Chapter-2

Finite Differences'

Operator: An operator in numerical method is a compact mathematical tool used to express operations like shifting, differentiating & averaging function values, which are central to numerical approximation techniques.

For example

Symbol		Operator
E	$\longrightarrow$	shift operator
Δ	$\longrightarrow$	Forward difference operator
∇	$\longrightarrow$	Backward difference operator
δ	$\longrightarrow$	Central difference operator
μ	$\longrightarrow$	mean operator
Δ^2	$\longrightarrow$	second difference.
E^{-1}	$\longrightarrow$	Anti-shift operator

i) $Ef(x) = f(x+h)$; where h is a constant

ii) $\Delta f(x) = f(x+h) - f(x)$.

iii) $\nabla f(x) = f(x) - f(x-h)$

$f(x+h)$

$\Delta f(x) = f(x+h) - f(x)$

$= \cancel{y_1 - y_0}$

iv) $\delta f(x) = f(x+h/2) - f(x-h/2)$

v) $\mu f(x) = \frac{1}{2}\{f(x+h/2) + f(x-h/2)\}$

vi) $\sigma f(x) = f(x+h/2) + f(x-h/2)$

Q] Establish the relation between following various operators.

i) $\Delta - \nabla \equiv \Delta\nabla$

ii) $(1+\Delta)(1-\nabla) \equiv 1$

iii) $(1+\Delta) \equiv (E-1)\nabla^{-1}$

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$\Delta y_0 = y_1 - y_0$

$\nabla y_n = y_n - y_{n-1}$

i) Solution

We know,

$$\Delta\nabla y_n = \Delta(y_n - y_{n-1}) = \Delta y_n - \Delta y_{n-1}$$

$$= \Delta y_n - (y_n - y_{n-1})$$

$$= \Delta y_n - \nabla y_n$$

$$= (\Delta - \nabla) y_n$$

$$\therefore \Delta - \nabla \equiv \Delta\nabla$$

$\Delta \rightarrow E-1$
$\nabla \rightarrow 1-E^{-1}$

ii) Solution:

$$(1+\Delta)(1-\nabla)y_n = (1+\Delta)(y_n - \nabla y_n)$$
$$= (1+\Delta)(y_n - y_n + y_{n-1})$$
$$= \cancel{y_{n-1}} (1+\Delta) y_{n-1}$$
$$= y_{n-1} + y_n - y_{n-1}$$
$$= 1 \cdot y_n$$

$\Delta y_{n-1} = y_n - y_{n-1}$

$$\therefore (1+\Delta)(1-\nabla)y_n \equiv 1$$

$(1-x)^{-1} = 1 + x + x^2$

iii) Solution:

$$(E-1)\nabla^{-1}y_n = (E-1)(1-E^{-1})^{-1}y_n$$
$$= (E-1)(1+E^{-1}+E^{-2}+E^{-3}+\dots)y_n$$
$$= (E+1+E^{-1}+E^{-2}+\dots -1-E^{-1} - E^{-2} - E^{-3} - \dots)y_n$$
$$= Ey_n$$
$$= (1+\Delta)y_n$$

$$\Rightarrow (1+\Delta) \equiv (E-1)\nabla^{-1}$$

$\Delta = E-1$
$\nabla = 1-E^{-1}$

Q/ $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Solution:

We know,

$\Delta = E-1$ and $\nabla = 1-E^{-1}$

Now, $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$= \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$= \frac{(E-1)^2 - (1-E^{-1})^2}{(E-1)(1-E^{-1})}$$

$$= \frac{(E-1+1-E^{-1})(E-1-1+E^{-1})}{\cancel{(E-EE^{-1})}(E-EE^{-1}-1+EE^{-1})}$$

$$= \frac{(E-E^{-1})(E+E^{-1}-2)}{(E-1)}$$

$$= \frac{E^2+EE^{-1}-2E-EE^{-1}-E^{-2}+2E^{-1}}{(E-1)}$$

$$= \frac{(E-1+1-E^{-1})(E-1-1+E^{-1})}{(E-1-1+E^{-1})}$$

$$= (E-1)+(1-E^{-1})$$

$$= \Delta + \nabla \Rightarrow \Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$

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Q: Show that nth difference of nth degree polynomial is constant.

