Hence the number of students who obtained marks between 40 to 45 is

$f(45) - 31$

$= 48 - 31$

$= 17$

Chapter- 3(B)

"Interpolation with unequal Intervals"

Divided difference: Let $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ are the functional values corresponding to any values (may or may not equidistant) $x_0, x_1, x_2, \dots, x_n$ of the argument. Then the divided difference of $f(x)$ in ascending order are defined as follows:

First order divided differences:

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \text{ etc}$$

Second order divided differences:

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}, \quad f(x_1, x_2, x_3) = \frac{f(x_1, x_2) - f(x_2, x_3)}{x_1 - x_3} \text{ etc.}$$

n-th order divided differences:

$$f(x_0, x_1, x_2, \ldots, x_n) = \frac{f(x_0, x_1, \ldots, x_{n-1}) - f(x_1, x_2, \ldots, x_n)}{x_0 - x_n}$$

Deduce Newton's formula for unequal intervals
OR, Establish Newton's divided difference formula for interpolation.
OR, Establish the Newton's general interpolation formula.

Solution: Let $f(x_0), f(x_1), \ldots \ldots, f(x_n)$ are values of $f(x)$ corresponding to any values (may or may not equidistant) $x_0, x_1, x_2, \ldots \ldots, x_n$ to the arguments. Then from definition of divided differences in asending order, we have,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \ldots \ldots \quad (1)$$

Again, $$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\Rightarrow f(x, x_0) = (x - x_1) f(x, x_0, x_1) + f(x_0, x_1)$$

(1) $\Rightarrow$ $f(x) = f(x_0) + (x-x_0)(x-x_1) f(x, x_0, x_1) + (x-x_0) f(x_0, x_1)$ ...... (2)

Also, $f(x, x_0, x_1, x_2) = \dfrac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$

$\Rightarrow f(x, x_0, x_1) = (x-x_2) f(x, x_0, x_1, x_2) + f(x_0, x_1, x_2)$

(2) $\Rightarrow$ $f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1)(x-x_2) f(x, x_0, x_1, x_2) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$

$= f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x, x_0, x_1, x_2)$

By continuing in this manner, we have

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1)\cancel{(x-x_2)} f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0, x_1, \dots, x_n) + R_n$$

where, $R_n = (x-x_0)(x-x_1)(x-x_2)\dots(x-x_n) f(x, x_0, \dots, x_n)$

The last term, R_n is the remainder.

Assuming that $f(x)$ is a polynomial of degree n.
So, $f(x, x_0, x_1, \dots, x_n) = 0$; since it is $(n+1)$th order divided difference.

i.e. $R_n = 0$

Thus $f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \cancel{(x - x_0)} \dots \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1}) f(x_0, x_1, x_2, \dots, x_n)$

This is the required formula.

Problem: 1

Using the following table find $f(x)$ as a polynomial in x:

x	-1	0	3	6	7
$f(x)$	3	-6	39	822	1611

Solution:

The divided difference table for the given data is as follows:

i	x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4})$
0	-1	3	$\frac{-6-3}{0-(-1)} = -9$	$\frac{15+9}{3+1} = 6$	$\frac{41-6}{6-(-1)} = 5$	$\frac{13-5}{7-(-1)} = 1$
1	0	-6	$\frac{39+6}{3-0} = 15$	$\frac{261-15}{6-0} = 41$	$\frac{132-41}{7-0} = 13$	
2	3	39	$\frac{822-39}{6-3} = 261$	$\frac{789-261}{7-3} = 132$		
3	6	822	$\frac{1611-822}{7-6} = 789$			
4	7	1611				

From the Newton's general interpolation formula we get,

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4) + \cdots$$

$$= 3 + (x+1)(-9) + (x+1)x(6) + x(x+1)(x-3)(5) + (x+1)x(x-3)(x-6)(1)$$

$$= 3 - 9x - 9 + 6x^2 + 6x + 5x(x^2 - 2x - 3) + x(x+1)(x^2 - 9x + 18)$$

$$= -9x - 6 + 6x^2 + 6x + 5x^3 - 10x^2 - 15x$$

$+ (x^2+x)(x^2-9x+18)$

$= 5x^3-4x^2-18x-6 + x^4-9x^3+18x^2+x^3-9x^2+18x$

$= x^4-3x^3+5x^2-6$

which is the required polynomial.

