Chapter 1

The Laplace Transform

DEFINITION OF THE LAPLACE TRANSFORM

Let $F(t)$ be a function of t specified for $t > 0$. Then the *Laplace transform* of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad (1)$$

where we assume at present that the parameter s is real. Later it will be found useful to consider s complex.

The Laplace transform of $F(t)$ is said to *exist* if the integral (1) *converges* for some value of s; otherwise it does not exist. For sufficient conditions under which the Laplace transform does exist, see Page 2.

NOTATION

If a function of t is indicated in terms of a capital letter, such as $F(t)$, $G(t)$, $Y(t)$, etc., the Laplace transform of the function is denoted by the corresponding lower case letter, i.e. $f(s)$, $g(s)$, $y(s)$, etc. In other cases, a tilde (~) can be used to denote the Laplace transform. Thus, for example, the Laplace transform of $u(t)$ is $\tilde{u}(s)$.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

The adjacent table shows Laplace transforms of various elementary functions. For details of evaluation using definition (1), see Problems 1 and 2. For a more extensive table see Appendix B, Pages 245 to 254.

	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}$ $\quad s > 0$
2.	t	$\frac{1}{s^2}$ $\quad s > 0$
3.	t^n $n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$ $\quad s > 0$ *Note.* Factorial $n = n! = 1 \cdot 2 \cdots n$ Also, by definition $0! = 1$.
4.	e^{at}	$\frac{1}{s-a}$ $\quad s > a$
5.	$\sin at$	$\frac{a}{s^2+a^2}$ $\quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}$ $\quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}$ $\quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}$ $\quad s > \lvert a \rvert$

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SOME IMPORTANT PROPERTIES OF LAPLACE TRANSFORMS

In the following list of theorems we assume, unless otherwise stated, that all functions satisfy the conditions of *Theorem 1-1* so that their Laplace transforms exist.

1. Linearity property.

Theorem 1-2. If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} = c_1f_1(s) + c_2f_2(s) \tag{2}$$

The result is easily extended to more than two functions.

Example.
$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$$
$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$
$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

The symbol $\mathcal{L}$, which transforms $F(t)$ into $f(s)$, is often called the *Laplace transformation operator*. Because of the property of $\mathcal{L}$ expressed in this theorem, we say that $\mathcal{L}$ is a *linear operator* or that it has the *linearity property*.

2. First translation or shifting property.

Theorem 1-3. If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a) \tag{3}$$

Example. Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Second translation or shifting property.

Theorem 1-4. If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then

$$\mathcal{L}\{G(t)\} = e^{-as}f(s) \tag{4}$$

Example. Since $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$, the Laplace transform of the function

$$G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$$

is $6e^{-2s}/s^4$.

4. Change of scale property.

Theorem 1-5. If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right) \tag{5}$$

Example. Since $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$, we have

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3}\,\frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$$

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(c) $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \lim_{P\to\infty}\int_0^P e^{-(s-a)t}\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^P = \lim_{P\to\infty}\frac{1-e^{-(s-a)P}}{s-a} = \frac{1}{s-a} \quad \text{if } s > a$$

For methods not employing direct integration, see Problem 15.

2. Prove that (a) $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$, (b) $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$ if $s > 0$.

(a) $$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\sin at\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\Big|_0^P$$

$$= \lim_{P\to\infty}\left\{\frac{a}{s^2+a^2} - \frac{e^{-sP}(s\sin aP + a\cos aP)}{s^2+a^2}\right\}$$

$$= \frac{a}{s^2+a^2} \quad \text{if } s > 0$$

(b) $$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt = \lim_{P\to\infty}\int_0^P e^{-st}\cos at\,dt$$

$$= \lim_{P\to\infty} \frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2}\Big|_0^P$$

$$= \lim_{P\to\infty}\left\{\frac{s}{s^2+a^2} - \frac{e^{-sP}(s\cos aP - a\sin aP)}{s^2+a^2}\right\}$$

$$= \frac{s}{s^2+a^2} \quad \text{if } s > 0$$

We have used here the results

$$\int e^{\alpha t}\sin\beta t\,dt = \frac{e^{\alpha t}(\alpha\sin\beta t - \beta\cos\beta t)}{\alpha^2+\beta^2} \tag{1}$$

$$\int e^{\alpha t}\cos\beta t\,dt = \frac{e^{\alpha t}(\alpha\cos\beta t + \beta\sin\beta t)}{\alpha^2+\beta^2} \tag{2}$$

Another method. Assuming that the result of Problem 1(c) holds for complex numbers (which can be proved), we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \tag{3}$$

But $e^{iat} = \cos at + i\sin at$. Hence

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st}(\cos at + i\sin at) \tag{4}$$

$$= \int_0^\infty e^{-st}\cos at\,dt + i\int_0^\infty e^{-st}\sin at\,dt = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}$$

From (3) and (4) we have on equating real and imaginary parts,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

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3. Prove that (a) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$, (b) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$ if $s > |a|$.