Statement: nth difference of nth degree polynomial is constant.

Proof:

Let, $y = f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots \dots + a_{n-1} x + a_n$

is nth degree polynomial.

1st difference of polynomial,

$\Delta y = f(x+h) - f(x)$

$= a_0\{(x+h)^n - x^n\} + a_1\{(x+h)^{n-1} - x^{n-1}\} + a_2\{(x+h)^{n-2} - x^{n-2}\}$

$+ \dots + a_{n-1}\{(x+h) - x\} + a_n - a_n$

$= a_0 [x^n + nhx^{n-1} + \frac{n(n-1)}{2!} h^2 x^{n-2} + \frac{n(n-1)(n-2)}{3!} h^3 x^{n-3}$

$+ \dots + h^n \} - x^n] + a_1 [\{x^{n-1} + (n-1) h x^{n-2}$

$+ \frac{(n-1)(n-2)}{2!} h^2 x^{n-3} + \dots + h^{n-1}\} - x^{n-1}]$

$+ a_2 [x^{n-2} + (n-2) h x^{n-3} + \frac{(n-2)(n-3)}{2!} h^2 x^{n-4} + \dots + h^{n-2}\}$

$$-x^{n-2}] + \cdots + a_{n-1}h$$

$$= a_0 n h x^{n-1} + a_1' x^{n-2} + a_2' x^{n-3} + \cdots + a_{n-2}' x + a_{n-1}'$$

where,

$$a_1' = \frac{a_0 n(n-1)}{2!} h^2 + a_1 (n-1) h$$

$$a_2' = a_0 \frac{n(n-1)(n-2)}{3!} h^3 + a_1 \frac{(n-1)(n-2)}{2!} h^2 + a_2 (n-2) h$$

2nd difference of the polynomial,

$$\Delta^2 y = a_0 n h \{(x+h)^{n-1} - x^{n-1}\} + a_1' \{(x+h)^{n-2} - x^{n-2}\} + a_2' \{(x+h)^{n-3} - x^{n-3}\} + \cdots + a_{n-2}' \{(x+h) - x\} + a_{n-1}'$$

$$= a_0 n (n-1) h^2 x^{n-2} + a_1'' x^{n-3} + a_2'' x^{n-4} + \cdots + a''_{n-2} \cancel{x + a''_{n-2}}$$

According the above statement we get,

$$\Delta^n y = a_0 n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 \, h^n x^{n-n}$$

$$= a_0 n! h^n \text{ (constant)}$$

So, nth difference of nth degree polynomial is constant.

Q Find the missing term in the following table:

x	10	15	20	25	30	35
$f(x)$	43	...	29	32	...	77

Solution:

As, here, given values of y are 4.

So, from the table we get 3rd degree polynomial of $f(x)$.

Let, $y_0 = 43$, $y_2 = 29$, $y_3 = 32$, $y_5 = 77$

Now, $\Delta^4 y_0 = 0$

$$\Rightarrow (E-1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\Rightarrow y_4 - (4 \times 32) + (6 \times 29) - 4y_1 + 43 = 0$$

$\Rightarrow y_4 - 4y_1 = -89 \quad \cdots\cdots (1)$

$$\begin{aligned} 4y_4 - 16y_1 &= -356 \\ 4y_4 - y_1 &= 153 \\ \hline -15y_1 &= -509 \\ \Rightarrow y_1 &= \frac{509}{15} \end{aligned}$$

Again, $\Delta^4 y_1 = 0$

$\Rightarrow \cancel{(E-1)^4 y_1 = 0}$

$\Rightarrow y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0$

$\Rightarrow 77 - 4y_4 + (6\times 32) - (4\times 29) + y_1 = 0$

$\Rightarrow 4y_4 - y_1 = 153 \quad \cdots\cdots (2)$

$$\begin{aligned} y_4 &= -89 + 4y_1 \\ &= -89 + \frac{4\times 509}{15} \\ &= \frac{701}{15} \end{aligned}$$

From (1) & (2) we get,

$y_1 = 509/15$ and $y_4 = \frac{701}{15\cancel{0}}$

So, our desired table:

x	10	15	20	25	30	35
$f(x)$	43	$509/15$	29	32	$701/15$	77