0 = 37$

multiplier, $\alpha = 7$

increment, $c = 29$

modulus, $m = 100$

Mixed congruential method,

$$\boxed{X_{i+1} = (aX_i + c) \bmod m}$$

① when, $i = 0$, $x_{0+1} = (a \cdot x_0 + c) \bmod m$

$x_1 = ((7 \times 37) + 29) \bmod 100$

$= (259 + 29) \bmod 100$

$= 288 \bmod 100 = 88$

② when $i = 1$, $x_2 = (a x_1 + c) \bmod m$

$= (7 \times 88 + 29) \bmod m$

$= 645 \bmod 100 = 45$

③ when $i = 2$, $x_3 = ((7 \times 45) + 29) \bmod 100$

$= 344 \bmod 100 = 44$

The three generated two ~~bit~~ digit random number 88, 45, 44 pdf 20

7 a From the Pdf given by Sir

For the following multiplicative generator, compute z_i for enough values of $i >= 1$ to cover entire cycle. [6]

i) $z_0 = 1$ $a = 11$ $m = 16$

ii) $z_0 = 2$ $a = 11$ $m = 16$

iii) $z_0 = 1$ $a = 2$ $m = 13$

iv) $z_0 = 2$ $a = 3$ $m = 13$

(i) We have $z_i = (a z_{i-1}) \bmod m = (11 \times z_{i-1}) \bmod 16$

$i = 0$, $z_0 = 1$

$i = 1$, $z_1 = (11 \times 1) \bmod 16 = 11$

$i = 2$, $z_2 = (11 \times 11) \% 16 = 121 \% 16 = 9$

for the first case,

$z_0 = 1$, $a = 11$ $m = 16$

i	0	1	2	3	4	5
z_i	1	11	9	3	1	11

As, it repeats after 4 value.

Period $= 4 = \frac{m}{4}$

Do same as it for ii, iii, iv

(ii) We have $z_i = (a z_{i-1}) \bmod m = (11 \times z_{i-1}) \bmod 16$

$z_0 = 2$, $a = 11$ $m = 16$

i	0	1	2	3	—	—	—
z_i	2	6	2	6	—	—	—

repeats after 2

$\therefore$ Periods $= 2$

(iii) Here . $Z_0 = 1$ $a = 2$ $m = 13$

$Z_i = (a\ Z_{i-1}) \bmod 13$

$= (2 \cdot Z_{i-1}) \bmod 13$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
Z_i	1	2	4	8	3	6	12	11	9	5	10	7	1

it repeats after 12 values.

Period = 12 = m−1

(iv) Here,

$Z_0 = 2$, $a = 3$, $m = 13$,

$Z_i = (a Z_{i-1}) \bmod 13$

$= (3\ Z_{i-1}) \bmod 13$

i	0	1	2	3
Z_i	2	6	5	2

Period = 3

7(b)

Find first three random variables in [0,1] using $x_0 = 27$, $a=8$, $c=47$ $m=10^2=100$

Ans: For generating random numbers we use mixed congruential method.

$$\boxed{X_{i+1} = (a X_i + c) \bmod m}$$

where, $x_0 = 27$ when $i=0$

multiplier, $a = 8$

increment, $c = 47$

modulus, $m = 100$

① Compute X_1 when $i=0$

$= (aX_0 + c) \bmod m$

$= (8 \times 27 + 47) \bmod 100$

$= (216 + 47) \bmod 100$

$= 263 \bmod 100$

$= 63$

as range[0,1] $\therefore X_1 = \frac{63}{100} = 0.63$

② Compute x_2

$(aX_1 + c) \bmod 100$

$= (8 \times 63 + 47) \bmod 100$

$= (504 + 47) \bmod 100$

$= 551 \bmod 100$

$= 51$

$\therefore x_2 = \frac{51}{100} = 0.51$

③ Compute x_3

$(a x_2 + c) \bmod 100$

$= ((8 \times 51) + 47) \bmod 100$

$= (408 + 47) \bmod 100$

$= 455 \bmod 100$

$= 55$

$\therefore x_3 = \frac{55}{100} = 0.55$

$\therefore$ The first three random variables in range [0,1]

is $x_1 = 0.63$, $x_2 = 0.51$, $x_3 = 0.55$

7(c)

The sequence of numbers 0.54, 0.73, 0.98, 0.11, and 0.68 has been generated. Use the Kolmogorov-Smirnov test with $a = 0.05$ to check uniformity.