(a)

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}e^{at}\,dt - \frac{1}{2}\int_0^\infty e^{-st}e^{-at}\,dt$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2} \quad \text{for } s > |a|$$

Another method. Using the linearity property of the Laplace transformation, we have at once

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2 - a^2} \quad \text{for } s > |a|$$

(b) As in part (a),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{s}{s^2 - a^2} \quad \text{for } s > |a|$$

4. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} 5 & 0 < t < 3 \\ 0 & t > 3 \end{cases}$

By definition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt = \int_0^3 e^{-st}(5)\,dt + \int_3^\infty e^{-st}(0)\,dt$$

$$= 5\int_0^3 e^{-st}\,dt = 5\left.\frac{e^{-st}}{-s}\right|_0^3 = \frac{5(1 - e^{-3s})}{s}$$

THE LINEARITY PROPERTY

5. Prove the *linearity property* [*Theorem 1-2*, Page 3].

Let $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st}F_1(t)\,dt$ and $\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st}F_2(t)\,dt$. Then if c_1 and c_2 are any constants,

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = \int_0^\infty e^{-st}\{c_1F_1(t) + c_2F_2(t)\}\,dt$$

$$= c_1\int_0^\infty e^{-st}F_1(t)\,dt + c_2\int_0^\infty e^{-st}F_2(t)\,dt$$

$$= c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$$

$$= c_1f_1(s) + c_2f_2(s)$$

The result is easily generalized [see Problem 61].

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9. Prove the *second translation or shifting property*:

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

$$\begin{aligned}
\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt &= \int_0^a e^{-st} G(t)\, dt + \int_a^\infty e^{-st} G(t)\, dt \\
&= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st} F(t-a)\, dt \\
&= \int_a^\infty e^{-st} F(t-a)\, dt \\
&= \int_0^\infty e^{-s(u+a)} F(u)\, du \\
&= e^{-as} \int_0^\infty e^{-su} F(u)\, du \\
&= e^{-as} f(s)
\end{aligned}$$

where we have used the substitution $t = u + a$.

10. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$.

Method 1.

$$\begin{aligned}
\mathcal{L}\{F(t)\} &= \int_0^{2\pi/3} e^{-st}(0)\, dt + \int_{2\pi/3}^\infty e^{-st} \cos(t - 2\pi/3)\, dt \\
&= \int_0^\infty e^{-s(u+2\pi/3)} \cos u\, du \\
&= e^{-2\pi s/3} \int_0^\infty e^{-su} \cos u\, du = \frac{s e^{-2\pi s/3}}{s^2+1}
\end{aligned}$$

Method 2. Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, it follows from Problem 9, with $a = 2\pi/3$, that

$$\mathcal{L}\{F(t)\} = \frac{s e^{-2\pi s/3}}{s^2+1}$$

11. Prove the *change of scale property*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$.

$$\begin{aligned}
\mathcal{L}\{F(at)\} &= \int_0^\infty e^{-st} F(at)\, dt \\
&= \int_0^\infty e^{-s(u/a)} F(u)\, d(u/a) \\
&= \frac{1}{a} \int_0^\infty e^{-su/a} F(u)\, du \\
&= \frac{1}{a} f\left(\frac{s}{a}\right)
\end{aligned}$$

using the transformation $t = u/a$.

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9. Prove the *second translation or shifting property*:

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$.

$$\begin{aligned} \mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} G(t)\,dt = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt \\ &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} F(t-a)\,dt \\ &= \int_a^\infty e^{-st} F(t-a)\,dt \\ &= \int_0^\infty e^{-s(u+a)} F(u)\,du \\ &= e^{-as}\int_0^\infty e^{-su} F(u)\,du \\ &= e^{-as} f(s) \end{aligned}$$

where we have used the substitution $t = u + a$.

10. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$.

Method 1.

$$\begin{aligned} \mathcal{L}\{F(t)\} &= \int_0^{2\pi/3} e^{-st}(0)\,dt + \int_{2\pi/3}^\infty e^{-st}\cos(t - 2\pi/3)\,dt \\ &= \int_0^\infty e^{-s(u+2\pi/3)} \cos u\,du \\ &= e^{-2\pi s/3}\int_0^\infty e^{-su}\cos u\,du = \frac{se^{-2\pi s/3}}{s^2+1} \end{aligned}$$

Method 2. Since $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$, it follows from Problem 9, with $a = 2\pi/3$, that

$$\mathcal{L}\{F(t)\} = \frac{se^{-2\pi s/3}}{s^2+1}$$

11. Prove the *change of scale property*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{s}{a}\right)$.

$$\begin{aligned} \mathcal{L}\{F(at)\} &= \int_0^\infty e^{-st} F(at)\,dt \\ &= \int_0^\infty e^{-s(u/a)} F(u)\,d(u/a) \\ &= \frac{1}{a}\int_0^\infty e^{-su/a} F(u)\,du \\ &= \frac{1}{a} f\left(\frac{s}{a}\right) \end{aligned}$$

using the transformation $t = u/a$.

