Markov Chain

A markov chain is a type of stochastic (random) Process that moves from one state to another, where the Probability of moving to the next state depends only on the current state, not on the Past history

The stochastic Process $\{x_n, n \geq 0\}$ is called a markov chain if for $j, k, j_1, \ldots, j_{n-1} \in N/I$

$$Pr[X_n = k \mid X_{n-1} = j ; x_{n-2} = j_1 \ldots x_0 = j_{n-1}]$$

$$= Pr[X_n = k \mid X_{n-1} = j] = P_{jk}$$

whenever the first number is defined.

The outcomes are called the states of markov chain; if x_n has the outcome $j (x_n = j)$ the Process is said to be at state j at nth trial. P_{jk} denote the transition property.

Different Kind of Markov Chain

By time Parameter:

① Discrete time Markov Chain: States change at discrete time steps.

Ex: A stock's value, measured only at the end of each day.

Characteristics: Uses a constant transition Probability matrix.

② Continuous time Markov Chain: states can change at any instant in time

Ex: Queue system (arrival, service)

Characteristics: Uses a time dependent transition matrix.

By Homogeneity

① Homogeneous Markov Chain: If the transition Probability P_{jk} is independent of n (time), then the markov Chain is called homogeneous.

Ex: Weather Model

Benefit: Simplifies analysis and Prediction of long term behaviour.

(2) Non homogeneous Markov Chain: If the transition Probability P_{jk} is dependent of n(time), the Markov Chain is said to be non homogeneous Markov Chain.

Ex: ~~Of~~ Financial Markets.

By State Space:

(1) Markov Chain on a countable state space: The set of possible states is finite or countable infinite.

(2) Markov chain on a general (or continuous) state space: The state space is continuous or a general measurable space.

By Behaviour of States:

(1) Ergodic Markov chain: Chains with both positive reccurence and aperiodicity, ensuring a unique stationary distribution exists.

(2) Absorving Markov chain: Contain one or more states, that, once entered, can not be left.

Finite Markov Chain: A markov chain $\{X_n, n \geq 0\}$ with K states, where K is finite, is said to be finite Markov Chain.

Infinite Markov Chain: A markov chain $\{X_n, n \geq 0\}$ with K states, where the possible values of X_n from an infinite set, then the Markov chain is said to be denumerable infinite

Transition probability

The conditional probability $P_n = [X_{n+1} = j \mid X_n = i] = P_{ij}$ is known as transition probability. That is, the transition probability referes to the probability is in state i and will be in state j is the next step. Here the transition is one step and P_{JK} is called one-step transition probability. Transition probability must satisfy

(i) $P_{ij} \geq 0$ and (ii) $\sum_{J} P_{iJ} = 1$

The conditional probability $P_r = [X_{n+m} = J / X_n = i] = P_{JK}^{m}$ is known as m step transition probability. That is, the transition probability that refers to the probability that the process is in state i and will be in state J in the next m step. Here the transition is m step and P_{JK}^{m} is called m-step transition probability.

(i) $P_{iJ}^{m} \geq 0$ and (ii) $\sum_{J} P_{JK}^{m} = 1$

Transition Matrix: The matrix of 1st order transition Probability of a Markov Chain ~~with~~ is called transition Probability matrix.

If $\{X_n, n \geq 0\}$ is a Markov Chain with transition probability P_{jk} then transition Probability matrix is given by

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} = [P_{ij}]$$

$i, j = 0, 1, 2 \cdots n$

where $P_{ij} \geq 0$ and $\sum_{k=1}^{n} P_{jk} = 1$ for all j

Stochastic Matrix: The transition Probability matrix P is said to be stochastic matrix if it has a square matrix with non-negative elements and unit row sums.

