Chapter 5

VECTOR INTEGRATION

ORDINARY INTEGRALS OF VECTORS. Let $\mathbf{R}(u) = R_1(u)\mathbf{i} + R_2(u)\mathbf{j} + R_3(u)\mathbf{k}$ be a vector depending on a single scalar variable u, where $R_1(u), R_2(u), R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \mathbf{R}(u)\,du = \mathbf{i}\int R_1(u)\,du + \mathbf{j}\int R_2(u)\,du + \mathbf{k}\int R_3(u)\,du$$

is called an *indefinite integral* of $\mathbf{R}(u)$. If there exists a vector $\mathbf{S}(u)$ such that $\mathbf{R}(u) = \frac{d}{du}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\,du = \int \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}$$

where $\mathbf{c}$ is an *arbitrary constant vector* independent of u. The *definite integral* between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \mathbf{R}(u)\,du = \int_a^b \frac{d}{du}(\mathbf{S}(u))\,du = \mathbf{S}(u) + \mathbf{c}\Big|_a^b = \mathbf{S}(b) - \mathbf{S}(a)$$

This integral can also be defined as a limit of a sum in a manner analogous to that of elementary integral calculus.

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral*. If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A}\cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In aerodynamics and fluid mechanics this integral is called the *circulation* of $\mathbf{A}$ about C, where $\mathbf{A}$ represents the velocity of a fluid.

In general, any integral which is to be evaluated along a curve is called a line integral. Such integrals can be defined in terms of limits of sums as are the integrals of elementary calculus.

For methods of evaluation of line integrals, see the Solved Problems.

The following theorem is important.

THEOREM. If $\mathbf{A} = \nabla\phi$ everywhere in a region R of space, defined by $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$, $c_1 \leqq z \leqq c_2$, where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

1. $\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r}$ is independent of the path C in R joining P_1 and P_2.

2. $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ around any closed curve C in R.

In such case $\mathbf{A}$ is called a *conservative vector field* and ϕ is its *scalar potential.*

A vector field $\mathbf{A}$ is conservative if and only if $\nabla\times\mathbf{A} = \mathbf{0}$, or equivalently $\mathbf{A} = \nabla\phi$. In such case $\mathbf{A}\cdot d\mathbf{r} = A_1\,dx + A_2\,dy + A_3\,dz = d\phi$, an exact differential. See Problems 10-14.

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of the positive side of S is called a *positive* or *outward drawn* unit normal.

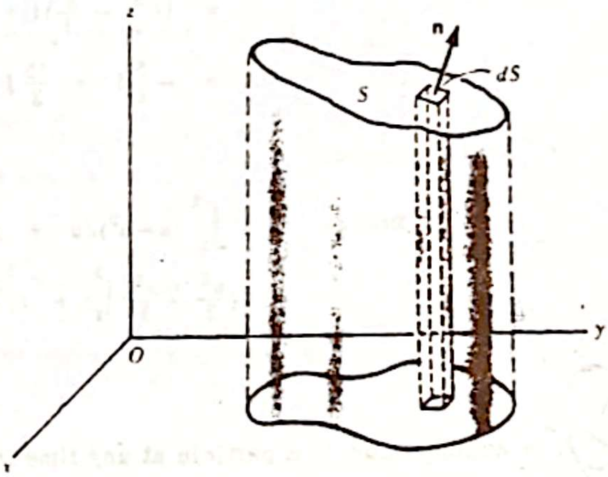
Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\,dS$. The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$

is an example of a surface integral called the *flux* of $\mathbf{A}$ over S. Other surface integrals are

$$\iint_S \phi\,dS, \quad \iint_S \phi\,\mathbf{n}\,dS, \quad \iint_S \mathbf{A}\times d\mathbf{S}$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus (see Problem 17).



The notation $\oiint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may also be used.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point. However, this does not pose any real problem since we can generally subdivide S into surfaces which do satisfy this restriction.