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12. Given that $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$, find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$.

By Problem 11,

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}(1/(s/a)) = \frac{1}{a}\tan^{-1}(a/s)$$

Then $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s)$.

LAPLACE TRANSFORM OF DERIVATIVES

13. Prove *Theorem 1-6*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0)$.

Using integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{F'(t)\} &= \int_0^\infty e^{-st}F'(t)\,dt = \lim_{P\to\infty}\int_0^P e^{-st}F'(t)\,dt \\
&= \lim_{P\to\infty}\left\{e^{-st}F(t)\Big|_0^P + s\int_0^P e^{-st}F(t)\,dt\right\} \\
&= \lim_{P\to\infty}\left\{e^{-sP}F(P) - F(0) + s\int_0^P e^{-st}F(t)\,dt\right\} \\
&= s\int_0^\infty e^{-st}F(t)\,dt - F(0) \\
&= s\,f(s) - F(0)
\end{aligned}$$

using the fact that $F(t)$ is of exponential order γ as $t\to\infty$, so that $\lim_{P\to\infty} e^{-sP}F(P) = 0$ for $s > \gamma$.

For cases where $F(t)$ is not continuous at $t=0$, see Problem 68.

14. Prove *Theorem 1-9*, Page 4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$.

By Problem 13,

$$\mathcal{L}\{G'(t)\} = s\,\mathcal{L}\{G(t)\} - G(0) = s\,g(s) - G(0)$$

Let $G(t) = F'(t)$. Then

$$\begin{aligned}
\mathcal{L}\{F''(t)\} &= s\,\mathcal{L}\{F'(t)\} - F'(0) \\
&= s\,[s\,\mathcal{L}\{F(t)\} - F(0)] - F'(0) \\
&= s^2\,\mathcal{L}\{F(t)\} - s\,F(0) - F'(0) \\
&= s^2 f(s) - s\,F(0) - F'(0)
\end{aligned}$$

The generalization to higher order derivatives can be proved by using mathematical induction [see Problem 65].

15. Use *Theorem 1-6*, Page 4, to derive each of the following Laplace transforms:

(*a*) $\mathcal{L}\{1\} = \frac{1}{s}$, (*b*) $\mathcal{L}\{t\} = \frac{1}{s^2}$, (*c*) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

Theorem 1-6 states, under suitable conditions given on Page 4, that

$$\mathcal{L}\{F'(t)\} = s\,\mathcal{L}\{F(t)\} - F(0) \tag{1}$$

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(a) Let $F(t) = 1$. Then $F'(t) = 0$, $F(0) = 1$, and (1) becomes

$$\mathcal{L}\{0\} = 0 = s\mathcal{L}\{1\} - 1 \qquad \text{or} \qquad \mathcal{L}\{1\} = 1/s \tag{2}$$

(b) Let $F(t) = t$. Then $F'(t) = 1$, $F(0) = 0$, and (1) becomes using part (a)

$$\mathcal{L}\{1\} = 1/s = s\mathcal{L}\{t\} - 0 \qquad \text{or} \qquad \mathcal{L}\{t\} = 1/s^2 \tag{3}$$

By using mathematical induction we can similarly show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$ for any positive integer n.

(c) Let $F(t) = e^{at}$. Then $F'(t) = ae^{at}$, $F(0) = 1$, and (1) becomes

$$\mathcal{L}\{ae^{at}\} = s\mathcal{L}\{e^{at}\} - 1, \quad \text{i.e.} \quad a\mathcal{L}\{e^{at}\} = s\mathcal{L}\{e^{at}\} - 1 \quad \text{or} \quad \mathcal{L}\{e^{at}\} = 1/(s-a)$$

16. Use *Theorem 1-9* to show that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$.

Let $F(t) = \sin at$. Then $F'(t) = a\cos at$, $F''(t) = -a^2 \sin at$, $F(0) = 0$, $F'(0) = a$. Hence from the result

$$\mathcal{L}\{F''(t)\} = s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0)$$

we have

$$\mathcal{L}\{-a^2 \sin at\} = s^2\mathcal{L}\{\sin at\} - s(0) - a$$

i.e.

$$-a^2\mathcal{L}\{\sin at\} = s^2\mathcal{L}\{\sin at\} - a$$

or

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

LAPLACE TRANSFORM OF INTEGRALS

17. Prove *Theorem 1-11*: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = f(s)/s$.

Let $G(t) = \int_0^t F(u)\,du$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace transform of both sides, we have

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) = s\mathcal{L}\{G(t)\} = f(s)$$

Thus

$$\mathcal{L}\{G(t)\} = \frac{f(s)}{s} \qquad \text{or} \qquad \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

18. Find $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\}$.

We have by the Example following *Theorem 1-13* on Page 5,

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$$

Thus by Problem 17,

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

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MULTIPLICATION BY POWERS OF t

19. Prove *Theorem 1-12*, Page 5:

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{t^n F(t)\} = (-1)^n \dfrac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s)$ where $n = 1, 2, 3, \ldots$.