□ Derive Lagrange's interpolation formula for unequal intervals.

Solution:

Let $y=f(x)$ be a polynomial of nth degree, which takes the values $f(x_0), f(x_1), f(x_2), \ldots\ldots, f(x_n)$ for any values (may or may not ~~equal~~ equidistant) $x_0, x_1, x_2, \cdots, x_n$ of the argument x. This polynomial may be written as:

$$f(x) = a_0(x-x_1)(x-x_2)\cdots(x-x_n) + a_1(x-x_0)(x-x_2)\cdots(x-x_n) + a_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad \cdots (1)$$

where, a's are constants.

To find the values of a's we put $x = x_0, x_1, \ldots, x_n$ in successively, then we get,

$$f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)}$$

$$f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)$$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)}$$

Similarly,

$$a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \ldots (x_2 - x_n)}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})}$$

Putting this value in (1) we get,

$$f(x) = \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) + \cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f(x_n)$$

This is the required formula.

Problem:

Using Lagrange's interpolation formula, find the form of the function $f(x)$ from the following table:

(i)

x	0	1	3	4
$f(x)$	-12	0	12	24

(ii)

x	-1	0	2	5
y	9	5	3	15

1) Solution:

Here we have four values of argument & four values of entry.

Suppose they are,

$x_0 = 0, \; x_1 = 1, \; x_2 = 3, \; x_3 = 4$

and $f(x_0) = -12, \; f(x_1) = 0, \; f(x_2) = 12, \; f(x_3) = 24$

From the Lagrange's interpolation formula we get,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdot\cancel{(x-x_n)}}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdot\cancel{(x_0-x_n)}} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cancel{(x-x_n)}}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_1)} f(x_3)$$

$$= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \frac{x(x-3)(x-4)}{(1-0)(1-3)(1-4)}(0)$$

$$+ \frac{x(x-1)(x-4)}{(3-0)(3-1)(3-4)}(12) + \frac{x(x-1)(x-3)}{(4-0)(4-1)(4-3)}(24)$$

~~3~~ 9 3

$$= (x-1)(x-3)(x-4) - 2x(x-1)(x-4) + 2x(x+1)(x-3)$$

$$= (x-1)(x^2-7x+12) - 2x(x^2-5x+4) + 2x(x^2-4x+3)$$

$$= x^3-7x^2+12x-x^2+7x-12-2x^3+10x^2-8x+2x^3-8x^2+6x$$

$$= x^3-6x^2+17x-12$$

11) Solution:

Here, we have four values of argument & four values of entry.

Suppose they are,

$x_0=-1,\ x_1=0,\ x_2=2,\ x_3=5$

and $f(x_0)=9,\ f(x_1)=5,\ f(x_2)=3,\ f(x_3)=15$

From the Lagrange's interpolation formula we get,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_3)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{x(x-2)(x-5)}{(-1)(-1-2)(-1-5)}(9) + \frac{(x+1)(x-2)(x-5)}{(1)(-2)(-5)}(5)$$

$$+ \frac{(x+1)(x)(x-5)}{(2+1)(2)(2-5)}(3) + \frac{(x+1)x(x-2)}{(5+1)(5)(5-2)}(15)$$

$$= -\frac{1}{2}\left(x(x^2-7x+10)\right) + \frac{1}{2}\left((x+1)(x^2-7x+10)\right)$$
$$-\frac{1}{6}\left(x(x^2-4x-5)\right) + \frac{1}{6}\left(x(x^2-x-2)\right)$$

$$= -\frac{1}{2}(x^3-7x^2+10x) + \frac{1}{2}(x^3-7x^2+10x+x^2-7x+10)$$
$$-\frac{1}{6}(x^3-4x^2-5x) + \frac{1}{6}(x^3-x^2-2x)$$

$$= \frac{-3x^3+21x^2-30x+3x^3-21x^2+30x+3x^2-21x+30-x^3+4x^2+5x+x^3-x^2-2x}{6}$$

$$= \frac{6x^2-18x+30}{6}$$

$$= x^2-3x+5$$

"Question Solve"