VOLUME INTEGRALS. Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \mathbf{A}\,dV \quad \text{and} \quad \iiint_V \phi\,dV$$

are examples of *volume integrals* or *space integrals* as they are sometimes called. For evaluation of such integrals, see the Solved Problems.

SOLVED PROBLEMS

1. If $\mathbf{R}(u) = (u-u^2)\mathbf{i} + 2u^3\mathbf{j} - 3\mathbf{k}$, find (*a*) $\int \mathbf{R}(u)\,du$ and (*b*) $\int_1^2 \mathbf{R}(u)\,du$.

(*a*)
$$\int \mathbf{R}(u)\,du = \int \left[(u-u^2)\mathbf{i} + 2u^3\mathbf{j} - 3\mathbf{k}\right]du$$
$$= \mathbf{i}\int (u-u^2)\,du + \mathbf{j}\int 2u^3\,du + \mathbf{k}\int -3\,du$$
$$= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + \mathbf{j}\left(\frac{u^4}{2} + c_2\right) + \mathbf{k}(-3u + c_3)$$
$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\mathbf{k} + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$
$$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\mathbf{k} + \mathbf{c}$$

where $\mathbf{c}$ is the constant vector $c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(*b*) From (*a*),
$$\int_1^2 \mathbf{R}(u)\,du = \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\mathbf{k} + \mathbf{c}\,\Big|_1^2$$
$$= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)\mathbf{i} + \frac{2^4}{2}\mathbf{j} - 3(2)\mathbf{k} + \mathbf{c}\right] - \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)\mathbf{i} + \frac{1^4}{2}\mathbf{j} - 3(1)\mathbf{k} + \mathbf{c}\right]$$
$$= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

Another Method.

$$\int_1^2 \mathbf{R}(u)\,du = \mathbf{i}\int_1^2 (u-u^2)\,du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du$$
$$= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Big|_1^2 + \mathbf{j}\left(\frac{u^4}{2}\right)\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2 = -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}$$

2. The acceleration of a particle at any time $t \geq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\cos 2t\,\mathbf{i} - 8\sin 2t\,\mathbf{j} + 16t\,\mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Integrating,
$$\mathbf{v} = \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt$$
$$= 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\mathbf{k} + \mathbf{c}_1$$

Putting $\mathbf{v}=0$ when $t=0$, we find $0 = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c}_1$ and $\mathbf{c}_1 = -4\mathbf{j}$.

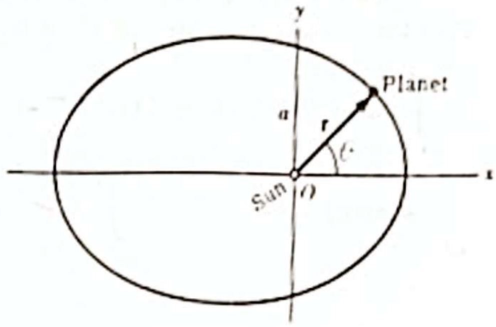
Then
$$\mathbf{v} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\mathbf{k}$$

so that
$$\frac{d\mathbf{r}}{dt} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\mathbf{k}.$$

Integrating,
$$\mathbf{r} = \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt$$
$$= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\mathbf{k} + \mathbf{c}_2$$

Putting $\mathbf{r}=0$ when $t=0$, $0 = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{c}_2$ and $\mathbf{c}_2 = 3\mathbf{i}$.

From analytical geometry, the polar equation of a conic section with focus at the origin and eccentricity ϵ is $r = \dfrac{a}{1+\epsilon\cos\theta}$ where a is a constant. Comparing this with the equation derived, it is seen that the required orbit is a conic section with eccentricity $\epsilon = p/GM$. The orbit is an ellipse, parabola or hyperbola according as ϵ is less than, equal to or greater than one. Since orbits of planets are closed curves it follows that they must be ellipses.