We have

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Then by Leibnitz's rule for differentiating under the integral sign,

$$\frac{df}{ds} = f'(s) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} F(t)\, dt$$

$$= \int_0^\infty -te^{-st} F(t)\, dt$$

$$= -\int_0^\infty e^{-st} \{t\,F(t)\}\, dt$$

$$= -\mathcal{L}\{t\,F(t)\}$$

Thus

$$\mathcal{L}\{t\,F(t)\} = -\frac{df}{ds} = -f'(s) \qquad (1)$$

which proves the theorem for $n = 1$.

To establish the theorem in general, we use *mathematical induction*. Assume the theorem true for $n = k$, i.e. assume

$$\int_0^\infty e^{-st} \{t^k F(t)\}\, dt = (-1)^k f^{(k)}(s) \qquad (2)$$

Then

$$\frac{d}{ds}\int_0^\infty e^{-st} \{t^k F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$$

or by Leibnitz's rule,

$$-\int_0^\infty e^{-st} \{t^{k+1} F(t)\}\, dt = (-1)^k f^{(k+1)}(s)$$

i.e.

$$\int_0^\infty e^{-st} \{t^{k+1} F(t)\}\, dt = (-1)^{k+1} f^{(k+1)}(s) \qquad (3)$$

It follows that *if* (2) is true, i.e. if the theorem holds for $n = k$, then (3) is true, i.e. the theorem holds for $n = k+1$. But by (1) the theorem is true for $n = 1$. Hence it is true for $n = 1+1 = 2$ and $n = 2+1 = 3$, etc., and thus for all positive integer values of n.

To be completely rigorous, it is necessary to prove that Leibnitz's rule can be applied. For this, see Problem 166.

20. Find (a) $\mathcal{L}\{t \sin at\}$, (b) $\mathcal{L}\{t^2 \cos at\}$.

(a) Since $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2 + a^2}$, we have by Problem 19

$$\mathcal{L}\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

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Another method.

Since $$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt = \frac{s}{s^2+a^2}$$

we have by differentiating with respect to the parameter a [using Leibnitz's rule],

$$\frac{d}{da}\int_0^\infty e^{-st}\cos at\,dt = \int_0^\infty e^{-st}(-t\sin at)\,dt = -\mathcal{L}\{t\sin at\}$$

$$= \frac{d}{da}\left(\frac{s}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2}$$

from which

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Note that the result is equivalent to $\frac{d}{da}\mathcal{L}\{\cos at\} = \mathcal{L}\left\{\frac{d}{da}\cos at\right\}$.

(*b*) Since $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, we have by Problem 19

$$\mathcal{L}\{t^2\cos at\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$

We can also use the second method of part (*a*) by writing

$$\mathcal{L}\{t^2\cos at\} = \mathcal{L}\left\{-\frac{d^2}{da^2}(\cos at)\right\} = -\frac{d^2}{da^2}\mathcal{L}\{\cos at\}$$

which gives the same result.

DIVISION BY t

21. Prove *Theorem 1-13*, Page 5: If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$.

Let $G(t) = \frac{F(t)}{t}$. Then $F(t) = t\,G(t)$. Taking the Laplace transform of both sides and using Problem 19, we have

$$\mathcal{L}\{F(t)\} = -\frac{d}{ds}\mathcal{L}\{G(t)\} \qquad \text{or} \qquad f(s) = -\frac{dg}{ds}$$

Then integrating, we have

$$g(s) = -\int_\infty^s f(u)\,du = \int_s^\infty f(u)\,du \tag{1}$$

i.e.

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

Note that in (*1*) we have chosen the "constant of integration" so that $\lim_{s\to\infty} g(s) = 0$ [see *Theorem 1-15*, Page 5].

22. (*a*) Prove that $\int_0^\infty \frac{F(t)}{t}\,dt = \int_0^\infty f(u)\,du$ provided that the integrals converge.

(*b*) Show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

(*a*) From Problem 21,

$$\int_0^\infty e^{-st}\frac{F(t)}{t}\,dt = \int_s^\infty f(u)\,du$$

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Then taking the limit as $s \to 0+$, assuming the integrals converge, the required result is obtained.

(b) Let $F(t) = \sin t$ so that $f(s) = 1/(s^2+1)$ in part (a). Then

$$\int_0^\infty \frac{\sin t}{t}\,dt \;=\; \int_0^\infty \frac{du}{u^2+1} \;=\; \tan^{-1} u \Big|_0^\infty \;=\; \frac{\pi}{2}$$

PERIODIC FUNCTIONS

23. Prove *Theorem 1-14*, Page 5: If $F(t)$ has period $T > 0$ then

$$\mathcal{L}\{F(t)\} \;=\; \frac{\int_0^T e^{-st} F(t)\,dt}{1-e^{-sT}}$$

We have

$$\mathcal{L}\{F(t)\} \;=\; \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^T e^{-st} F(t)\,dt \;+\; \int_T^{2T} e^{-st} F(t)\,dt \;+\; \int_{2T}^{3T} e^{-st} F(t)\,dt \;+\; \cdots$$