FINAL (18-19)

4(c) The values of x and $f(x)$ are given below. With the help of given data construct a divided difference table and hence the values of $f(8)$ and $f(15)$.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution:

The divided difference table of the given data ~~as~~ is as follows:

i	x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3})$
0	4	48	$\frac{100-48}{5-4}=52$	$\frac{97-52}{7-4}=15$	$\frac{21-15}{10-4}=1$
1	5	100	$\frac{294-100}{7-5}=97$	$\frac{202-97}{10-5}=21$	$\frac{27-21}{11-5}=1$
2	7	294	$\frac{900-294}{10-7}=202$	$\frac{310-202}{11-7}=27$	$\frac{33-27}{13-7}=1$
3	10	900	$\frac{1210-900}{11-10}=310$	$\frac{409-310}{13-10}=33$	
4	11	1210	$\frac{2028-1210}{13-11}=409$		
5	13	2028			

From the Newton's general Interpolation formula we get,

$$f(x) = f(x_0) + (x-x_0)\, f(x_0,x_1) + (x-x_0)(x-x_1)\, f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\, \cancel{f(x_3)}\, f(x_0,x_1,x_2,x_3) + \cdots \cdots$$

$$\therefore f(8) = 48 + (8-4)(52) + (8-4)(8-5)(15)$$
$$+ (8-4)(8-5)(8-7)(1)$$
$$= 48 + 208 + 180 + 12$$
$$= 448$$

and $$f(15) = 48 + (15-4)(52) + (15-4)(15-5)(15)$$
$$+ (15-4)(15-5)(15-7)(1)$$
$$= 48 + 572 + 1650 + 880$$
$$= 3150$$

5] (c) Find the cubic polynomial which takes the following values $y(0)=1$, $y(1)=3$, $y(3)=31$, $y(6)=223$ and $y(10)=1011$ then find $y(2.5)$ using Newton's interpolation formula

Solution:

The divided difference table of the given data is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	$\frac{3-1}{1-0}=2$	$\frac{14-2}{3-0}=4$	$\frac{10-4}{6-0}=1$	$\frac{1-1}{10-1}$
1	3	$\frac{31-3}{3-1}=14$	$\frac{64-14}{6-1}=10$	$\frac{19-10}{10-1}=1$	$=0$
3	31	$\frac{223-31}{6-3}=64$	$\frac{197-64}{10-3}=19$		
6	223	$\frac{1011-223}{10-6}=197$			
10	1011				

From Newtons general interpolation formula we get,

$$y(x)=y_0+(x-x_0)\Delta y_0+(x-x_0)(x-x_1)\Delta^2 y_0+(x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0+(x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0$$

$$y(2.5) = 1 + (2.5-0)2 + (2.5-0)(2.5-1)4$$
$$+ (2.5-0)(2.5-1)(2.5-3)(1) +$$
$$(2.5-0)(2.5-1)(2.5-3)(2.5-6)\cancel{0.04}(0)$$

$$= 1 + 5 + 15 - 1.875 \cancel{+0.2625}$$

$$= \cancel{19.4 \text{ (almost)}} \text{ (approx)} = 19.125$$

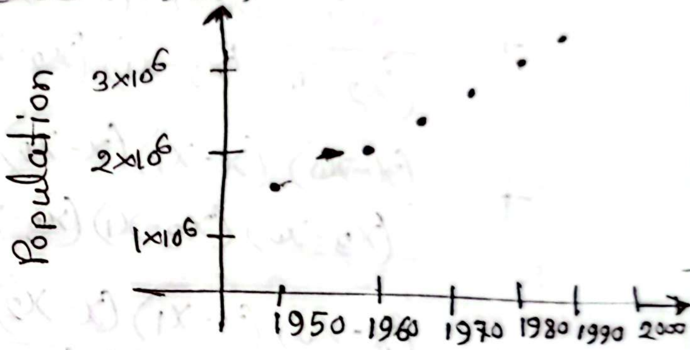
FINAL (16-17)