Ellipse $r = \dfrac{a}{1+\epsilon\cos\theta}$

LINE INTEGRALS

6. If $\mathbf{A} = (3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:
(a) $x = t,\ y = t^2,\ z = t^3$.
(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).
(c) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C [(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a) If $x=t,\ y=t^2,\ z=t^3$, points (0,0,0) and (1,1,1) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$
$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$
$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$
$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(b) Along the straight line from (0,0,0) to (1,0,0) $y=0,\ z=0,\ dy=0,\ dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \big(3x^2+6(0)\big)\,dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

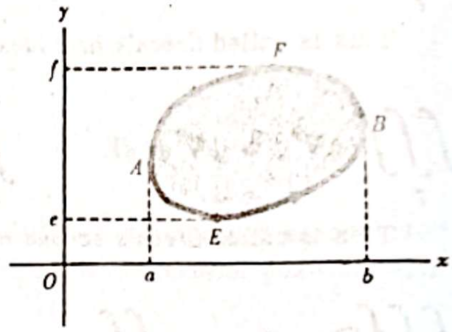
Along the straight line from (1,0,0) to (1,1,0) $x=1,\ z=0,\ dx=0,\ dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2+6y)\,0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

SOLVED PROBLEMS

GREEN'S THEOREM IN THE PLANE

1. Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.



Let the equations of the curves AEB and AFB (see adjoining figure) be $y = Y_1(x)$ and $y = Y_2(x)$ respectively. If R is the region bounded by C, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx = \int_{x=a}^{b} M(x,y)\Big|_{y=Y_1(x)}^{Y_2(x)}\,dx = \int_a^b \left[M(x,Y_2) - M(x,Y_1)\right]dx$$

$$= -\int_a^b M(x,Y_1)\,dx - \int_b^a M(x,Y_2)\,dx = -\oint_C M\,dx$$

Then

$$(1) \qquad \oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

Similarly let the equations of curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy = \int_e^f \left[N(X_2,y) - N(X_1,y)\right]dy$$

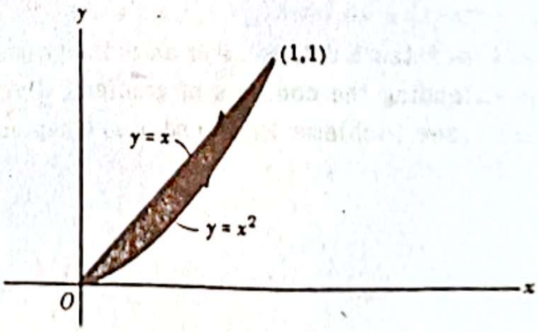
$$= \int_f^e N(X_1,y)\,dy + \int_e^f N(X_2,y)\,dy = \oint_C N\,dy$$

Then

$$(2) \qquad \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

Adding (1) and (2), $\displaystyle\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$

2. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.



$y = x$ and $y = x^2$ intersect at (0,0) and (1,1). The positive direction in traversing C is as

Along $y = x^2$, the line integral equals

$$\int_0^1 \left((x)(x^2) + x^4\right) dx + (x^2)(2x)\, dx = \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}$$

Along $y = x$ from (1,1) to (0,0) the line integral equals

$$\int_1^0 \left((x)(x) + x^2\right) dx + x^2\, dx = \int_1^0 3x^2\, dx = -1$$

Then the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\, dy$$

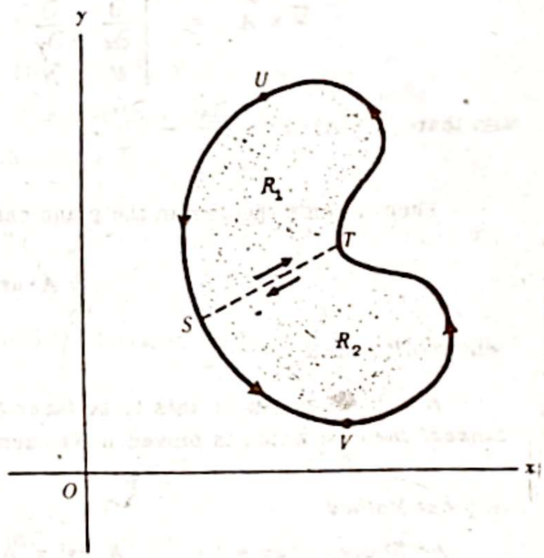
$$= \iint_R (x - 2y)\, dx\, dy = \int_{x=0}^1 \int_{y=x^2}^x (x-2y)\, dy\, dx$$

