Mid 1 (20-21)

Why do we need to study numerical methods in CSE? Briefly describe different types of iteration methods and the limitation of each.

~~Ans~~ **Ans:** As same as 18-19(1(a)

Common iterative method for root finding:

1. **Bisection Method:** Divides intervals repeatedly.

$$c = \frac{a+b}{2}$$ where $[a,b]$ is interval and c is root.

Limitation: slow convergence, require sign change.

2. **False Position Method:**

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

where, $[a,b]$ is interval, x is root and $f(a) \cdot f(b) < 0$.

Limitation: Can be slow if one end remains fixed.

3. **Newton - Rapshon Method:**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Limitation: Need derivatives, may diverge if initial guess is poor.

4. Secant Method: (এখানে derivate লাগে না)

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Limitation: May converge slowly,
less stable than Newton-Raphson.

5. Fixed-Position Method:

$$x = g(x) \quad \text{or} \quad x_{n+1} = g(x_n)$$

converge only if $|g'(x)| < 1$. otherwise diverge.

Limitation:

(4) Explain the Newton-Raphson iterative method formula? What are the necessities using Newton-Raphson method in numerical analysis? Explain all the steps to find out the roots of a function using this method.

Ans: For a function $f(x)$ ~~newton R~~ whose root we want to ~~be~~ find, the Newton-Raphson formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where,

x_n = current approximation

x_{n+1} = Next approximation

$f'(x_n)$ = derivative of $f(x_n)$ at x_n

Necessity of Newton-Raphson Method:

1. Provides rapid convergence for finding real roots.
2. ~~when~~ Works effeciently when $f'(x)$ is known.
3. Suitable for polynomials and transcendental ~~fu~~ function and reduces the number of iterations.

Steps of Newton-Raphson Method to find root:

1. Find point ~~on~~ a and b so that $f(a) \cdot f(b) < 0$ and $a < b$.
2. ~~b~~ Take interval $[a,b]$ and find $x_0 = \frac{a+b}{2}$
3. Find $f(x_0)$ and $f'(x_0)$ then ~~can~~ calculate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

4. If $f(x_1) = 0$ then x_1 is an exact root

else $x_0 = x_1$

5. Repeat step 3 and 4 until $f(x_i) = 0$

or ~~$f(x_i)$~~ $\leq$ 0.001 or very lower difference.

$|x_{n+1} - x_n| \leq 0.001$

Note

Example: ~~Unit~~ Using Newton Raphson method find the root of $f(x) = x^3 - x - 1$

Ans: ~~Let $a = 1$ and~~

Given, $f(x) = x^3 - x - 1$

$f'(x) = 3x^2 - 1$

Let, $a = 1$ and $b = 2$

$\therefore f(a) = f(1) = -1$ and $f(b) = f(2) = 5$

so $f(a) \cdot f(b) < 1$, means root lies between the interval $[a, b] = [1, 2]$.

$\therefore x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$

Example:

$x = \sqrt{12}$ Newton Rapshon method

Given that, $x = \sqrt{12}$

$\Rightarrow x^2 = 12$

$\Rightarrow x^2 - 12 = 0$

$\therefore f(x) = x^2 - 12$

$\therefore f'(x) = 2x$

Let, $a = 3$ and $b = 4$

$f(a) = -3$

$f(b) = 4$

$\therefore f(a) * f(b) < 0$: that means root lies in [3, 4]

$x_0 = \frac{3+4}{2} = 3.5$

$f(x_0) = 0.25$

$f'(x_0) = 7$

iteration 1:

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \frac{0.25}{7} = 3.4642$

tolerance: $|x_1 - x_0| = 0.036 \nless 0.001$

So, $f(x_1) = -0.000704$ and $f'(x_1) = 6.928$

iteration 2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.464 - \frac{-0.000704}{6.928} = 3.469$$

tolerance $= |x_2 - x_1| = |3.464 - 3.464| = 0 \le 0.001$

~~So~~ As torelance is 0, so the final root is

$= 3.464$.

Mid (21-22)

Newton Rapshon method:

Given that

$$f(x) = 2x^3 - 2x - 5$$

$$f'(x) = 6x^2 - 2$$

Let $a = 1$ and $b = 2$

$f(a) = -5$ and $f(b) = 7$

So $f(a) \cdot f(b) < 0$, that means root lies in the interval $[1, 2]$.

$$x_0 = \frac{1+2}{2} = 1.5$$

iteration 1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.5 - \frac{-1.25}{11.5}$$

$\therefore x_1 = 1.6086$

tolerance: $|x_1 - x_0| = |1.6086 - 1.5| = 0.1086 \nleq 0.001$.

So, $f(x_1) = 0.1076$

iteration 2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.6086 - \frac{0.1076}{13.5255} = 1.6006.$$

tolerance: $|x_2 - x_1| = |1.6086 - 1.6006| = 0.0062 \nleq 0.001$.