In the second integral let $t = u+T$, in the third integral let $t = u + 2T$, etc. Then

$$\mathcal{L}\{F(t)\} \;=\; \int_0^T e^{-su} F(u)\,du + \int_0^T e^{-s(u+T)} F(u+T)\,du \;+\; \int_0^T e^{-s(u+2T)} F(u+2T)\,du \;+\; \cdots$$

$$= \int_0^T e^{-su} F(u)\,du \;+\; e^{-sT}\int_0^T e^{-su} F(u)\,du \;+\; e^{-2sT}\int_0^T e^{-su} F(u)\,du \;+\; \cdots$$

$$= (1 + e^{-sT} + e^{-2sT} + \cdots)\int_0^T e^{-su} F(u)\,du$$

$$= \frac{\int_0^T e^{-su} F(u)\,du}{1-e^{-sT}}$$

where we have used the periodicity to write $F(u+T) = F(u)$, $F(u+2T) = F(u)$, ..., and the fact that

$$1 + r + r^2 + r^3 + \cdots \;=\; \frac{1}{1-r}, \qquad |r| < 1$$

24. (a) Graph the function

$$F(t) \;=\; \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

extended periodically with period 2π.

(b) Find $\mathcal{L}\{F(t)\}$.

(a) The graph appears in Fig. 1-5.

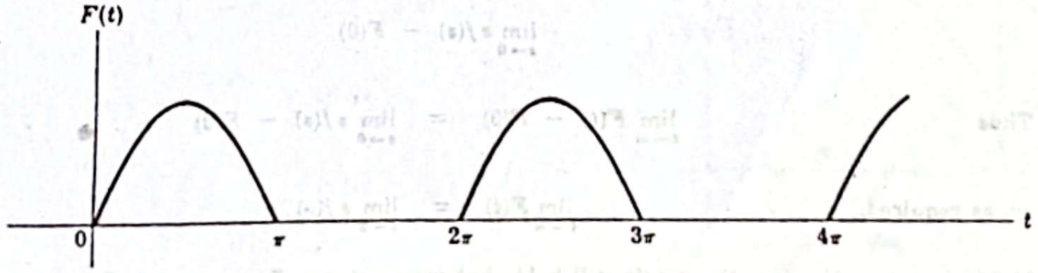

Fig. 1-5

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Since the integrand is positive, we have

$$\iint_{\mathcal{R}_1} e^{-(x^2+y^2)}\,dx\,dy \quad \le \quad I_P^2 \quad \le \quad \iint_{\mathcal{R}_2} e^{-(x^2+y^2)}\,dx\,dy \qquad (1)$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the regions in the first quadrant bounded by the circles having radii P and $P\sqrt{2}$ respectively.

Using polar coordinates (r, θ) we have from (1),

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{P} e^{-r^2}\,r\,dr\,d\theta \quad \le \quad I_P^2 \quad \le \quad \int_{\theta=0}^{\pi/2}\int_{r=0}^{P\sqrt{2}} e^{-r^2}\,r\,dr\,d\theta \qquad (2)$$

or

$$\frac{\pi}{4}(1-e^{-P^2}) \quad \le \quad I_P^2 \quad \le \quad \frac{\pi}{4}(1-e^{-2P^2}) \qquad (3)$$

Then taking the limit as $P \to \infty$ in (3), we find $\lim_{P\to\infty} I_P^2 = I^2 = \pi/4$ and $I = \sqrt{\pi}/2$.

30. Prove: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$\Gamma(\frac{1}{2}) = \int_0^\infty u^{-1/2}e^{-u}\,du$. Letting $u = v^2$, this integral becomes on using Problem 29

$$2\int_0^\infty e^{-v^2}\,dv \quad = \quad 2\left(\frac{\sqrt{\pi}}{2}\right) \quad = \quad \sqrt{\pi}$$

31. Prove: $\mathcal{L}\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1$, $s > 0$.

$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}t^n\,dt$. Letting $st = u$, assuming $s > 0$, this becomes

$$\mathcal{L}\{t^n\} \quad = \quad \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n d\left(\frac{u}{s}\right) \quad = \quad \frac{1}{s^{n+1}}\int_0^\infty u^n e^{-u}\,du \quad = \quad \frac{\Gamma(n+1)}{s^{n+1}}$$

32. Prove: $\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}$, $s > 0$.

Let $n = -1/2$ in Problem 31. Then

$$\mathcal{L}\{t^{-1/2}\} \quad = \quad \frac{\Gamma(\frac{1}{2})}{s^{1/2}} \quad = \quad \frac{\sqrt{\pi}}{s^{1/2}} \quad = \quad \sqrt{\frac{\pi}{s}}$$

Note that although $F(t) = t^{-1/2}$ does not satisfy the sufficient conditions of *Theorem 1-1*, Page 2, the Laplace transform does exist. The function does satisfy the conditions of the theorem in Prob. 145.

