Chapter-5

"Numerical Solution of Ordinary Differential Equations"

Differential Equation:

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Example:

i) $\frac{dy}{dx} + xy = 0$ [First order 1st degree differential equation]

ii) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ [Partial differential equation]

Ordinary differential equation → from book

Order & degree → ,,

Ordinary linear & non-linear equation

Initial value problem

Boundary value problem

☐ Describe the Picard's Method of Successive approximations for the solution of a differential equation.

$y = f(x)$
$\underline{y} = f(\underline{x_0}) = \underline{y_0}$

Solution:

We consider the intial value problem

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0 \qquad \cancel{f(x) = 2x+3}$$

Integrating the above differential ~~equation~~ between the corresponding limits for x and y, we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x,y)\, dx \quad \dots \dots (1)$$

In the equation (1) the unknown function y appears under the integral sign. Thus equation (1) is an integral equation. This integral equation can be solved by the process of successive approximation.

The first approximation to y is the obtained by putting y_0 for y on the right hand side of (1)

We have,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Now resulting y_1 is substituting for y in the integrand of (1), we get the 2nd approximation.

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Proceeding in this way, we obtain $y_3, y_4, \dots, y_{n-1}, y_n$ are successive approximation of y.

The n-th approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx.$$

The process is repeated until the

value of y is obtained to desired accuracy. The process is stopped when the two values of y, viz. y_{n+1} and y_n are same to desired degree of accuracy.

Example - 01

Using Picard's method, obtain the solution of $dy/dx = x(1+x^3y)$, $y(0)=3$. Tabulate the values of $y(0.1)$ and $y(0.2)$.

$y' = f(x,y)$

Solution:

Given, $\frac{dy}{dx} = x(1+x^3y)$

$= x + x^4y$

$= f(x,y)$

and $y(0) = 3$

Here, ~~$y_0 = 0$~~ $x_0 = 0$, $y_0 = 3$

From the Picard's method of successive approximation we get,

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

1st approx, $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

$$\cancel{= 3 + \int_0^x f(0,3)\, dx}$$

$$\cancel{= y_0 + \int_{x_0}^{x} f}$$

$$= 3 + \int_0^x f(x,3)\, dx$$

$$= 3 + \int_0^x (x + 3x^4)\, dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5}$$

2nd approx, $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$

$$= 3 + \int_0^x f\left(x, \left(3 + \frac{x^2}{2} + \frac{3x^5}{5}\right)\right) dx$$

$$= 3 + \int_0^x \left(x + 3x^4 + \frac{x^6}{2} + \frac{3x^9}{5}\right) dx$$

$$= 3 + \frac{x^2}{2} + \frac{3}{5}x^5 + \frac{1}{14}x^7 + \frac{3}{50}x^{10}$$

Thus, $y(x) = 3 + \frac{x^2}{2} + \frac{3}{5}x^5 + \frac{x^7}{14} + \frac{3}{50}x^{10} + \cdots\cdots$

Now, $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{3}{5}(0.1)^5 + \frac{(0.1)^7}{14} + \frac{3}{50}(0.1)^{10} + \cdots$

$= 3.005006007$

≈ 3.005 (approx)

$\cancel{y(0.2)}$ $y(0.2) = 3 + \frac{(0.2)^2}{2} + \frac{3}{5}(0.2)^5 + \frac{(0.2)^7}{14} + \frac{3}{5}(0.2)^{10} + \cdots$

$= 3.02019292$

≈ 3.020 (approx)

□ Describe the Taylor series Method for the solution of ordinary differential equation.

Solution:

We consider the differential equation (or initial value problem or ordinary differential equation)

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

or, $y'(x) = f(x,y), \quad y(x_0) = y_0 \quad \cdots \cdots (1)$

If $y(x)$ is the exact solution of (1), then we can expand $y(x)$ by Taylor's series about $x = x_0$ in the form,

$$y(x) = y_0 + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots\cdots$$

Differentiating (1) with respect to x, we get,

$$y''(x) = y'' = \frac{\delta f}{\delta x} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta x}$$

$$= f_x + f_y y'$$

Differentiating with respect to x we get,

$$y''' = f_{xx} + f_{xy} f + f\,[f_{yx} + f_{yy} f] + f_y [f_x + f_y f]$$

$= f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$

Proceeding in this way, we obtain other higher derivatives of y i.e. $y^{iv}, y^{v}, \ldots \ldots$

Using initial condition $y = y_0$ at $x = x_0$ we get y_0', y_0'', y_0'''

Putting $x = x_0 + h$ in (2), we get,

$$y_1 = y(x_0+h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots \quad (3)$$

$$\therefore y_1 = y(x_0+h) = y(x_1)$$

Here the values of $y_0, y_0', y_0'' \ldots$ are known. So (3) gives a power series for y_1.

Now we expand y by Taylor's series about $x = x_1 = x_0 + h$ in the form,

$$y_2 = y(x_1+h) = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \ldots \quad (4)$$

The values of $y_1, y_1', y_1'', \ldots$ can be found from (3).

Then (4) gives a power series for y_2.
Proceeding in this way, we obtain y_3, y_4, ..., y_n

where, $y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \cdots$

Example-01

Using Taylor's series, find the solution of the differential equation $y' = y^2$, $y(0) = 1$ at $x = 0.01$

Solution:

Here, we have,

$y'(0) = [y(0)]^2 = 1$

$y'' = 2yy' \Rightarrow y''(0) = 2y(0)\, y'(0)$
$= 2(1)(1) = 2$

$y''' = 2(y')^2 + 2yy''$

$\Rightarrow y'''(0) = 2[y'(0)]^2 + 2y(0)\, y''(0)$
$= 2(1) + 2(1)(2)$
$= 2 + 4 = 6$ and so on

From the Taylor's series we get,

$$y(x_0+h) = y_0 + \cancel{hy'_0}\ hy'_0 + \frac{h^2}{2!} y''_0 + \cdots$$

$$\Rightarrow y(0+0.01) = \cancel{\ } y(0) + (0.01) y'(0) + \frac{(0.01)^2}{2!} y''(0) + \frac{(0.01)^3}{3!} y'''(0) + \cdots \cdots$$

$$\Rightarrow y(0.01) \approx 1.010101$$

Here, the step-length $h = 0.01$.

Question Solve

8(b)

Use Taylor's series method to solve the equation.

$$dy/dx = -xy, \quad y(0) = 1$$

Solution:

Given, $\frac{dy}{dx} = -xy = f(x, y)$, $y(0)=1$, $x_0 = 0, y_0 = 1$

we know from the Taylor's series,

$$y(x) = y_0 + (x-x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots \quad (1)$$

$y'(0) = -x\,y(0)$

$= -x$

$y''(0) = -y - xy'$

$\Rightarrow y''(0) = -y(0) - x\,y'(0)$

$= -1 + x^2$

$y'(x) = -xy$

$\Rightarrow y'(0) = -(0)(1) = 0$

$y''(x) = -y - xy'$

$\Rightarrow y''(0) = -y(0) - 0$

$= -1$

$y'''(x) = -y' - y' - xy''$

$= -2y' - xy''$

$\Rightarrow y'''(0) = -2y'(0) - 0 \cdot y''(0)$

$= 0$

$y^{iv}(x) = -2y'' - y'' - xy'''$

$\Rightarrow y^{iv}(0) = -2(-1) + 1 + 0$

$= \cancel{2+}\ 3$

$\cancel{=1}$

From (1) we get,

$$y(x) \approx y(0) + y'(0)x + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0)$$

$$y(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{8}$$