So, $f(x_2) = \cancel{0.0159}\ 0.000019$

iteration 3:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6006 - \frac{0.000019}{13.3715} = 1.6006$$

~~tolerance: $|x_3 - x_2| = |1.6017 - 1.6006| = 0.0011 \nleq 0.001$~~

~~So $f(x_3) = 0.0147$~~

~~$f'(x_3) = 13.3926$~~

tolerance: $|x_3 - x_2| = |1.6006 - 1.6006|$

$= 0 \leq 0.001$

$\therefore$ the root is 1.6006.

Mid 21-22

False Position Method:

Given that

$f(x) = e^x - 3x$; interval $[0, 1]$ (a = 0, b = 1)

$\therefore f(a) = 1$

$\therefore f(b) = -0.28$

So, $f(a) \cdot f(b) < 0$; means root in the $[0, 1]$

iteration 1:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-0 \times 0.28 - 1 \times 1}{-0.28 - 1} = 0.7812$$

$\therefore f(x_1) = -0.1595$

So, $f(a) \cdot f(x_1) < 0$ replaced b by x_1

iteration 2:

$$x_2 = \frac{af(x_1) - x_1 f(a)}{f(x_1) - f(a)} = \frac{-0 \times 0.1595 - 0.7812 \times 1}{-0.1595 - 1} = 0.6737$$

$\therefore f(x_2) = -0.0596$

So, $f(a) \cdot f(x_2) < 0$; replaced b by x_2.

iteration 3:

$$x_3 = \frac{af(x_2) - x_2 f(a)}{f(x_2) - f(a)} = \frac{-0 \times 0.0596 - 0.6737 \times 1}{-0.0596 - 1} = 0.6358$$

$\therefore f(x_3) = -0.0188$

So, $f(a) \cdot f(x_3) < 0$; replaced b by x_3

iteration 4:

$$x_4 = \frac{af(x_3) - x_3 f(a)}{f(x_3) - f(a)} = \frac{-0.6358}{-1.0188} = 0.6240$$

$\therefore f(x_4) = -0.0056$

So, $f(a) \cdot f(x_4) < 0$; replaced b by x_4

iteration 5:

$$x_5 = \frac{af(x_4) - x_4 f(a)}{f(x_4) - f(a)} = \frac{-0.6240}{-1.0056} = 0.6205$$

$f(x_5) = -0.0016$

So, $f(a) \cdot f(x_5) < 0$; replaced ~~by~~ b by x_5

iteration 6:

$$x_6 = \frac{af(x_5) - x_5 f(a)}{f(x_5) - f(a)} = \frac{-0.6205}{-1.0016} = 0.6205$$

tolerance: $|x_6 - x_5| = |0.6205 - 0.6205| = 0 \leq 0.001$.

So the root is approximately 0.6205.

<u>Mid 21-22</u>

Fixed Position Method (Normal iteration):

Given that

$$\cancel{f(x) = x^2 - 4x + 10}$$

$$\cancel{f'(x) = 2x - 4}$$

$$x^2 - 4x - 10 = 0$$

for interval finding :

x	0	1	2	3	4	5	6
$f(x)$	-10	-13	-14	-13	-10	-5	2

So, $a = 5$ and $b = 6$ because $f(a) \cdot f(b) < 0$.

~~So~~ & Means root lies into interval $[5, 6]$.

Now, $x^2 - 4x - 10 = 0$

$$\Rightarrow x^2 = 4x + 10$$

$$x = \sqrt{4x + 10}$$

$$\therefore g(x) = \sqrt{4x + 10}$$

$$\therefore g'(x) = \frac{2}{\sqrt{4x + 10}}$$

$g'(5) = \frac{2}{\sqrt{4 \times 5 + 10}} = 0.365 < 1$ and $g'(6) = \frac{2}{\sqrt{4 \times 6 + 10}} = 0.342 < 1$

$\therefore |g'(x)| < 1$ method converges.

let initial value $x_0 = 5.5$

iterations:

$$x_1 = \sqrt{4x_0 + 10} = \sqrt{4\times5.5+10} = 5.656$$

iteration 2:

$$x_2 = \sqrt{4x_1 + 10} = \sqrt{4\times5.656+10} = 5.711$$

iteration 3:

$$x_3 = \sqrt{4\times 5.711 + 10} = 5.730$$

iteration 4:

$$x_4 = \sqrt{4\times5.730 + 10} = 5.737$$

iteration 5:

$$x_5 = \sqrt{4\times5.737 + 10} = 5.740$$

iteration 6:

$$x_6 = \sqrt{4\times5.740 + 0} = 5.741$$

tolerance: $|x_6 - x_5| = |5.741 - 5.740| = 0.001 \leq 0.001$

So the root is 5.741