33. By assuming $\Gamma(n+1) = n\Gamma(n)$ holds for all n, find:

(*a*) $\Gamma(-\frac{1}{2})$, (*b*) $\Gamma(-\frac{3}{2})$, (*c*) $\Gamma(-\frac{5}{2})$, (*d*) $\Gamma(0)$, (*e*) $\Gamma(-1)$, (*f*) $\Gamma(-2)$.

(*a*) Letting $n = -\frac{1}{2}$, $\Gamma(\frac{1}{2}) = -\frac{1}{2}\Gamma(-\frac{1}{2})$. Then $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$.

(*b*) Letting $n = -\frac{3}{2}$, $\Gamma(-\frac{1}{2}) = -\frac{3}{2}\Gamma(-\frac{3}{2})$. Then $\Gamma(-\frac{3}{2}) = -\frac{2}{3}\Gamma(-\frac{1}{2}) = (2)(\frac{2}{3})\sqrt{\pi} = \frac{4}{3}\sqrt{\pi}$ by part (*a*).

(*c*) Letting $n = -\frac{5}{2}$, $\Gamma(-\frac{3}{2}) = -\frac{5}{2}\Gamma(-\frac{5}{2})$. Then $\Gamma(-\frac{5}{2}) = -\frac{2}{5}\Gamma(-\frac{3}{2}) = -(2)(\frac{2}{3})(\frac{2}{5})\sqrt{\pi} = -\frac{8}{15}\sqrt{\pi}$ by part (*b*).

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35. Find $\mathcal{L}\{J_1(t)\}$, where $J_1(t)$ is Bessel's function of order one.

From Property 3 for Bessel functions, Page 7, we have $J_0'(t) = -J_1(t)$. Hence

$$\mathcal{L}\{J_1(t)\} = -\mathcal{L}\{J_0'(t)\} = -[s\,\mathcal{L}\{J_0(t)\} - 1]$$

$$= 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}$$

The methods of infinite series and differential equations can also be used [see Problem 178, Page 41].

THE SINE, COSINE AND EXPONENTIAL INTEGRALS

36. Prove: $\mathcal{L}\{\mathrm{Si}(t)\} = \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$.

Method 1. Let $F(t) = \int_0^t \frac{\sin u}{u}\,du$. Then $F(0) = 0$ and $F'(t) = \frac{\sin t}{t}$ or $tF'(t) = \sin t$.

Taking the Laplace transform,

$$\mathcal{L}\{tF'(t)\} = \mathcal{L}\{\sin t\} \quad \text{or} \quad -\frac{d}{ds}\{s f(s) - F(0)\} = \frac{1}{s^2+1}$$

i.e.

$$\frac{d}{ds}\{s f(s)\} = \frac{-1}{s^2+1}$$

Integrating,

$$s f(s) = -\tan^{-1}s + c$$

By the initial value theorem, $\lim_{s\to\infty} s f(s) = \lim_{t\to 0} F(t) = F(0) = 0$ so that $c = \pi/2$. Thus

$$s f(s) = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s} \quad \text{or} \quad f(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Method 2. See Problem 18.

Method 3. Using infinite series, we have

$$\int_0^t \frac{\sin u}{u}\,du = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right)du$$

$$= t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots$$

Then

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\right\} = \mathcal{L}\left\{t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3\cdot 3!}\cdot\frac{3!}{s^4} + \frac{1}{5\cdot 5!}\cdot\frac{5!}{s^6} - \frac{1}{7\cdot 7!}\cdot\frac{7!}{s^8} + \cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \cdots$$

$$= \frac{1}{s}\left\{\frac{(1/s)}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \cdots\right\}$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}$$

using the series $\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \cdots$, $|x| < 1$.

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THE ERROR FUNCTION

39. Prove: $\mathcal{L}\{\text{erf}\sqrt{t}\} = \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} = \frac{1}{s\sqrt{s+1}}$.

Using infinite series, we have

$$\begin{aligned}
\mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du\right\} &= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \cdots\right)du\right\} \\
&= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5\cdot 2!} - \frac{t^{7/2}}{7\cdot 3!} + \cdots\right)\right\} \\
&= \frac{2}{\sqrt{\pi}}\left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5\cdot 2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7\cdot 3!\,s^{9/2}} + \cdots\right\} \\
&= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^{7/2}} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^{9/2}} + \cdots \\
&= \frac{1}{s^{3/2}}\left\{1 - \frac{1}{2}\frac{1}{s} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{s^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{s^3} + \cdots\right\} \\
&= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{s+1}}
\end{aligned}$$

using the binomial theorem [see Problem 172].

For another method, see Problem 175(a).

IMPULSE FUNCTIONS. THE DIRAC DELTA FUNCTION.

40. Prove that $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$ where $\mathcal{U}(t-a)$ is Heaviside's unit step function.