Ans: Given data

0.54, 0.73, 0.98, 0.11, 0.68

Size, $N = 5$

Sort data => 0.11, 0.54, 0.68, 0.73, 0.98

$a = 0.05$

i	1	2	3	4	5	
R_i	0.11	0.54	0.68	0.73	0.98	
$\frac{i}{N}$	0.20	0.40	0.60	0.80	1.0	
$\frac{i}{N} - R_i$	0.09	−0.14	−0.08	0.07	0.02	$-(D^+)$
$R_i - \frac{i-1}{N}$	0.11	0.34	0.28	0.13	0.18	$+(D^-)$

$\max(D^+) = 0.09$, $\text{Max}(D^-) = 0.34$

$D = \max(D^+, D^-) = (0.09, 0.34) = 0.34$

$D_\alpha \Big|_{\alpha = 0.05, \; N = 5} = 0.565$ (From Kolmogorov Smirnov Critical values table)

এটা মুখস্থ রাখার No idea

as $D < D_\alpha$

So, the hypothesis is not rejected

so, the numbers passed the uniformity test.

At a grocery store with one counter customers arrive at random from 1 to 8 ~~mines~~ min apart (each inter-arrival times has the same probability occurrence) The service times vary from 1 to 6 minutes with the probabilities as 0.10, 0.20, 0.30, 0.25, 0.10 and 0.05 respectively.

Analyze the system by simulating the arrival and service of 15 customers. (Justifying your situation and requirements, you can choose your required random values) [12] [এখানে 10 customer এর জন্য করা হলো]

Ans: Random digit for Inter arrival time

IAT = 64, 112, 678, 289, 871, 583, 139, 423, 39.

Random digit for service time = 84, 18, 87, 81, 6, 91, 79, 9, 64, 38.

given,

Service time:	1	2	3	4	5	6
Probability:	0.10	0.20	0.30	0.25	0.10	0.05

The Probabili of Inter arrival time [IAT] = $\frac{1}{8}$ = 0.125

Numbers of customer = 10

Service time = 1 to 6 min.

Table-1. (Distribution of service time)

Time between arrival (min)	Probability	Cumulative Probability	Random digit Assignment
1	0.125	0.125	001 - 125
2	0.125	0.250	126 - 250
3	0.125	0.375	251 - 375
4	0.125	0.500	376 - 500
5	0.125	0.625	501 - 625
6	0.125	0.750	626 - 750
7	0.125	0.875	751 - 875
8	0.125	1.000	876 - 000

Note: এখানে 10 customer এর জন্য করা হলো।

15 customer এর জন্য।

IAT = (2-15) - 14 টি। কারন customer-1 arrive at time 0.

এবং value গুলো 99 এর মধ্যে নিলে range : ~~00~~ 01- 99 বা

এর মধ্যে হবে। range 00 বা 01 যেকোনো ~~থ~~ থেকে শুরু করা যাবে।

Table two - Service time distribution

Service time	Probability	Cumulative Probability	Random digit Assignment
1	0.10	0.10	01 - 10
2	0.20	0.30	11 - 30
3	0.30	0.60	31 - 60
4	0.25	0.85	61 - 85
5	0.10	0.95	86 - 95
6	0.05	1.00	96 - 00

Table - 3 Arrival time distribution

Customer	Random digit	IAT
1	—	—
2	64	1
3	112	1
4	678	6
5	289	3
6	871	7
7	583	5
8	139	2
9	423	4
10	39	1

এটা table 1 থেকে আসবে, random digit এর range এ
পাবে তার arrival time in min.

Table-4 Service time distribution

এটা table two থেকে আসবে,

Customer	Random digit	Service time
1	84	4
2	18	2
3	87	5
4	81	4
5	6	1
6	91	5
7	79	4
8	9	1
9	64	4
10	38	3

Table-5. Simulation table

customer	IAT	Arrival time	Service time	Time service begins	Waiting time in queue	Time service ends	Time customer spends in the system	Idle time of servers
1	–	0	4	0	0	4	4	0
2	1	1	2	4	3	6	5	0
3	1	2	5	6	4	11	9	0
4	6	8	4	11	3	15	7	0
5	3	11	1	15	4	16	5	0
6	7	18	5	18	0	23	5	2
7	5	23	4	23	0	27	4	0
8	2	25	1	27	2	28	3	0
9	4	29	4	29	0	33	4	1
10	1	30	3	33	3	36	6	0
	30		33		19			0.3

Total IAT = 30

Total service time = 33

Total waiting time = 19

Total idle time = 3

Total time customer spend in the system = 52

Average waiting time $= \frac{19}{10} = 1.9$

Average service time $= \frac{33}{10} = 3.3$

Waiting time for those who wait

$= \frac{19}{6} = 3.166$

Average time customer spends in system

$= \frac{52}{10} = 5.2$

Average Inter arival time $= \frac{\text{total IAT}}{\text{Number of IAT}}$

$= \frac{30}{9}$

$= 3.33$

Probability of customer has to wait in queue $= \frac{6}{10} = 0.6$

The probability of idle time of server

$$= \frac{\text{total idle time of server}}{\text{time service ends at last customer}}$$

$$= \frac{3}{36} = 0.033$$