5] (b) Derive ~~Lagrange's~~ Interpolation formula for unequal intervals. (~~Rep~~ Previous ~~6~~ note).

(c) Find the cubic polynomial which takes the following values $y(0)=1$, $y(1)=3$, $y(3)=31$, $y(6)=223$ and $y(10)=1011$ then find $y(2.5)$ using Newton's interpolation formula. [Repeat]

FINAL-(17-18)

2) (d) — The figure included a table depicts the population of the USA. Use Labrange interpolation to approximate the population in the years 1930, 1965 and 2010



Year	1950	1960	1970	1980	1990	2000
Population	1,51,326	1,79,323	203,302	226,542	249,653	3,41,427

~~So~~

~~Here, we~~

Solution:

Here, we have 6 values of argument & six values of entry.

Suppose they are,

$x_0 = 1950$, $x_1 = 1960$, $x_2 = 1970$, $x_3 = 1980$, $x_4 = 1990$, $x_5 = 2000$ and $f(x_0) = 1,51,326$, $f(x_1) = 1,79,323$, $f(x_2) = 2,03,302$, $f(x_3) = 2,26,542$, $f(x_4) = 2,49,653$, $f(x_5) = 3,41,427$

From the Lagrange's interpolation formula we get,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

$$\therefore f(1930) = \frac{(1930-1960)(1930-1970)(1930-1980)(1930-1990)(1930-2000)(451{,}326)}{(1950-1960)(1950-1970)(1950-1980)(1950-1990)(1950-2000)}$$

$$+ \frac{(1930-1950)(1930-1970)(1930-1980)(1930-1990)(1930-200)(1{,}79{,}323)}{(1960-1950)(1960-1970)(1960-1980)(1960-1990)(1960-2000)}$$

$$+ \frac{(1930-1950)(1930-1960)}{}$$

$\vdots$

FINAL—(19-20)

4) (d) The values of x and $f(x)$ are given below. With the help of given data construct a divided difference table and hence find the values of $f(3)$ and $f(10)$.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution:

The divided difference table of the given data is as follows:

i	x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3})$
0	4	48	$\frac{100-48}{5-4}=52$	$\frac{97-52}{7-4}=15$	$\frac{21-15}{10-4}=1$
1	5	100	$\frac{294-100}{7-5}=97$	$\frac{202-97}{10-5}=21$	$\frac{27-21}{11-5}=1$
2	7	294	$\frac{900-294}{10-7}=202$	$\frac{310-202}{11-7}=27$	$\frac{33-27}{13-7}=1$
3	10	900	$\frac{1210-900}{11-10}=310$	$\frac{409-310}{13-10}=33$	
4	11	1210	$\frac{2028-1210}{13-11}=409$		
5	13	2028			

From the Newtons general interpolation formula we get,

$$f(x) = f(x_0) + (x-x_0)\cancel{f(x_1,x_0)}\, f(x_0,x_1) + (x-x_0)(x-x_1)\, f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\, f(x_0,x_1,x_2,x_3)$$

$$\therefore f(3) = 48 + (3-4)(52) + (3-4)(3-5)(15) + (3-4)(3-5)(3-7)(1)$$

$$= 48 - 52 + 30 - 4$$

$$= 22$$

$$\therefore f(10) = 48 + (10-4)(52) + (10-4)(10-5)(15) + (10-4)(10-5)(10-7)(1)$$

$$= 48 + 312 + 450 + 90$$

$$= 900$$

1