We have $\mathcal{U}(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$. Then

$$\begin{aligned}
\mathcal{L}\{\mathcal{U}(t-a)\} &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt \\
&= \lim_{P\to\infty}\int_a^P e^{-st}\,dt = \lim_{P\to\infty}\left.\frac{e^{-st}}{-s}\right|_a^P \\
&= \lim_{P\to\infty}\frac{e^{-as}-e^{-sP}}{s} = \frac{e^{-as}}{s}
\end{aligned}$$

Another method.

Since $\mathcal{L}\{1\} = 1/s$, we have by Problem 9, $\mathcal{L}\{\mathcal{U}(t-a)\} = e^{-as}/s$.

41. Find $\mathcal{L}\{F_\epsilon(t)\}$ where $F_\epsilon(t)$ is defined by (30), Page 8.

We have $F_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \leq t \leq \epsilon \\ 0 & t > \epsilon \end{cases}$. Then

$$\begin{aligned}
\mathcal{L}\{F_\epsilon(t)\} &= \int_0^\infty e^{-st}F_\epsilon(t)\,dt \\
&= \int_0^\epsilon e^{-st}(1/\epsilon)\,dt + \int_\epsilon^\infty e^{-st}(0)\,dt \\
&= \frac{1}{\epsilon}\int_0^\epsilon e^{-st}\,dt = \frac{1-e^{-s\epsilon}}{\epsilon s}
\end{aligned}$$

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42. (*a*) Show that $\lim_{\epsilon \to 0} \mathcal{L}\{F_\epsilon(t)\} = 1$ in Problem 41.

(*b*) Is the result in (*a*) the same as $\mathcal{L}\left\{\lim_{\epsilon \to 0} F_\epsilon(t)\right\}$? Explain.

(*a*) This follows at once since

$$\lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon} = \lim_{\epsilon \to 0} \frac{1 - (1 - s\epsilon + s^2\epsilon^2/2! - \cdots)}{s\epsilon} = \lim_{\epsilon \to 0}\left(1 - \frac{s\epsilon}{2!} + \cdots\right) = 1$$

It also follows by use of L'Hospital's rule.

(*b*) Mathematically speaking, $\lim_{\epsilon \to 0} F_\epsilon(t)$ does not exist, so that $\mathcal{L}\left\{\lim_{\epsilon \to 0} F_\epsilon(t)\right\}$ is not defined. Nevertheless it proves useful to consider $\delta(t) = \lim_{\epsilon \to 0} F_\epsilon(t)$ to be such that $\mathcal{L}\{\delta(t)\} = 1$. We call $\delta(t)$ the *Dirac delta function* or *impulse function*.

43. Show that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$, where $\delta(t)$ is the Dirac delta function.

This follows from Problem 9 and the fact that $\mathcal{L}\{\delta(t)\} = 1$.

44. Indicate which of the following are null functions.

(*a*) $F(t) = \begin{cases} 1 & t = 1 \\ 0 & \text{otherwise} \end{cases}$, (*b*) $F(t) = \begin{cases} 1 & 1 \leqq t \leqq 2 \\ 0 & \text{otherwise} \end{cases}$, (*c*) $F(t) = \delta(t)$.

(*a*) $F(t)$ is a null function, since $\int_0^t F(u)\,du = 0$ for all $t > 0$.

(*b*) If $t < 1$, we have $\int_0^t F(u)\,du = 0$.

If $1 \leqq t \leqq 2$, we have $\int_0^t F(u)\,du = \int_1^t (1)\,du = t - 1$.

If $t > 2$, we have $\int_0^t F(u)\,du = \int_1^2 (1)\,du = 1$.

Since $\int_0^t F(u)\,du \neq 0$ for all $t > 0$, $F(t)$ is not a null function.

(*c*) Since $\int_0^t \delta(u)\,du = 1$ for all $t > 0$, $\delta(t)$ is not a null function.

EVALUATION OF INTEGRALS

45. Evaluate (*a*) $\int_0^\infty t\,e^{-2t} \cos t\,dt$, (*b*) $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt$.

(*a*) By Problem 19,

$$\mathcal{L}\{t \cos t\} = \int_0^\infty t\,e^{-st} \cos t\,dt$$

$$= -\frac{d}{ds}\mathcal{L}\{\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

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48. Find $\mathcal{L}\{\sin\sqrt{t}\}$.

Method 1, using series.

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

Then the Laplace transform is

$$\begin{aligned}\mathcal{L}\{\sin\sqrt{t}\} &= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - \frac{\Gamma(9/2)}{7!\,s^{9/2}} + \cdots \\ &= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{(1/2^2 s)^2}{2!} - \frac{(1/2^2 s)^3}{3!} + \cdots\right\} \\ &= \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/2^2 s} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}\end{aligned}$$

Method 2, using differential equations.

Let $Y(t) = \sin\sqrt{t}$. Then by differentiating twice we find

$$4tY'' + 2Y' + Y = 0$$

Taking the Laplace transform, we have if $y = \mathcal{L}\{Y(t)\}$

$$-4\frac{d}{ds}\{s^2 y - s\,Y(0) - Y'(0)\} + 2\{s y - Y(0)\} + y = 0$$

or
$$4s^2 y' + (6s - 1)y = 0$$

Solving,
$$y = \frac{c}{s^{3/2}}\,e^{-1/4s}$$

For small values of t, we have $\sin\sqrt{t} \sim \sqrt{t}$ and $\mathcal{L}\{\sqrt{t}\} = \sqrt{\pi}/2s^{3/2}$. For large s, $y \sim c/s^{3/2}$. It follows by comparison that $c = \sqrt{\pi}/2$. Thus

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}$$

49. Find $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$.

Let $F(t) = \sin\sqrt{t}$. Then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$, $F(0) = 0$. Hence by Problem 48,

$$\mathcal{L}\{F'(t)\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\,f(s) - F(0) = \frac{\sqrt{\pi}}{2\,s^{1/2}}\,e^{-1/4s}$$

from which
$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}\,e^{-1/4s}$$

The method of series can also be used [see Problem 175(*b*)].

50. Show that

$$\mathcal{L}\{\ln t\} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where $\gamma = .5772156\ldots$ is *Euler's constant*.

We have
$$\Gamma(r) = \int_0^\infty u^{r-1}\,e^{-u}\,du$$

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Method 1, using series.

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

Then the Laplace transform is

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - \frac{\Gamma(9/2)}{7!\,s^{9/2}} + \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \left(\frac{1}{2^2 s}\right) + \frac{(1/2^2 s)^2}{2!} - \frac{(1/2^2 s)^3}{3!} + \cdots\right\}$$

$$= \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/2^2 s} = \frac{\sqrt{\pi}}{2\,s^{3/2}}\,e^{-1/4s}$$

Method 2, using differential equations.

Let $Y(t) = \sin\sqrt{t}$. Then by differentiating twice we find

$$4tY'' + 2Y' + Y = 0$$

Taking the Laplace transform, we have if $y = \mathcal{L}\{Y(t)\}$

$$-4\frac{d}{ds}\{s^2 y - s\,Y(0) - Y'(0)\} + 2\{s\,y - Y(0)\} + y = 0$$

or

$$4s^2 y' + (6s - 1)y = 0$$

Solving,

$$y = \frac{c}{s^{3/2}}\,e^{-1/4s}$$

For small values of t, we have $\sin\sqrt{t} \sim \sqrt{t}$ and $\mathcal{L}\{\sqrt{t}\} = \sqrt{\pi}/2s^{3/2}$. For large s, $y \sim c/s^{3/2}$. It follows by comparison that $c = \sqrt{\pi}/2$. Thus

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Let $F(t) = \sin\sqrt{t}$. Then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$, $F(0) = 0$. Hence by Problem 48,

$$\mathcal{L}\{F'(t)\} = \frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\,f(s) - F(0) = \frac{\sqrt{\pi}}{2\,s^{1/2}}\,e^{-1/4s}$$

from which

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}\,e^{-1/4s}$$

The method of series can also be used [see Problem 175(*b*)].

50. Show that

$$\mathcal{L}\{\ln t\} = \frac{\Gamma'(1) - \ln s}{s} = -\frac{\gamma + \ln s}{s}$$

where $\gamma = .5772156\ldots$ is *Euler's constant.*

We have

$$\Gamma(r) = \int_0^\infty u^{r-1}e^{-u}\,du$$

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Chapter 2

The Inverse Laplace Transform

DEFINITION OF INVERSE LAPLACE TRANSFORM

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an *inverse Laplace transform* of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the *inverse Laplace transformation operator.*

Example. Since $\mathcal{L}\{e^{-3t}\} = \dfrac{1}{s+3}$ we can write

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

UNIQUENESS OF INVERSE LAPLACE TRANSFORMS. LERCH'S THEOREM

Since the Laplace transform of a null function $\mathcal{N}(t)$ is zero [see Chapter 1, Page 9], it is clear that if $\mathcal{L}\{F(t)\} = f(s)$ then also $\mathcal{L}\{F(t) + \mathcal{N}(t)\} = f(s)$. From this it follows that we can have two different functions with the same Laplace transform.

Example. The two different functions $F_1(t) = e^{-3t}$ and $F_2(t) = \begin{cases} 0 & t = 1 \\ e^{-3t} & \text{otherwise} \end{cases}$ have the same Laplace transform, i.e. $1/(s+3)$.

If we allow null functions, we see that the inverse Laplace transform is not unique. It is unique, however, if we disallow null functions [which do not in general arise in cases of physical interest]. This result is indicated in

Theorem 2-1. *Lerch's theorem.* If we restrict ourselves to functions $F(t)$ which are sectionally continuous in every finite interval $0 \leqq t \leqq N$ and of exponential order for $t > N$, then the inverse Laplace transform of $f(s)$, i.e. $\mathcal{L}^{-1}\{f(s)\} = F(t)$, is unique. We shall always assume such uniqueness unless otherwise stated.

SOME INVERSE LAPLACE TRANSFORMS

The following results follow at once from corresponding entries on Page 1.

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