$$= \int_0^1 \left[\int_{x^2}^x (x-2y)\, dy\right] dx = \int_0^1 (xy - y^2)\Big|_{x^2}^x\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx = -\frac{1}{20}$$

so that the theorem is verified.

3. Extend the proof of Green's theorem in the plane given in Problem 1 to the curves C for which lines parallel to the coordinate axes may cut C in more than two points.



Consider a closed curve C such as shown in the adjoining figure, in which lines parallel to the axes may meet C in more than two points. By constructing line ST the region is divided into two regions R_1 and R_2 which are of the type considered in Problem 1 and for which Green's theorem applies, i.e.,

$$(1) \quad \int_{STUS} M\,dx + N\,dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$(2) \quad \int_{SVTS} M\,dx + N\,dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

Adding the left hand sides of (1) and (2), we have, omitting the integrand $M\,dx + N\,dy$ in each case,

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

using the fact that $\int_{ST} = -\int_{TS}$

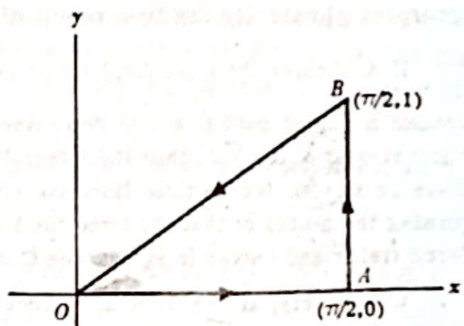
Adding the right hand sides of (1) and (2), omitting the integrand,

8. Find the area of the ellipse $x = a\cos\theta,\ y = b\sin\theta$.

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi} ab\,d\theta = \pi ab$$

9. Evaluate $\oint_C (y - \sin x)\,dx + \cos x\,dy$, where C is the triangle of the adjoining figure:
(*a*) directly,
(*b*) by using Green's theorem in the plane.



(*a*) Along OA, $y = 0$, $dy = 0$ and the integral equals

$$\int_0^{\pi/2} (0 - \sin x)\,dx + (\cos x)(0) = \int_0^{\pi/2} -\sin x\,dx$$

$$= \cos x\Big|_0^{\pi/2} = -1$$

Along AB, $x = \frac{\pi}{2}$, $dx = 0$ and the integral equals

$$\int_0^1 (y-1)0 + 0\,dy = 0$$

Along BO, $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi}dx$ and the integral equals

$$\int_{\pi/2}^0 \left(\frac{2x}{\pi} - \sin x\right)dx + \frac{2}{\pi}\cos x\,dx = \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi}\sin x\right)\Big|_{\pi/2}^0 = 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

Then the integral along $C = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$.

(*b*) $M = y - \sin x$, $N = \cos x$, $\frac{\partial N}{\partial x} = -\sin x$, $\frac{\partial M}{\partial y} = 1$ and

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_R (-\sin x - 1)\,dy\,dx$$

$$= \int_{x=0}^{\pi/2}\left[\int_{y=0}^{2x/\pi} (-\sin x - 1)\,dy\right]dx = \int_{x=0}^{\pi/2} (-y\sin x - y)\Big|_0^{2x/\pi} dx$$

$$= \int_0^{\pi/2} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi}\right)dx = -\frac{2}{\pi}(-x\cos x + \sin x) - \frac{x^2}{\pi}\Big|_0^{\pi/2} = -\frac{2}{\pi} - \frac{\pi}{4}$$

in agreement with part (*a*).

Note that although there exist lines parallel to the coordinate axes (coincident with the coordinate axes in this case) which meet C in an *infinite* number of points, Green's theorem in the plane still holds. In general the theorem is valid when C is composed of a finite number of straight line segments.