Not very important

Example-1: Consider a Markov Chain with the following TPM (Transition Probability Matrix)

$$P = \begin{bmatrix} 3/4 & \frac{1}{4} & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

with Initial Distribution $a_i = P[X_0 = i] = \frac{1}{3}$; $i = 0, 1, 2 \ldots$

Find (1) P_{01}^{2} . P_{02}^{2}

(2) $P[X_n = 1, X_0 = 0]$

(3) $P(0,1,1) = P(X_0 = 0, X_1 = 1, X_2 = 1)$

(4) $P[X_i = 2]$

(5) $P(0,0,1,1) = P(X_0 = 0, X_1 = 0, X_2 = 1, X_3 = 1)$

Ans: $P^2 = P \cdot P$

$$= \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \times \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 2/4 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{16} + \frac{1}{16} + 0 & \frac{3}{16} + \frac{2}{16} + 0 & 0 + \frac{1}{16} + 0 \\ 3/16 + 2/16 + 0 & 1/16 + \frac{4}{16} \; \frac{3}{16} & 0 + \frac{2}{16} + \frac{1}{16} \\ 0 + 3/16 + 0 & 0 + \frac{6}{16} + \frac{3}{16} & 0 + \frac{3}{16} + 1/16 \end{bmatrix}$$

$$= \begin{bmatrix} 10/16 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix}$$

$P(A,B) = P(A|B) \cdot P(B)$

① $P_{01}^{(2)} = P[x_2=1 | x_0=0] = \frac{5}{16}$

$P_{02}^{(2)} = P[x_2=2 | x_0=0] = 1/16$

② $P[X_n=1, x_0=0]$

$P[X_2=1, X_0=0] = [X_2=1 | X_0=0] \times P[X_0=0]$

$= 5/16 \times 1/3$

$= \frac{5}{48}$

③ $P(0,1,1) = P(x_0=0, x_1=1, x_2=1)$

$= P[x_1=1 | x_0=0] \; P[x_2=1 | x_1=1] \; P[x_0=0]$

$= \frac{5}{16} \times \frac{1}{4} \times 1/3 = \frac{5}{192}$

④ $P[x_1=2] = \sum_{i=0}^{2} P_{i2}^{(1)} a_i$

$= P_{02} a_0 + P_{12} a_1 + P_{22} a_2$

$= 0 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{3}$

$= \frac{1}{6}$

⑤ $P(0,0,1,1) = P(x_0=0, x_1=0, x_2=1, x_3=1)$

$= P[X_1=0 | x_0=0] \; P[x_2=1 | x_1=0] \; P[X_3=1 | x_2=1]$
$P[X_0=0]$

$= \frac{5}{16} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{3} = \frac{5}{768}$

Chapman-Colmogorov Equation:

The m-step transition Probability is denoted by

$$P_r[X_{n+m}=K \mid X_n=J] = P_{JK}^{(m)} \quad \text{---} (1)$$

If $P_{JK}^{(m)}$ gives the Probability that from the state J at nth trial, the state K is reached at $(m+n)^{th}$ trial in m steps;

the Probability of transition from the state J to the state K in exactly m steps.

Since the right side of the equation (1) is independent of n. so, the chain is homogeneous.

The one-step transition Probabilities $P_{JK}^{(1)}$ are denoted by P_{JK} for simplicity.

Let's consider for, two step transition Probability matrix $P_{JK}^{(2)} = P_r[X_{n+2}=K \mid X_n=J]$

The state K can be reached from the state J in two steps through some intermediate state r.

Consider a fixed value of r, we have,

$$P_r[X_{n+2}=k \, ; X_{n+1}=r / X_n=j\}$$

$$= P_r\{X_{n+2}=k \mid X_{n+1}=r,\ X_n=j\}\ P_r[X_{n+1}=r \mid X_n=j]$$

$$= P_r\{X_{n+2}=k \mid X_{n+1}=r\}\cdot P_r\{X_{n+1}=r \mid X_n=j\}$$

$$= P_{rk}\cdot P_{jr} \qquad r=1,2$$

Now we have,

$$P_{jk}^{(2)} = P_r[X_{n+2}=k \mid X_n=j]$$

$$= \sum_r P_r[X_{n+2}=k \, , X_{n+1}=r \mid X_n=j]$$

$$= \sum_r P_{rk}\cdot P_{jr}$$

By induction, we have,

$$P_{jk}^{(m+1)} = P_r\{X_{n+m+1}=k \mid X_n=j\}$$

$$= \sum_r P_r[X_{n+m+1}=k \mid X_{n+m}=r]\ P_r[X_{n+m}=r \mid X_n=j]$$

$$= \sum_r P_{rk}\cdot P_{jr}^{m}$$

Similarly we get,

$$P_{jk}^{m+1} = \sum_r P_{rk}^{m}\cdot P_{jr}$$

In general we can write

$$P_{jk}^{m+n} = \sum_r P_{rk}^{(m)}\cdot P_{jr}^{(n)} = \sum_r P_{jr}^{(m)}\cdot P_{rk}^{(n)} \longrightarrow (2)$$

This equation is a special case of Chapman Kolmogorov equation, which satisfied by the transition Probabilities of a Markov-Chain.

From the equation (2) we can write,

$$P_{JK}^{(m+2)} \geq P_{rK}^{(m)} \cdot P_{Jr}^{(n)} \quad \text{for any } r$$

The Chapman Kolmogorov equation provides a method for computing these transition probabilities.

let $P = (P_{JK})$ denote the transition matrix of the unit step transitions and $P^{(m)} = \left(P_{JK}^{(m)}\right)$ denote the transition matrix of the m-step transitions.

for $m=2$ we have,

$$P^{(2)} = P \cdot P = P^2$$

Similarly $P^{m+1} = P^m \cdot P = P \cdot P^m$ and by indication $P^{(m)} = P^{(m-1+1)} = P^{m-1} \cdot P = P^m$

That is the m-step transition matrix may be obtained by multiplying the matrix P by itself m times.

Classification of States According to Communication:

i. Accessible

ii. Not accessible

iii. Communicate

(i) Accessible states: state j of a Markov Chain is said to be accessible from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 1$. That is state j is accessible from state i if and only if, starting in i, it is possible that the process will ever enter state j.

The relation is $i \rightarrow j$

(ii) Non-accessible: If for all n, $p_{ij}^{(n)} = 0$, then j is not accessible from i. $i \not\rightarrow j$

(iii) Communicate state: Two state i and j are said to be communicate state if each is accessible from other. $i \leftrightarrow j$

Properties of Communicate state

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Properties

(1) State i communicate with itself.

$P_{ii}^{(m)} > 0$. Notation : $i \leftrightarrow i$

self communicate.

(2) The relation is clearly systomatic.

if state i communicate with state j then state j also communicate with i.

$i \leftrightarrow j$, $j \leftrightarrow i$

(3) Transivity property.

if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

Theorem: Prove that, if state i communicate with state J, and state J communicate with state K then state i communicate with state K. $i \leftrightarrow J$ and $J \leftrightarrow K$ then $i \leftrightarrow K$

Proof:

Since i and j communicate with each other,

that is $i \leftrightarrow J$

$$P_{iJ}^{(m)} > 0 \text{ and } P_{Ji}^{(s)} > 0$$

and $j \leftrightarrow k$

$$P_{JK}^{(n)} > 0 \text{ and } P_{KJ}^{(t)} > 0$$

Now $P_{iK}^{(m+n)} = \sum_{l=0}^{\infty} P_{il}^{(m)} \cdot P_{lK}^{(n)}$

$$= \sum_{l \neq J} P_{il}^{(m)} \cdot P_{lK}^{(n)} + P_{iJ}^{(m)} \cdot P_{JK}^{(n)}$$

$$\therefore P_{iK}^{(m+n)} \geq P_{iJ}^{(m)} \cdot P_{JK}^{(n)}$$

$$\therefore P_{iK}^{(m+n)} > 0 \quad \text{as } P_{iJ}^{(m)} > 0 \text{ and } P_{JK}^{(n)} > 0$$

