$x-x_0 = hv$
$x-x_1 =$

Chapter – 3(c)

"Central Difference Interpolation Formulas"

☐ Derive Gauss's forward formula for equal intervals.

Solution:

We have, Newton's general interpolation formula:

$$y = f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$
$$+ \ldots \ldots$$

Putting $x_0 = x_0$, $x_1 = x_0+h$, $x_2 = x_0-h$, $x_3 = x_0+2h$, $x_4 = x_0-2h$ etc we get,

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_0+h) + (x-x_0)(x-x_0-h)$$
$$f(x_0, x_0+h, x_0-h) + (x-x_0)(x-x_0-h)(x-x_0+h)$$
$$f(x_0, x_0+h, x_0-h, x_0+2h) + (x-x_0)(x-x_0-h)$$
$$(x-x_0+h)(x-x_0-2h) f(x_0, x_0+h, x_0-h, x_0+2h, x_0-2h)$$
$$+ \ldots \quad \ldots$$

Now put, $v = \frac{x-x_0}{h}$ i.e. $x-x_0 = hv$, Then we have,

$$f(x) = f(x_0) + hv\, f(x_0, x_0+h) + hv(hv-h)$$
$$f(x_0-h, x_0, x_0+h) + hv(hv-h)(hv+h)$$
$$f(\cancel{x_0}-h, x_0\cancel{+2h}, x_0+h, x_0+2h) + hv(hv-h)(hv+h)$$
$$(hv-2h) f(\cancel{x_0, x_0+h}, x_0-h, x_0, x_0+h, x_0+2h, x_0-2h)$$
$$+ \ldots \quad (1)$$

But we have, from the relation between simple differences & divided differences,

$$f(x_0, x_0+h) = \frac{\Delta y_0}{h}$$

$$f(x_0-h, x_0, x_0+h) = \frac{\Delta^2 y_{-1}}{2!h^2}$$

$$f(x_0-h, x_0, x_0+h, x_0+2h) = \frac{\Delta^3 y_{-1}}{3!h^3}$$

$$f(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h) = \frac{\Delta^4 y_{-2}}{4!h^4} \text{ etc.}$$

Substituting these values in (1) we get,

$$f(x) = y_0 + hu \cdot \frac{\Delta y_0}{h} + hu(hu-h) \cdot \frac{\Delta^2 y_{-1}}{2!h^2} + hu(hu-h)(hu+h)\frac{\Delta^3 y_{-1}}{3!h^3} + hu(hu-h)(hu+h)(hu-2h) \cdot \frac{\Delta^4 y_{-1}}{4!h^4} + \cdots$$

$$\Rightarrow f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)\cancel{(u+1)}}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \cdots$$

This is the Gauss forward formula for equal intervals.

☐ Derive Gauss's backward formula for equal intervals.

Solution:

We have Newton's general interpolation formula:

$$y = f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0,x_1,x_2,x_3,x_4)$$
$$+ \dots\dots$$

Putting $x_0 = x_0$, $x_1 = x_0 - h$, $x_2 = x_0 + h$, $x_3 = x_0 - 2h$, $x_4 = x_0 + 2h$ etc we get,

$$f(x) = f(x_0) + (x-x_0)\cancel{+h} f(x_0, x_0-h) + (x-x_0)(x-x_0+h) f(x_0, x_0-h, x_0+h) + (x-x_0)(x-x_0+h)(x-x_0-h) f(x_0, x_0-h, x_0+h, x_0-2h) + (x-x_0)(x-x_0+h)(x-x_0-h)(x_0-x_0+2h) f(x_0, x_0-h, x_0+h, x_0-2h, x_0+2h) + \dots\dots$$

Now put, $u = \frac{x-x_0}{h}$ i.e. $x - x_0 = hu$. Then we have,

$$y = f(x) = f(x_0) + \cancel{hu f(x_0-h)}\; hu f(x_0, x_0-h)$$
$$+ hu(hu+h) f(x_0-h, x_0, x_0+h)$$
$$+ hu(hu+h)(hu-h) f(x_0-2h, x_0-h, x_0, x_0+h)$$
$$+ hu(hu+h)(hu-h)(hu+2h) f(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h) + \dots \quad \dots (1)$$

But we have from the relation between simple differences & divided differences:

$$f(x_0-h, x_0) = \frac{\Delta y_{-1}}{h}$$

$$f(x_0-h, x_0, x_0+h) = \frac{\Delta^2 y_{-1}}{2!\,h^2}$$

$$f(x_0-2h, x_0-h, x_0, x_0+h) = \frac{\Delta^3 y_{-2}}{3!\,h^3}$$

$$f(x_0-2h, x_0-h, x_0, x_0+h, x_0+2h) = \frac{\Delta^4 y_{-2}}{4!\,h^4} \text{ etc}$$

Substituting these values in (1) we get,

$$y = f(x) = y_0 + hu\frac{\Delta y_{-1}}{h} + \cancel{b}\, hu(hu+h)\cdot\frac{\Delta^2 y_{-1}}{2!\,h^2}$$
$$+ hu(hu+h)(hu-h)\cdot\frac{\Delta^3 y_{-2}}{3!\,h^3} +$$
$$hu(hu+h)(hu-h)(hu+2h)\frac{\Delta^4 y_{-2}}{4!\,h^4} + \dots$$

$$\cancel{= y_0 + u\Delta y_{-1}}$$

$$= y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-2}$$
$$+ \frac{u(u^2-1)(u+2)}{4!}\Delta^4 y_{-2} + \dots$$

This is the Gauss backward formula for equal intervals.

~~Forward~~ forward $\rightarrow x_1 = x_0+h,\ x_2 = x_0-h,\ x_3 = x_0+2h$ etc

Backward $\rightarrow x_1 = x_0-h,\ x_2 = x_0+h,\ x_3 = x_0-2h$ etc

[ii] Derive third formula due to Gauss.

Solution:

We have Gauss's backward formula for equal intervals:

$$y = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^2 y_{-2} + \frac{u(u^2-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \cdots \quad (1)$$

Here, $u = \frac{x-x_0}{h}$

$$\therefore \frac{x-x_k}{h} = \frac{x-(x_0+kh)}{h}$$

$$= u-k$$

To derive the third formula we start with y_1 and run parallel to (1) i.e. we advance the subscripts of x and y in (1) by one unit.

To get these we have to replace u by $(u-1)$ in (1) and advancing all subscripts of y by one unit, we have

$$y = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-1} + \cdots \cdots$$

This is the third formula due to Gauss.

Example-01

Use Gauss's forward formula to find the value of y when $x = 3.75$ given the following table.

Solution:

First we form a central difference table taking $x = 3.5$ as the origin:

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$
−2	2.5	24.145	−2.102	0.284	−0.047	$\boxed{0.009}$	$\boxed{-0.003}$
−1	3.0	22.043	−1.818	$\boxed{0.237}$	$\boxed{-0.038}$	0.006	
0	3.5	$\boxed{20.225}$	$\boxed{-1.581}$	0.199	−0.032		
1	4.0	18.644	−1.382	0.167			
2	4.5	17.262	−1.215				
3	5.0	16.047					

~~we k~~ From the Gauss's forward formula we get,

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}\Delta^5 y_{-2} + \cdots$$

where, $u = \frac{x - x_0}{h}$

Here, $x = 3.75$, $x_0 = 3.5$, $h = 0.5$

$\therefore u = \frac{3.75 - 3.5}{0.5} = 0.5$

$$\therefore y(3.75) = 20.225 + (0.5)(-1.581) + \frac{(0.5)(0.5-1)}{2!}(0.237) + \frac{\cancel{(0.5-1)}\, 0.5\,((0.5)^2 - 1)}{3!}(-0.038) +$$

$$\frac{(0.5)((0.5)^2-1)(0.5-2)}{4!}(0.009) + \frac{(0.5)((0.5)^2-1)((0.5)^2-4)}{5!}(-0.003)$$

$$= \cancel{20.225 - 0.7905 - 0.2963 + 0.0024 + 0.0002 - 0.00009}$$

$$= 20.225 - 0.7905 - 0.029625 + 0.00237 + 0.00210938 - 0.0000352$$

$$= 19.40 \text{ (approx)}$$

Hence, the value of y is 19.90 approx when $x = 3.75$.

Derive stirlings interpolation formula.

Solution:

We have Gauss's forward interpolation and backward interpolation formulas are respectively,

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-2} + \dots \dots \quad (1)$$

and $$y(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-2} + \frac{u(u^2-1)(u+2)}{4!}\Delta^4 y_{-2} + \dots \dots \quad (2)$$

Adding (1) and (2) we get,

$$2y(x) = 2y_0 + u\left[\Delta y_0 + \Delta y_{-1}\right] + \left[\frac{u(u-1)}{2!} + \frac{u(u+1)}{2!}\right]\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \left[\frac{u(u^2-1)(u-2)}{4!} + \frac{u(u^2-1)(u+2)}{4!}\right]\Delta^4 y_{-2} + \dots \dots$$

$(u^3-u)(u+2)$
$= u^4-2u^3-u^2+2u$
$u^4+2u^3-u^2-2u$

$$= 2y_0 + u\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{u^2-u+u^2+u}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right] + \left[\frac{u^4-2u^3-u^2+2u+u^4+2u^3-u^2-2u}{4!}\right]\Delta^4 y_{-2} + \dots \dots$$

$$= 2y_0 + u\left[\Delta y_0 + \Delta y_{-1}\right] + \frac{2u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\Delta^3 y_{-1} + \Delta^3 y_{-2}\right]$$
$$+ \frac{2u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$\therefore y(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} +$$
$$\frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

where, $u = \frac{x - x_0}{h}$

which is Stirling's formula.

☐ Derive Bessel's interpolation formula.

Solution:

We have Gauss's forward interpolation formula and the third formula due to Gauss are respectively:

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} +$$
$$\frac{u(u^2-1)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{3!}\Delta^4 y_{-2} + \cdots \quad (1)$$

and

$$y(x) = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\Delta^4 y_{-1} + \cdots \cdots \cdots \quad (2)$$

Adding (1) and (2) we get,

$$2y(x) = y_0 + y_1 + (2u-1)\Delta y_0 + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \left[\frac{u(u^2-1)}{3!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1} + \left[\frac{u(u^2-1)(u-2)}{4!}\right]\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right] + \cdots\cdots$$

$$\therefore y(x) = \frac{y_0 + y_1}{2} + \left(u - \tfrac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] \cancel{+ \frac{u(u^2-1)(u-2)}{4!}} + \frac{u\left(u-\frac{1}{2}\right)(u-1)}{3!}\Delta^3 y_{-1} + \left[\frac{u(u^2-1)(u-2)}{4!}\right]\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \cdots$$

where, $u = \frac{x - x_0}{h}$

which is the Bessel's formula.

FINAL (17-18)

Example-01

Use Stirling's formula to find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$

Solution:

We construct a central difference table:

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
~~20~~ -1	20	512	$\boxed{-73}$	$\boxed{-20}$	$\boxed{10}$
0	30	$\boxed{439}$	$\boxed{-93}$	-10	
1	40	346	-103		
2	50	243			

~~From~~ ~~The~~ The Stirling's formula is,

$$y(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_1}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\cdot\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \cdots$$

where, $u = \frac{x - x_0}{h}$

Taking 30 as origin i.e $x_0 = 30$, we get,

$$u = \frac{35-30}{10} = 0.5$$

$$\therefore y(35) = 439 + (0.5)\left[\frac{-93-73}{2}\right] + \frac{(0.5)^2}{2}(-20) + \frac{(0.5)((0.5)^2-1)}{3!}\cdot\frac{10}{2}$$

$= 439 \cancel{+70} - 41.5 - 2.5 - 0.3125$

$= 394.6875$

≈ 395

Thus, $y_{35} = 395$ approximately.

Example-2

Given, $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$.
Find y_{25} by Bessel's formula.

Solution:
First we form a central difference table taking ~~the~~ $x = 24$ as the origin:

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
-1	20	24	8	$\boxed{-5}$	$\boxed{7}$
0	24	$\boxed{32}$	$\boxed{3}$	$\boxed{2}$	
1	28	35	5		
2	32	40			

Here, $u = \frac{x - x_0}{h}$; where $x = 25$

$\therefore u = \frac{25-24}{4} = 0.25$

The Bessel's formula is

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{u(u-\frac{1}{2})(u-1)}{3!}\,\Delta^3 y_{-1} + \cdots\cdots$$

$$\therefore\ y(25) = 32 + (0.25)(3) + \frac{(0.25)(0.25-1)}{2!}\cdot\frac{(-5+2)}{2} + \frac{(0.25)(0.25-\frac{1}{2})(0.25-1)}{3!}(7)$$

$$= 32 + \cancel{0.1406}\ 0.75 + 0.140625 + 0.054687$$

$$= 32.9453125$$

Thus, $y_{25} = 32.9453125$.