10. Show that Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the figure below.

The shaded region R, shown in the figure below, is multiply-connected since not every closed curve

$$(2) \qquad \iiint_V \frac{\partial A_1}{\partial x}\, dV = \iint_S A_1\, \mathbf{i}\cdot\mathbf{n}\, dS$$

$$(3) \qquad \iiint_V \frac{\partial A_2}{\partial y}\, dV = \iint_S A_2\, \mathbf{j}\cdot\mathbf{n}\, dS$$

Adding (*1*), (*2*) and (*3*),

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\mathbf{n}\, dS$$

or

$$\iiint_V \nabla\cdot\mathbf{A}\, dV = \iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points. To establish this extension, subdivide the region bounded by S into subregions whose surfaces do satisfy this condition. The procedure is analogous to that used in Green's theorem for the plane.

17. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\, dz\, dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz \Big|_{z=0}^{1} dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\, dy\, dx = \frac{3}{2}$$

The surface integral may also be evaluated directly as in Problem 23, Chapter 5.

18. Verify the divergence theorem for $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4,\ z=0$ and $z=3$.

$$\text{Volume integral} = \iiint_V \nabla\cdot\mathbf{A}\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\, dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4 - 4y + 2z)\, dz\, dy\, dx = 84\pi$$

The surface S of the cylinder consists of a base $S_1\ (z=0)$, the top $S_2\ (z=3)$ and the convex portion $S_3\ (x^2+y^2=4)$. Then

Surface integral $= \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 + \iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 + \iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3$

<u>On S_1 $(z=0)$</u>, $\mathbf{n} = -\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 = 0$.

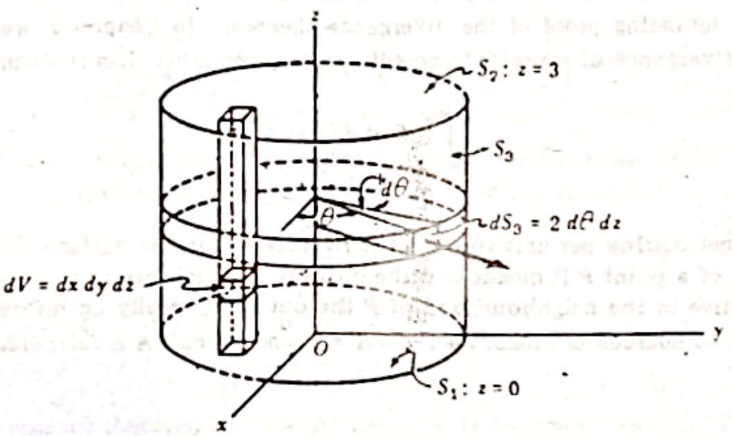
<u>On S_2 $(z=3)$</u>, $\mathbf{n} = \mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 9$, so that

$$\iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 = 9\iint_{S_2} dS_2 = 36\pi, \quad \text{since area of } S_2 = 4\pi$$

<u>On S_3 $(x^2+y^2=4)$</u>. A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Then a unit normal is $\mathbf{n} = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{2}$ since $x^2+y^2=4$.

$$\mathbf{A}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{2}\right) = 2x^2 - y^3$$



From the figure above, $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2\,d\theta\,dz$ and so

$$\iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection of S_3 on the xz or yz coordinate planes.

19. If div **A** denotes the divergence of a vector field **A** at a point P, show that

$$\operatorname{div}\mathbf{A} = \lim_{\Delta V\to 0} \frac{\iint_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

$$c\rho \frac{\partial U}{\partial t} = \nabla\cdot(\kappa \nabla U)$$

or if κ, c, ρ are constants,

$$\frac{\partial U}{\partial t} = \frac{\kappa}{c\rho}\nabla\cdot\nabla U = k\,\nabla^2 U$$

The quantity k is called the *diffusivity*. For steady-state heat flow (i.e. $\frac{\partial U}{\partial t} = 0$ or U is independent of time) the equation reduces to Laplace's equation $\nabla^2 U = 0$.