again,

$$P_{Ki}^{(s+t)} = \sum_{q=0}^{q} P_{Kq}^{(t)} \cdot P_{qi}^{(s)}$$

$$= \sum_{q \neq J} P_{Kq}^{(t)} \cdot P_{qi}^{(s)} + P_{KJ}^{(t)} \cdot P_{Ji}^{(s)}$$

$$\because P_{Ki}^{(s+t)} \geq P_{KJ}^{(t)} \cdot P_{Ji}^{(s)}$$

$\therefore P_{Ki}^{(s+t)} > 0$ as $P_{KJ}^{(t)} > 0$ and $P_{Ji}^{(s)} > 0$

since $P_{iK}^{(m+n)} > 0$ and $P_{Ki}^{(s+t)} > 0$

so that $i \leftrightarrow K$

Return state . Periodicity - $\boxed{\text{Problem}}$

Closed set: Closed set is a set of states such that no state of that set is reachable by any state outside the set.

Absorving State: If a closed set contains only one state then the state is known as absorving state.

First Entrance Decomposition Formula:

For a markov chain $\{X_n\}_{n=0}^{\alpha}$,

$$P_{ij}^{(n)} = \sum_{r=0}^{n} f_{ij}^{(r)} \cdot P_{jj}^{(n-r)} \; ; \; n \geq r$$

Proof: Starting from state i, the Process will be in state j not necessarily for the first time at the n^{th} step ($P_{ij}^{(n)}$ is the union of the following mutually exclusive events.

Starting from state i the Process will enter to the state j for the first time -

(1) For the 1st time at 1st step and again in state j at $(n-1)^{st}$ step, that is $f_{ij}^{1} P_{jj}^{n-1}$

(2) For the first time at 2nd step and again in ~~step~~ state j at $(n-2)^{nd}$ step; $f_{ij}^{(2)} P_{jj}^{(n-2)}$.,

similarly, for the 1st time at rth

step and again in state j at $(n-r)^{th}$ step

that is $f_{ij}^{(r)} \cdot P_{jj}^{(n-r)}$

$$P_{ij}^{n} = f_{ij}^{(1)} P_{jj}^{(n-1)} + f_{ij}^{(2)} \cdot P_{jj}^{(n-2)} + \cdots + f_{ij}^{(r)} \cdot P_{jj}^{(n-r)}$$

$$\cdots + f_{ij}^{n} P_{jj}^{(0)}$$

$$\Rightarrow P_{ij}^{n} = \sum_{r=1}^{n} f_{ij}^{(r)} \; P_{jj}^{(n-r)}$$

$$P_{ij}^{n} = \sum_{r=0}^{n} f_{ij}^{(r)} \cdot P_{jj}^{(n-r)} \qquad \left[f_{ij}^{(0)} = 0 \right]$$

(Prooved)

<u>Theorem:</u> Show that for any state j $\sum_{n=0}^{\alpha} P_{jj}^{(n)} = \frac{1}{1-f_{jj}}$

and if state j is recurrent then $\sum_{n=0}^{\alpha} P_{jj}^{(n)} = \alpha$

<u>Proof:</u> We know that,

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} \; P_{jj}^{(n-r)}$$

multiply both sides by t^{n} and adding for all $n \geq 1$ we get,

$$\sum_{n=1}^{\alpha} P_{jj}^{(n)} t^{n} = \sum_{n=1}^{\alpha} \sum_{r=0}^{n} f_{jj}^{(r)} \cdot P_{jj}^{(n-r)} \cdot t^{n}$$

$$\Rightarrow \sum_{n=1}^{\alpha} P_{jj}^{(n)} t^{n} = \sum_{r=0}^{n} \sum_{n=1}^{\alpha} f_{jj}^{(r)} \cdot P_{jj}^{(n-r)} \cdot t^{n}$$

theorem: From State j the mean number of returns to state j is $\frac{F_{jj}}{1-F_{jj}}$

theorem; If state i is reccurent and $i \longleftrightarrow j$ then state j is reccurent.