STOKES' THEOREM

30. (*a*) Express Stokes' theorem in words and (*b*) write it in rectangular form.

(*a*) The line integral of the tangential component of a vector **A** taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of **A** taken over any surface S having C as its boundary.

(*b*) As in Problem 14(*b*),

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}, \qquad \mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

Then

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}$$

$$(\nabla\times\mathbf{A})\cdot\mathbf{n} = \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma$$

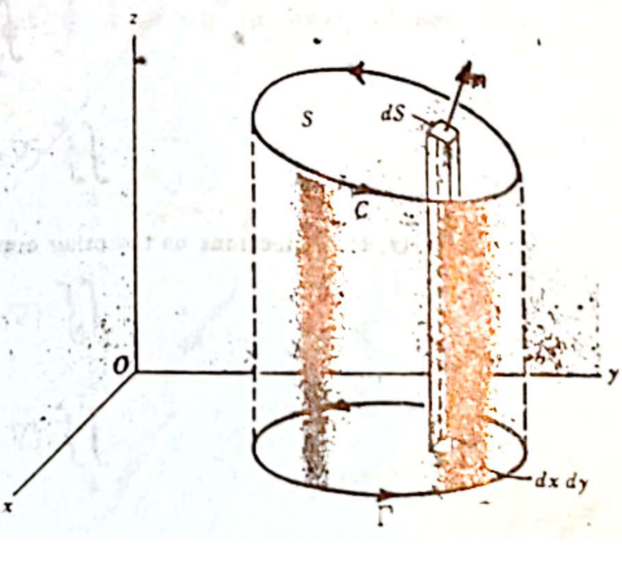
$$\mathbf{A}\cdot d\mathbf{r} = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = A_1dx + A_2dy + A_3dz$$

and Stokes' theorem becomes

$$\iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\cos\beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\cos\gamma\right]dS = \oint_C A_1dx + A_2dy + A_3dz$$

31. Prove Stokes' theorem.

Let S be a surface which is such that its projections on the xy, yz and xz planes are regions bounded by simple closed curves, as indicated in the adjoining figure. Assume S to have representation $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$, where f, g, h are single-valued, continuous and differentiable functions. We must show that



$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S [\nabla\times(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})]\cdot\mathbf{n}\,dS$$

$$= \oint_C \mathbf{A}\cdot d\mathbf{r}$$

where C is the boundary of S.

Thus by addition,

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, \dots S_k$ with boundaries $C_1, C_2, \dots C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, \dots C_k$, the line integral over C is obtained.

32. Verify Stokes' theorem for $\mathbf{A} = (2x - y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and centre at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C (2x - y)\, dx - yz^2\, dy - y^2z\, dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt = \pi$$

Also,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

Then

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{k} \cdot \mathbf{n}\, dS = \iint_R dx\, dy$$

since $\mathbf{n} \cdot \mathbf{k}\, dS = dx\, dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx = 4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = 4\int_0^1 \sqrt{1 - x^2}\, dx = \pi$$

and Stokes' theorem is verified.

33. Prove that a necessary and sufficient condition that $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ for every closed curve C is that $\nabla \times \mathbf{A} = \mathbf{0}$ identically.

Sufficiency. Suppose $\nabla \times \mathbf{A} = \mathbf{0}$. Then by Stokes' theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = 0$$

Necessity. Suppose $\oint_C \mathbf{A} \cdot d\mathbf{r} = 0$ around every closed path C, and assume $\nabla \times \mathbf{A} \neq \mathbf{0}$ at some point P. Then assuming $\nabla \times \mathbf{A}$ is continuous there will be a region with P as an interior point, where $\nabla \times \mathbf{A} \neq \mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point has the same direction as $\nabla \times \mathbf{A}$, i.e. $\nabla \times \mathbf{A} = \alpha\mathbf{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem