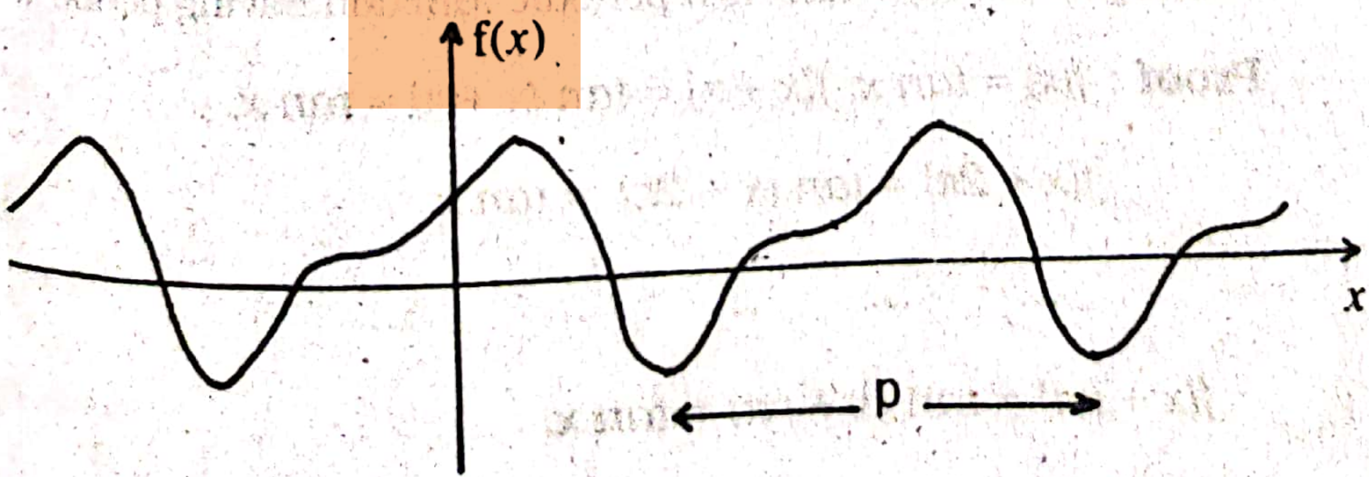
FOURIER SERIES AND FOURIER INTEGRALS

3.1 Introduction

Periodic functions occur frequently in engineering problems. The representation of these engineering problems in terms of simple periodic functions, such as sine and cosine's is a matter of great practical importance, which leads to **Fourier series.** These series, named after the French physicist **JOSEPH FOURIER** (1768—1830), are a very powerful tool in connection with various problems involving ordinary and partial differential equations. Here we shall discuss basic concepts, facts and techniques in connection with Fourier series. Some illustrative examples and also some important engineering applications of these series will be included.

3.2 Periodic functions and Trigonometric series

Definition : The function $f(x)$ of a real variable x is said to be **periodic** if there exists a non-zero number p, independent of x, such that the equation $f(x+p) = f(x)$ holds for all values of x. The least value of p>0 is called the **least period** or simply the **period** of f(x).



Graph of periodic function.

Examples of periodic functions

Example 1. $f(x) = \sin x$, is a periodic function having period 2π.

Proof : $f(x) = \sin x$, $f(x + 2\pi) = \sin (x + 2\pi) = \sin x = f(x)$

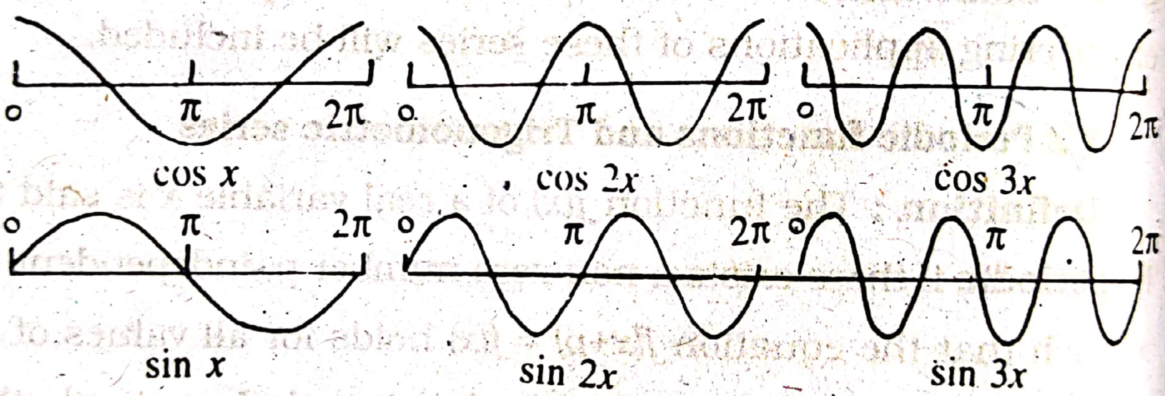
$f(x + 4\pi) = \sin (x + 4x) = \sin x = f(x)$

... ...

$\therefore f(x) = f(x + 2\pi) = f(x + 4\pi) = \dots \dots = f(x + 2n\pi)$

Thus $f(x) = \sin x$ is a periodic function having period 2π.

Similarly, we can show that $f(x) = \cos x$ is also a periodic function having period 2π.



Graphs of cosine and sine functions having period 2π.

Example 2. $f(x) = \tan x$ is a periodic function having period π.

Proof : $f(x) = \tan x$, $f(x + \pi) = \tan (x + \pi) = \tan x$,

$f(x + 2\pi) = \tan (x + 2\pi) = \tan x$

... ...

$f(x + n\pi) = \tan (x + n\pi) = \tan x$.

$\therefore f(x) = f(x + \pi) = f(x + 2\pi) = \dots = f(x + n\pi)$.

Thus $f(x) = \tan x$ is a periodic function having period π.

Definition By a **Trigonometric series** we shall mean any series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where the coefficients a_n and b_n are constants.

3.3 Fourier series

Definition : The trigonometric series $f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

is a **Fourier series** if its coefficients a_0, a_n and b_n are given by the following formulas :

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)dv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \cos nv dv$$

$$(n = 1, 2, 3, \ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \sin nv dv$$

$$(n = 1, 2, 3, \ldots)$$

where $f(x)$ is any single–valued function defined on the interval $(-\pi, \pi)$.

The Fourier series can also be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \ (n = (0, 1, 2, 3, \ldots)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx, \ (n = 1, 2, 3. \ldots)$

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3.4 Process of determining the coefficients a_0, a_n and b_n.

(i) Let us first determine a_0.

$$\text{Given } f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

Integrating both sides of (1) with respect to x from $-\pi$, to π.

we get.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \left[a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx$$

By term–by–term integration of the series we obtain

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nxdx + b_n \int_{-\pi}^{\pi} \sin nxdx \right] \quad (2)$$

$$\text{Now} \int_{-\pi}^{\pi} a_o dx = 2\pi a_o \text{ and} \int_{-\pi}^{\pi} \cos nxdx = \int_{-\pi}^{\pi} \sin nx\, dx = 0.$$

$$\therefore \int_{-\pi}^{\pi} f(x)dx = 2\pi a_o, \text{ or, } a_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v)dv.$$

(ii) To determine a_1, a_2, ... in (1), we multiply both sides of (1) by cos mx where m is any fixed positive integer and then integrate with respect to x from —π to π finding.

$$\int_{-\pi}^{\pi} f(x) \cos mxdx = a_o \int_{-\pi}^{\pi} \cos mxdx +$$

$$\sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos nx \cos mxdx + b_n \int_{-\pi}^{\pi} \sin nx \cos mxdx \right] \quad (3)$$

Now $\int_{-\pi}^{\pi} \cos mx dx = 0, \int_{-\pi}^{\pi} \sin nx \cos mx dx = 0.$

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx$$

where $\frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx = 0$ and

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx = \begin{cases} 0 \text{ if } m \neq n \\ \pi \text{ if } m = n \end{cases}$$

$$\therefore \int_{-\pi}^{\pi} f(x) \cos mx dx = a_m \pi \text{ or, } a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx dx.$$

Now writing n for m, we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \cos nv dv.$$

(iii) Finally, we determine $b_1, b_2, \ldots \ldots$ in (1).

If we multiply (1) by sin mx, where m is any fixed positive integer and then integrate with respect to x from $-\pi$ to π, finding

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = a_o \int_{-\pi}^{\pi} \sin mx dx +$$

$$\sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cos nx \sin mx dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx dx\right] \quad (4)$$

Now $\int_{-\pi}^{\pi} \sin mx dx = 0, \int_{-\pi}^{\pi} \cos nx \sin mx dx = 0,$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx$$

$$-\frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\,xdx \text{ where } \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\,xdx = 0$$

$$\text{and } \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)xdx = \begin{cases} 0 \text{ if } n \neq m \\ \pi \text{ if } n = m \end{cases}$$

Thus from (4), we have

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = b_m\pi \text{ or, } b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mxdx$$

Writing n for m, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin nvdv.$$

$$\text{Therefore, } f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)dv +$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left\{\int_{-\pi}^{\pi} f(v)\cos nvdv\right\}\cos nx + \left\{\int_{-\pi}^{\pi} f(v)\sin nvdv\right\}\sin nx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} f(v)\{\cos nv\cos nx + \sin nv\sin nx\}\,dv\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(v)\cos n(x-v)dv.$$

3.5 The Fourier cosine and sine series.

Definition Even function

A function f(x) is called **even** if $f(-x) = f(x)$.

Graphically, an even function is symmetrical about the y-axis.

If f(x) is an even function, then

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{\pi}^{0} f(-x)\, d(-x) + \int_{0}^{\pi} f(x)dx$$

$$= -\int_{\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{0}^{\pi} f(x)dx + \int_{0}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx.$$

Thus if f(x) is even, we have

$$a_0 = \frac{1}{2\pi} \cdot 2\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(v)dv.$$

Also if $f(x)$ is even i, e $f(-x) = f(x)$

then $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad (n = 1, 2, 3, \ldots)$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x) \cos n(-x)\, d(-x)$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(v) \cos nv\, dv.$$

but $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x) \sin n(-x)\, d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x).\ -\sin nx.\ -dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = -b_n$$

$\therefore 2b_n = 0$ or, $b_n = 0$.

Therefore, if f(x) is even, then we have

$$f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(v)dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} f(v) \cos nv dv\right\} \cos nx$$

which represents the function f(x) in a series of cosines and therefore it is known as **cosine series** in the interval $(0, \pi)$.

Definition odd function

A function f(x) is called **odd** if $f(-x) = -f(x)$.

Graphically, an odd function is symmetrical about the origin.

When f(x) is odd, we have

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)\, d(-x)$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x)dx = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = -a_o$$

$\therefore 2a_o = 0$ or, $a_o = 0$.

Also if $f(x)$ is odd i.e $f(-x) = -f(x)$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_{\pi}^{-\pi} f(-x) \cos n(-x)\, d(-x)$$

$$= \frac{1}{\pi} \int_{\pi}^{-\pi} -f(x) \cos nx. - dx$$

$$= \frac{1}{\pi} \int_{\pi}^{-\pi} f(x) \cos nx\, dx = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = -a_n$$

$\therefore 2a_n = 0$, or, $a_n = 0$.

Finally, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$$= \frac{1}{\pi} \int_{\pi}^{-\pi} f(-x) \sin n(-x)\, d(-x)$$

$$= \frac{1}{\pi} \int_{\pi}^{-\pi} -f(x). -\sin nx. - dx$$

$$= -\frac{1}{\pi} \int_{\pi}^{-\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(v) \sin nv dv.$$

Therefore, if f(x) is odd, then we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \int_{0}^{\pi} f(v) \sin nv dv \right\} \sin nx$$

which represents the function f(x) in a series of sines in the interval (o,π) and therefore, it is known as **sine series** in the interval (o, π).

3.6 Half Range Fourier Cosine and Sine Series.

When the Fourier series has only the cosine terms or only the sine terms in the expansion, we call the series **Half range Fourier cosine series** or **Half range Fourier sine series** respectively. When we are interested to find out a half range series corresponding to a given function, the function must be defined in the interval $(0, \pi)$, which is the half of the interval $(-\pi, \pi)$ and then the funtcion is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(-\pi, 0)$ In such a case we have

$$b_n = 0,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \text{ for half range cosine series}$$

and $a_n = 0,\ b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\,dx$ for half range sine series.

3.7 Dirichlet's conditions

Any function $f(x)$ can be developed as a Fourier series $a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where a_o, a_n and b_n are constants, provided.

(i) $f(x)$ is periodic, single-valued and finite;

(ii) $f(x)$ has a finite number of discontinuities in any one period;

(iii) $f(x)$ has at the most a finite number of maxima and minima.

In fact the problem of expressing any function $f(x)$ as a Fourier series depends upon the evaluations of the integrals

$\frac{1}{\pi}\int f(x)\cos nx\,dx$; $\frac{1}{\pi}\int f(x)\sin nx\,dx$. within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$ according as f(x) is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha + 2\pi)$.

3.8 Change of intervals

(A) If f(x) is defined in the interval (– c, c) having period 2c, the Fourier series of f(x) in the interval (– c, c) is given by

$$f(x) = a_o + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

where $a_o = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx\ (n = o)$.

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots).$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx\ (n = 1, 2, 3, \ldots)$$

(B) If f(x) be defined in the inerval $(\alpha, \alpha + 2c)$ having period 2c, the Fourier series of f(x) in the interval $(\alpha, \alpha + 2c)$ is given by

$$f(x) = a_n + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c}\right)$$

where $a_o = \frac{1}{2c}\int_{\alpha}^{\alpha+2c} f(x)\,dx\ (n = 0)$

$$a_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\cos\frac{n\pi x}{c}\,dx\ (n = 1, 2, \ldots)$$

$$b_n = \frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\sin\frac{n\pi x}{c}\,dx\ (n = 1, 2, \ldots).$$

Again the Fourier series $f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ may be integrated term-by-term from $-\pi$ to x and the resulting series will converge uniformly to $\int_{-\pi}^{x} f(x)dx$ provided that f(x) is sectionally continuous in $-\pi \le x \le \pi$.

$$\text{Thus} \int_{-\pi}^{x} f(x)\, dx = a_n(x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n} [\, a_o (\sin nx - \sin n\pi)$$

$$- b_n(\cos nx - \cos n\pi)].$$

$$= a_o (x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n} [a_n \sin nx - b_n (\cos nx - \cos n\pi)].$$

3.12 Complex form of the Fourier series,

From Trigonometry, we have

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ and } \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Now the complex form of the Fourier series is obtained by expressing $\cos\left(\frac{n\pi x}{c}\right)$ and $\sin\left(\frac{n\pi x}{c}\right)$ in exponential form, that is, the Fourier series.

$$f(x) = a_o + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin \frac{n\pi x}{c} \right]$$

$$-c \angle x \angle c \quad (1)$$

can be written in the form

$$f(x) = a_o + \sum_{n=1}^{\infty} \left[\frac{a_n}{2}\left(e^{\frac{in\pi x}{c}} + e^{\frac{-in\pi x}{c}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{c}} - e^{\frac{-in\pi x}{c}}\right)\right]$$

$$= a_o + \sum_{n=1}^{\infty} \left[\frac{a_a}{2}\left(e^{\frac{in\pi x}{c}} + e^{\frac{-in\pi x}{c}}\right) - \frac{ib_n}{2}\left(e^{\frac{in\pi x}{c}} - e^{\frac{-in\pi x}{c}}\right)\right]$$

$$= a_o + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - ib_n)\, e^{\frac{in\pi x}{c}} + \frac{1}{2}(a_n + ib_n)\, e^{\frac{-in\pi x}{c}}\right]$$

$$= c_o + \sum_{n=1}^{\infty} \left[c_n\, e^{\frac{in\pi x}{c}} + c_{-n}\, e^{\frac{-in\pi x}{c}}\right] \qquad (2)$$

where $c_o = a_o$, $c_n = \frac{1}{2}(a_n - ib_n)$ and $c_{-n} = \frac{1}{2}(a_n + ib_n)$

Equation no (2) is known as the **complex form of the Fourier series** which can also be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{c}} \quad -c \angle x \angle c. \; .$$

$$\text{where } c_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-\frac{in\pi x}{c}}\, dx \text{ and } n = 0, \pm 1, \pm 2, \dots \dots .$$

3.13 Parseval's Formula.

One of the most important properties of Fourier series is **Parseval's Formula** or the **completeness relation** which gives a relation between the average of the square (or absolute square) of the function f(x) and the co-efficients in Fourier series of f(x).

(A) Particular case

Let f(x) be a real-valued function of period 2π whose Fourier series is

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{where } a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx, \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Now the average of the square of f(x) over

$$(-\pi, \pi) \text{ is } \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx$$

Thus we have average of $[f(x)]^2$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[a_o+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right]^2 dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}a_o^2\,dx+\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}a_n^2\cos^2 nx\,dx+$$

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi}b_n^2\sin^2 nx\,dx$$

+ othere terms (which vanish when average is taken).

$$=\frac{1}{2\pi}\cdot a_o^2\cdot[x]_{-\pi}^{\pi}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2\int_{-\pi}^{\pi}(1+\cos 2nx)\,dx$$

$$+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2\int_{-\pi}^{\pi}(1-\cos 2nx)\,dx+0$$

$$=a_o^2+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2\,[x]_{-\pi}^{\pi}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2\,[x]_{-\pi}^{\pi}$$

$$=a_o^2+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}a_n^2.2\pi+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2}b_n^2.2\pi$$

$$=a_o^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Hence $\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx=a_o^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$

This is one form of **Parseval's formula.** One can easily verify that this formula is unchanged if $f(x)$ has period $2c$ in place of 2π and its square is averaged over any period of length $2c$. Then we have

$$\frac{1}{2c}\int_{-c}^{c}[f(x)]^2\,dx=a_o^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

Let f(x) be a complex-valued function of period 2π, whose Fourier series is $\sum_{n-\infty}^{\infty} c_n e^{inx}$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}$ $((n = 0, \pm 1, \pm 1, \pm 2, \ldots)$

Then the average square of f(x)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

(B) **General case.**

A general form of Parseval's Formula states that if f(x) and g(x) are two real-valued functions of period 2π, whose Fourier series are $a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ and

$a_o' + \sum_{n=1}^{\infty} (a'_n \cos nx + b_n' \sin nx)$ respectively

where $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.$

Similarly, a_o', a_n' and b_n' are defined in terms of g. Then where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-nix}\,dx \quad (n = 0, \pm 1, \pm 2, \ldots)$

and $c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\, e^{-inx}dx \quad (n = 0, \pm 1, \pm 2, \ldots)$

Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx \quad \sum_{n=-\infty}^{\infty} c_n c_n'.$$

WORKED OUT EXAMPLES

Example 1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution : Let $f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (1)

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx$

$$= \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1}{\pi}\left(1 - e^{-2\pi}\right).$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$

$$= \frac{1}{\pi}\left[-\cos nx;\, e^{-x}\right]_0^{2\pi} - \frac{n}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}(1 - e^{-2\pi}) + \frac{n}{\pi}\left[\sin nx.\, e^{-x}\right]_0^{2\pi} - \frac{n^2}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}(1 - e^{-2\pi}) + 0 - n^2 a_n.$$

$$\therefore (n^2 + 1)a_n = \frac{1}{\pi}(1 - e^{-2\pi}),$$

or, $a_n = \frac{1}{\pi}(1 - e^{-2\pi}) \cdot \frac{1}{n^2 + 1}$

Finally, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx$

$$= \frac{1}{\pi}\Big[-e^{-x}\sin nx\Big]_0^{2\pi} + \frac{n}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx$$

$$= 0 + \frac{n}{\pi}\Big[-e^{-x}\cos nx\Big]_0^{2\pi} - \frac{n^2}{\pi}\int_0^{2\pi} e^{-x}\sin nx dx$$

$$= \frac{n}{\pi}(1 - e^{-2\pi}) - n^2 b_n .$$

or, $(n^2 + 1)b_n = \frac{n}{\pi}(1 - e^{-2\pi})$

$\therefore b_n = \frac{1}{\pi}(1 - e^{-2\pi}) . \frac{n}{n^2 + 1}$

Now substituting the values of a_0, a_n and b_n in(1), we get

$$f(x) = \frac{1}{2\pi}(1 - e^{-2\pi}) + \sum_{n=1}^{\infty}\Big[\frac{1}{\pi}(1 - e^{-2\pi})\frac{1}{n^2 + 1}\cos nx + \frac{1}{\pi}(1 - e^{-2\pi}) . \frac{n}{n^2 + 1} . \sin nx\Big]$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2 + 1} . \cos nx + \sum_{n=1}^{\infty}\frac{n}{n^2 + 1} . \cos nx\right]$$

$$= \frac{1}{\pi}(1 - e^{-2\pi})$$

$$\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\ldots\right)\right]$$

Example 2. Find the Fourier series for the function $f(x) = e^x$ in the interval $-\pi < x < \pi$.

Solution : Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (1)

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx$

$$= \frac{1}{\pi}(e^{\pi} - e^{-\pi})$$

$$\text{or, } a_o = \frac{2}{\pi}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right) = \frac{2\sin h\pi}{\pi}.$$

$$\text{and } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[e^x \cos nx\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx.$$

$$= \frac{1}{\pi}(e^{\pi} - e^{-\pi})\cos n\pi + \frac{n}{\pi}\left[e^x \sin nx\right]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx$$

$$= \frac{2}{\pi}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right).(-1)^n + o - n^2 a_n$$

$$\text{or, } (n^2 + 1)a_n = (-1)^n \frac{2\sin h\pi}{\pi}$$

$$\therefore \quad a_n = \frac{2\sin h\pi}{\pi}.\frac{(-1)^n}{n^2 + 1}.$$

$$\text{Finally, } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nxdx.$$

$$= \frac{1}{\pi}\left[e^x \sin nx\right]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} e^x \cos nxdx$$

$$= 0 - \frac{n}{\pi}\left[e^x \cos nx\right]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x \sin nxdx$$

$$= -n.(-1)^n.\frac{2\sin h\pi}{\pi} - n^2 b_n.$$

$$\text{or, } (n^2 + 1)b_n = -n.(-1)^n.\frac{2\sin h\pi}{\pi}$$

$$b_n = (-1)^n \frac{2\sin h\pi}{\pi}.-\frac{n}{n^2 + 1}.$$

Now substituting the values of a_o, a_n, and b_n in (1) we get

$$f(x) = \frac{1}{2\pi}.2\sinh\pi$$

$$+\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2 + 1}.\frac{2\sinh\pi}{\pi}.\cos nx - n.\frac{(-1)^n}{n^2 + 1}\frac{2\sinh\pi}{\pi}.\sin nx\right]$$

$$=\frac{1}{\pi}.\ 2\sinh\pi\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right]$$

$$=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}+\left(-\frac{1}{2}\cos x+\frac{1}{5}\cos 2x-\frac{1}{10}\cos 3x+\ldots\right)\right.$$

$$\left.+\left(\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{20}\sin 3x-\ldots\right)\right]$$

Note : $\cos(-n\pi)=\cos n\pi$ and

$\cos n\pi = \cos\pi+\cos 2\pi+\cos 3\pi+\cos 4\pi+\cos 5\pi+\ldots$

$=-1+1-1+1-1+1\ldots\ldots$

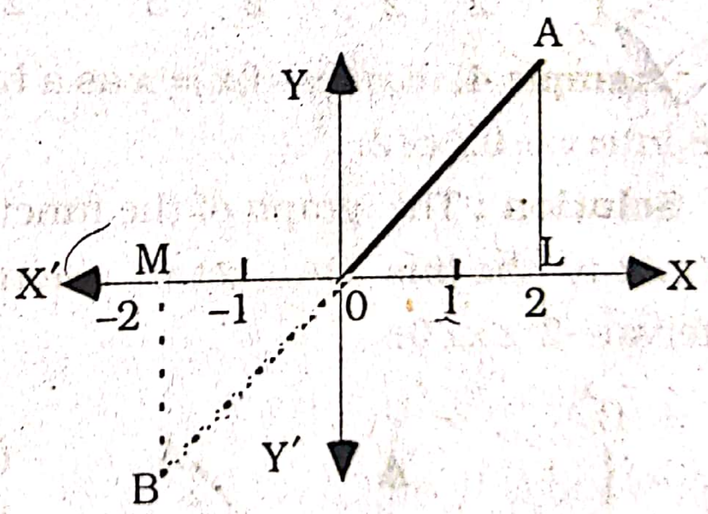
Now $\cos\pi=-1,\ \cos 2\pi=1=(-1)^2.$

$\cos 3\pi=-1=(-1)^3,\ \cos 4\pi=1=(-1)^4\ldots$

$\therefore \cos n\pi=(-1)^n.$

Example 3. Express $f(x)=x$ as a half range sine series in the interval $0<x<2$.

The graph of $f(x)=x$ in $0<x<2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2<x<0$ (shown by the line BO) so that the new function is symmtrical about the origin and therefore represents an odd function in the interval $(-2, 2)$.



Hence the Fourier series for $f(x)$ will have only the sine terms given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}.\ (a_0=0,\ a_n=0)$$

$$\text{where } b_n=\frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}\,dx$$

$$=\frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx.$$

$$= \left[- x . \frac{2}{n\pi} . \cos \frac{n\pi x}{2} \right]_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2}$$

$$= - \frac{4}{nx} \cos n\pi + 0 + \frac{4}{n^2\pi^2} \left[\sin \frac{n\pi x}{2} \right]_0^2$$

$$= - \frac{4}{n\pi} . (-1)^n + 0 = \frac{-4(-1)^n}{n\pi} .$$

Hence the Fourier sine series for $f(x) = x$ over the half range (0 , 2) is

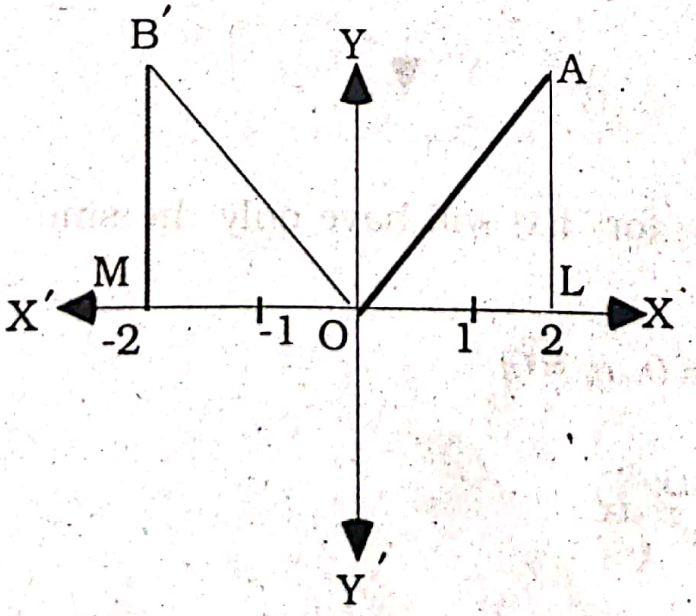
$$f(x) = \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi} . \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi} . \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} . \sin \frac{n\pi x}{2}$$

$$= \frac{4}{\pi} \left[\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \frac{1}{4} \sin \frac{4\pi x}{2} + \ldots \right] .$$

Example 4. Express $f(x) = x$ as a half range cosine series in the interval $0 \angle x \angle 2$.

Solution : The graph of the function $f(x) = x$ in the interval $0 \angle x \angle 2$ is the line OA. Let us extend the function $f(x)$ in the interval $-2 \angle x \angle 0$.



(shown by the line OB′) so that the new function is symmetrical about the y–axis and therefore, represents an even function in the interval $-2 \angle x \angle 2$.

Hence the Fourier series for $f(x)$ over the full period (– 2, 2) will have only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} \quad (b_n = 0)$$

where $a_0 = \frac{2}{c}\int_0^c f(x)dx = \frac{2}{2}\int_0^2 xdx = \left[\frac{x^2}{2}\right]_0^2 = 2.$

and $a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx.$

$= \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}dx.$

$= 0 + \frac{4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^2$

$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}\{(-1)^n - 1\}.$

Thus $f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2}\right]$

$= 1 + \left[-\frac{8}{1^2\cdot\pi^2}\cos\frac{\pi x}{2} + 0 - \frac{8}{3^2\pi^2}\cdot\cos\frac{3\pi x}{2} + 0 - \frac{8}{5^2\pi^2}\cos\frac{5\pi x}{2} + \dots\right]$

$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \dots\right].$

Example 5. Find the Fourier series expansion of the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$ and hence evaluate $\sum_{n=1}^{\infty}\frac{1}{n^2}$. **[R. U. P. 1980, D. U. S 1987]**

Solution : By definition we have

$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$

Now $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2dx$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{6\pi}\{\pi^3-(-\pi^3)\} = \frac{\pi^2}{3}.$$

Again, since $f(x) = x^2$ and $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ $f(-x) = f(x)$. Thus $f(x) = x^2$ is an even function and so sine terms will vanish i. e. $b_n = 0$.

Finally, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx$

$(n \neq o)$

$$= \frac{1}{\pi}\left[x^2.\frac{1}{n}\sin nx\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x.\frac{1}{n}\sin nx dx.$$

$$= 0 - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{2}{n\pi}\left[x.\frac{1}{n}\cos nx\right]_{-\pi}^{\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi} \cos nx dx$$

$$= \frac{2}{\pi n^2}[\pi\cos n\pi - (-\pi)\cos n(-\pi)] - \frac{2}{\pi n^3}[\sin nx]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2}[\pi\cos n\pi + \pi\cos n\pi] - 0$$

$$= \frac{4}{n^2}.\cos n\pi = \frac{4}{n^2}.(-1)^n. \text{ Since } [\cos n\pi = (-1)^n]$$

Now putting the values of a_o, a_n and b_n in (1) we get

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\cos nx + 0\right\}$$

$$\text{or, } f(x) = \frac{\pi^2}{3} + 4\left\{\frac{-\cos x}{1^2} + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x - \dots\dots\right\}$$

$$= \frac{\pi^2}{3} - 4\left\{\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x + \dots\dots\right\}$$

When $x = 0$, we have

$$0=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$=\frac{\pi^2}{3}-4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots+\right)$$

$$=\frac{\pi^2}{3}-4\left\{\left(\frac{1}{1^2}+\frac{1}{2^2}\ \frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)-2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}\ldots\right)\right\}$$

$$=\frac{\pi^2}{3}-4\left\{\left(\frac{1}{1^2}+\frac{1}{2^2}\ \frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)-\frac{2}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}\ \frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)\right\}$$

$$=\frac{\pi^2}{3}-4\left\{\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right\}$$

$$=\frac{\pi^2}{3}-2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\text{or, } 2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

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Example 6. (a) Obtain the Fourier series of the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ 1, & 0\le x\le \pi\end{cases}$$

M. Sc. Final 1991

(b) Verify the result found in 6(a) by assuming the complex form of the Fourier series.

Solution of (a) : By definition we have

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\quad(1)$$

where $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$.

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$.

Now $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1.\,dx\right]$$

$$= \frac{1}{2\pi}[0 + \pi] = \frac{1}{2}.$$

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\ (n \neq 0)$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nx\,dx + \int_{\pi}^{0} 1.\cos nx\,dx\right]$$

$$= 0 + \frac{1}{\pi n}[\sin nx]_0^{\pi} = 0 + 0 = 0.$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} 1.\,\sin nx\,dx]$$

$$= \frac{1}{\pi}\left[0 - \frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= -\frac{1}{\pi n}[\cos n\pi - \cos 0]$$

$$= -\frac{1}{\pi n}[(-1)^n - 1]$$

$$= \frac{1}{\pi n}[1 - (-1)^n] = \begin{cases} 0 \text{ when } n \text{ is even} \\ \frac{2}{\pi n} \text{ when } n \text{ is odd} \end{cases}$$

Now putting the values of a_o, a_n and b_n in (1) we get

$$f(x)=\frac{1}{2}+0+\sum_{n=1}^{\infty} b_n \sin nx.$$

$$\text{Or, } f(x)=\frac{1}{2}+b_1 \sin x+b_2 \sin 2x+b_3 \sin 3x+b_4 \sin 4x + b_5 \sin 5x+\ldots\ldots$$

$$=\frac{1}{2}+\frac{2}{\pi}\sin x+0+\frac{2}{3\pi}\sin 3x+\frac{2}{5\pi}\sin 5x+\ldots.$$

$$=\frac{1}{2}+\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\ldots\right).$$

Solution of (b) : Here we have to expand the function f(x) in the complex Fourier series.

By definition we have

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{c}} \quad -c \angle x \angle c$$

$$\text{where } c_n=\frac{1}{2c}\int_{-c}^{c} f(x) e^{\frac{-in\pi x}{c}} dx.$$

$$n=0,\ \pm 1, \pm 2, \pm 3,\ldots\ldots$$

Here in our given problem $c = \pi$.

$$\therefore\ f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx},\ -\pi \angle x \angle \pi$$

$$\text{where } c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}\ dx,\ n=0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\text{Now } c_o=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx.$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\ dx+\int_{0}^{\pi} f(x) dx\right]$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0} 0 dx+\int_{0}^{\pi} 1 dx\right]$$

$= 0 + \frac{1}{2\pi}[x]^{\pi}_{0} = \frac{1}{2}.$

and $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \ (n \neq 0)$

$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)e^{-inx}\,dx + \int_{0}^{\pi} f(x)\,e^{-inx}\,dx\right]$

$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0e^{-inx}dx + \int_{0}^{\pi} 1.e^{-inx}dx\right]$

$= 0 + \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}$

$= \frac{1}{-2\pi ni}[e^{-in\pi} - e^{o}]$

$= \frac{1}{-2\pi ni}[\cos n\pi - i\sin n\pi - 1]$

$= \begin{cases} \frac{1}{n\pi i} & \text{when } n = \pm 1, \pm 3, \pm 5, \ldots \\ 0 & \text{when } n = \pm 2, \pm 4, \pm 6, \ldots \end{cases}$

Thus $f(x) = \frac{1}{2} + \frac{1}{\pi i}\left(\frac{e^{ix}}{1} + 0 + \frac{e^{i3x}}{3} + 0 + \frac{e^{i5x}}{5} + \ldots\ldots\right)$

$+ \frac{1}{\pi i}\left(\frac{e^{-ix}}{-1} + 0 + \frac{e^{-i3x}}{-3} + 0 + \frac{e^{-i5x}}{-5} + \ldots\right)$

$= \left[\frac{1}{2} + \frac{1}{\pi i}(e^{ix} - e^{-ix}) + \frac{1}{3}(e^{i3x} - e^{-i3x}) + \frac{1}{5}(e^{i5x} - e^{-i5x}) + \right]$

$= \frac{1}{2} + \frac{1}{\pi i}\left[2i\sin x + \frac{1}{3}2i\sin 3x + \frac{1}{5}2i\sin 5x + \ldots\ldots\right]$

$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\ldots\right]$

which is same as in 6 (a).

Example 7. Find the Fourier series expansion of the function. $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x \le \pi \end{cases}$

Hence evaluate the sum $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

[C. U. P. 1977, D. U. P. 1986, D. U. S' 1986]

Solution : By definition of Fourier series we have

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \cos nx) \quad (1)$$

where $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$

Now $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{o} f(x)\,dx + \int_{o}^{\pi} f(x)\,dx\right]$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{o} o.dx + \int_{o}^{\pi} x\,dx\right]$$

$$=o+\frac{1}{2\pi}\left[\frac{x^2}{2}\right]_o^{\pi} = \frac{1}{4\pi}(\pi^2-o) = \frac{\pi}{4}.$$

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \ (n \neq o)$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{o} f(x)\cos nx\,dx + \int_{o}^{\pi} f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{o} o.\cos nx\,dx + \int_{o}^{\pi} x\cos nx\,dx\right]$$

$$=0+\frac{1}{\pi n}[x\sin nx]_{\pi}^{o} - \frac{1}{\pi n}\int_{o}^{\pi}\sin nx\,dx$$

$$=0+\frac{1}{\pi n^2}[\cos nx]_o^{\pi}$$

$$=\frac{1}{\pi n^2}[\cos n\pi - \cos o] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

(since $[\cos n\pi = (-1)^n]$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} f(x) \sin nx\, dx + \int_{o}^{\pi} f(x) \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o} o. \sin nx dx + \int_{o}^{\pi} x \sin nx dx\right]$$

$$= o - \frac{1}{\pi n}[x \cos nx]_o^{\pi} + \frac{1}{\pi n}\int_{o}^{\pi} \cos nx dx$$

$$= -\frac{1}{\pi n}(\pi \cos n\pi - o) + \frac{1}{\pi n^2}[\sin nx]_o^{\pi}$$

$$= -\frac{1}{n}\cos n\pi + o \text{ since } \sin n\pi = o,\ \sin 0 = 0$$

$$= -\frac{1}{n}.(-1)^n = \frac{(-1)^{n+1}}{n}.$$

Now putting the values of a_o, a_n and b_n in (1), we get

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\{(-1)^n - 1\}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$= \frac{\pi}{4} + \left[\left(-\frac{2}{\pi 1^2}\cos x + o - \frac{2}{\pi 3^2}\cos 3x + o - \frac{2}{\pi 5^2}\cos 5x + o - \ldots\right)\right.$$

$$\left. + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \ldots \ldots\right)\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots \ldots\right)$$

$$+ \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots \ldots\right) \quad (2)$$

Putting $x = o$ in the above equation (2), we get

$$o = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \ldots \ldots\right)$$

$$+ \left(\frac{\sin 0}{1} - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \frac{\sin 0}{5} + \ldots \ldots\right)$$

Or, $o=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ \ldots\right)+o$

Or, $\frac{2}{\pi}\left\{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{(2n-1)^2}+\ldots\right\}=\frac{\pi}{4}$

Or, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots+\frac{1}{(2n-1)^2}+\ldots=\frac{\pi^2}{8}$

Or, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

Example 8. Find the Fourier series of the function of period 2π given below in the interval $-\pi < x < \pi$ as follows :

$$f(x)=\begin{cases}0 \text{ when } -\pi < x \le 0\\ \sin x \text{ when } 0 < x \le \pi\end{cases}$$ **[D.U.S 1985]**

Solution : By definition of the Fourier series

we have $f(x)=a_o+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ (1)

where $a_o=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$.

Now $a_o=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right]$$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0.dx+\int_{0}^{\pi}\sin xdx\right]$$

$$=0+\frac{1}{2\pi}[-\cos x]_o^{\pi}$$

$$=-\frac{1}{2\pi}(\cos\pi-1)=-\frac{1}{2\pi}(-1-1)=\frac{1}{\pi}.$$

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx dx + \int_{0}^{\pi} (f)x) \cos nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx dx + \int_{0}^{\pi} \sin x \cos nx dx\right]$$

$$= 0 + \frac{1}{2\pi}\int_{0}^{\pi} 2\cos nx \sin x dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} [\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}[\cos(n+1)x]_0^{\pi} + \frac{1}{n-1}[\cos(n-1)x]_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}\{\cos(n+1)\pi - \cos 0\} + \frac{1}{n-1}\{\cos(n-1)\pi - \cos o\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos(n\pi+\pi)}{n+1} + \frac{\cos(n\pi-\pi)-1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(n-1-n-1)(I+\cos n\pi)}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2(1+\cos n\pi)}{n^2-1}\right]$$

$$= -\frac{(1+\cos n\pi)}{\pi(n^2-1)} = \frac{\{1+(-1)^n\}}{\pi(n^2-1)} \quad (n \neq 1)$$

When $n = 2$, $a_2 = -\frac{(1+1)}{\pi(2^2-1)} = -\frac{2}{3\pi}$.

When $n = 3$, $a_3 = 0$

When $n = 4$, $a_4 = -\frac{(1+1)}{\pi(4^2-1)} = -\frac{2}{15\pi}$.

When $n = 5$, $a_5 = 0$

When $n = 6$, $a_6 = -\frac{(1+1)}{\pi(6^2-1)} = -\frac{2}{35\pi}$ and so on.

But when $n = 1$, $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\cos x dx \ \frac{1}{\pi}\left[\frac{(\sin x)^2}{2}\right]_0^{\pi} = 0.$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$

When $n = 1$, $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx$

Or, $b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin x dx + \int_0^{\pi} f(x)\sin x dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin x dx + \int_0^{\pi} \sin x \sin x dx\right]$$

$$= 0 + \frac{1}{2\pi}\int_0^{\pi} 2\sin^2 x dx.$$

$$= \frac{1}{2\pi}\int_0^{\pi} (1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}.$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \ (n \neq 1,\ n = 2, 3, 4, \ldots)$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx dx + \int_0^{\pi} f(x)\sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx dx + \int_0^{\pi} \sin x \sin nx dx\right]$$

$$= 0 + \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \sin x dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\, dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin(n-1)x - \frac{1}{n+1}\sin(n+1)x\right]_0^{\pi}$$

$$= \frac{1}{2\pi} \cdot 0 = 0 \text{ for } (n \neq 1, n = 2, 3, 4).$$

Thus $f(x) = \frac{1}{\pi} + \left(-\frac{2}{3\pi}\cos 2x - \frac{2}{15\pi}\cos 4x - \frac{2}{35\pi}\cos 6x - \ldots\ldots\right)$

$+ \frac{1}{2}\sin x + 0$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right).$$

Example 9. Find the Fourier series expansion of the function $f(x) = |x|$ in the interval $[-\pi, \pi]$. Hence evaluate

the sum $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. **[D. U. S. 1986]**

Solution : By definition of the Fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ (n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx.$

Now by definition $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

Hence the given function $f(x) = |x|$ is an even function and for the even function $b_n = 0$, $(n = 1, 2, 3, \ldots)$ in the Fourier series expansion (1) of $f(x)$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ (n \neq 0)$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} |x| \cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi n}[x \sin nx]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi} \sin nx dx$$

$$= 0 + \frac{2}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - \cos o)$$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\}$$

$$= \begin{cases} \frac{-4}{\pi n^2} & \text{when } n = 1, 3, 5, \ldots \\ 0 & \text{when } n = 2, 4, 6, \ldots \end{cases}$$

Now substituting the values of a_o, a_n, and b_n in (1) we get

$$f(x) = \frac{\pi}{2} + \left[\left(-\frac{4}{\pi.1^2} + 0 - \frac{4}{\pi.3^2}\cos 3x + 0 - \frac{4}{\pi.5^2}\cos 5x + \ldots\right) + 0\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right] \quad (2)$$

Putting $x = 0$ in (2), we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \ldots\right\}$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right)$$

Or, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi}{2} \times \frac{\pi}{4} = \frac{\pi^2}{8}$

Or, $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

[**Note** : The values of a_0 and a_n can also be found in the following way :

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-x)dx + \int_{0}^{\pi} xdx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} xdx + \int_{0}^{\pi} xdx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} xdx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

$$a_n = \int_{-\pi}^{\pi} f(x) \cos nxdx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \cos nxdx + \int_{0}^{\pi} f(x) \cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-x) \cos nxdx + \int_{0}^{\pi} x\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} x\cos xdx + \int_{0}^{\pi} x\cos nxdx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nxdx = \begin{cases} \dfrac{-4}{\pi n^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

Example 10. Find a series of sines and cosines of multiples of x which will reperesent $x + x^2$ in the interval $-\pi \angle x \angle \pi$.

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

[D.U.P. 1982, C.U.P. 1982, R.U.P. 1983

Solution : By definition of the Fourier series we have

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx \ (n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nxdx$

Now given in question $f(x) = x + x^2$.

$$\therefore\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx.$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi n}\left[(x + x^2)\sin nx\right]_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi} (1 + 2x)\sin nx\,dx$$

$$= 0 + \frac{1}{\pi n^2}\left[\cos nx\right]_{-\pi}^{\pi} + \frac{2}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi}\cos nx\,dx$$

$$= \frac{1}{\pi n^2}(\cos n\pi - \cos n\pi) + \frac{2}{\pi n^2}(\pi\cos n\pi + \pi\cos n\pi) - \frac{2}{\pi n^3}\left[\sin nx\right]_{-\pi}^{\pi}$$

$$= o + \frac{4\pi}{\pi n^2}\cos n\pi - o$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n.\ [\cos n\pi = (-1)^n]$$

$$\text{Finally, } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\sin nx\,dx$$

$$= -\frac{1}{\pi n}\left[(x + x^2)\cos nx\right]_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi} (1 + 2x)\cos nx\,dx$$

$$= -\frac{1}{\pi n}[\pi \cos n\pi + \pi^2 \cos n\pi + \pi \cos n\pi - \pi^2 \cos n\pi]$$

$$+\frac{1}{\pi n^2}[\sin nx]\int_{-\pi}^{\pi} + \frac{2}{\pi n^2}\left[x \sin nx\right]_{-\pi}^{\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi} \sin nx dx$$

$$= -\frac{1}{\pi n}. 2\pi \cos n\pi + 0 + 0 + \frac{2}{\pi n^3}[\cos nx]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\cos n\pi + o = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}. \; [\cos n\pi = (-1)^n]$$

Now putting the values of a_o, a_n and b_n in (1) we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n \cos nx + \frac{2}{n}(-1)^{n+1}\sin nx\right]$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \frac{1}{4^2}\cos 4x - \ldots\right]$$

$$+ 2\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right] \quad (2)$$

Now at extremum π and $-\pi$, the sum of the series

$$= f(\pi) = \frac{1}{2}\{f(-\pi + o) + f(\pi - 0)\} = \frac{1}{2}\{-\pi + \pi^2 + \pi + \pi^2\} = \pi^2.$$

Putting $x = \pi$ in the above series (2); we get

$$f(\pi) = \pi^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \ldots\right]$$

$$+ 2\left[\frac{\sin\pi}{1} - \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} - \frac{\sin 4x}{4} + \ldots\right]$$

$$\text{Or, } \pi^2 = \frac{\pi^2}{3} + 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) + 0$$

$$\text{Or, } \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\text{Or, } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Example 11. Expand in Fourier series the function

$f(x) = x \sin x$ in the interval $-\pi < x < \pi$.

Hence deduce that $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ....$

[C. U. P. 1973]

Solution : By the definition of Fourier series, we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \ (n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx.$

The given function in question is $f(x) = x \sin x$

$\therefore \ f(-x) = -x \sin(-x) = -x. -\sin x = x \sin x = f(x).$

So $f(x) = x \sin x$ is an even function and in this case $b_n = 0$

and $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx.$

Or, $a_o = \frac{2}{\pi}\int_{0}^{\pi} x \sin x dx$

$= \frac{2}{\pi}[-x \cos x]_o^{\pi} + \frac{2}{\pi}\int_{0}^{\pi} \cos x\, dx$

$= -\frac{2}{\pi}(\pi \cos\pi - 0) + \frac{2}{\pi}[\sin x]_0^{\pi}$

$= -2.(-1) = 2.$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx dx$

Or, $a_n = \frac{2}{\pi}\int_{0}^{\pi} x \sin x \cos nx\, dx.$

when $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos 2x}{2}\right]_o^{\pi} + \frac{1}{2\pi}\int_0^{\pi} \cos 2x\, dx$$

$$= \frac{1}{-2\pi}(\pi\cos 2\pi - o) + \frac{1}{4\pi}[\sin 2x]_o^{\pi}$$

$$= -\frac{1}{2} + o = -\frac{1}{2}.$$

Also $a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx dx\ (n \neq 1),\ n = 2, 3, 4, \ldots$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin (n+1)x - \sin (n-1)x\}\, dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin (n+1)\, xdx - \frac{1}{\pi}\int_0^{\pi} x\sin (n-1)\, xdx.$$

$$= \frac{1}{\pi}\left[-\frac{x\cos (n+1)x}{n+1}\right]_0^{\pi} + \frac{1}{(n+1)\pi}\int_0^{\pi} \cos (n+1)\, xdx$$

$$+ \frac{1}{(n-1)\pi}\left[x\cos (n-1)x\right]_0^{\pi} - \frac{1}{(n-1)\pi}\int_0^{\pi} \cos (n-1)\, xdx$$

$$= -\frac{1}{(n+1)\pi}.\ \pi\cos (n+1)\pi + 0 + \frac{1}{(n+1)^2\pi}\left[\sin (n+1)\, x\right]_0^{\pi}$$

$$+ \frac{1}{(n-1)\pi}.\ \pi \cos(n-1)\pi + 0 - \frac{1}{(n-1)^2\pi}\left[\sin (n-1)x\right]_0^{\pi}$$

$$= -\frac{1}{(n+1)}\cos (n+1)\pi + \frac{1}{(n-1)}\cos (n-1)\pi + 0$$

$$= \frac{1}{n+1}\cos n\pi - \frac{1}{n-1}\cos n\pi$$

$$= \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\cos n\pi$$

$$= -\frac{2}{n^2-1}\cos n\pi = -\frac{2}{n^2-1}.(-1)^n = \frac{2}{n^2-1}(-1)^{n+1}.$$

Now Putting the values of a_0, a_1, a_n and b_n in

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we get $f(x) = \frac{2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx + 0$

$$= 1 - \frac{1}{2}\cos x - \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x - \frac{2}{4^2-1}\cos 4x$$

$$+ \frac{2}{5^2-1}\cos 5x - \frac{2}{6^2-1}\cos 6x + \ldots\ldots$$

$$= 1 - \frac{2}{4}\cos x - \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x - \frac{2}{15}\cos 4x$$

$$+ \frac{2}{24}\cos 5x - \frac{2}{35}\cos 6x + \ldots\ldots$$

$$= 1 - 2\left[\frac{1}{4}\cos x + \frac{1}{3}\cos 2x - \frac{1}{8}\cos 3x + \frac{1}{15}\cos 4x\right.$$

$$\left. - \frac{1}{24}\cos 5x + \frac{1}{35}\cos 6x - \ldots\right] \quad (2)$$

Now putting $x = \pi/2$ in (2). we get

$$f(\pi/2) = \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left[\frac{1}{4}\cos\frac{\pi}{2} + \frac{1}{5}\cos\pi - \frac{1}{8}\cos 3\frac{\pi}{2}\right.$$

$$\left. + \frac{1}{15}\cos 2\pi - \frac{1}{24}\cos\frac{5\pi}{2} + \frac{1}{35}\cos 3\pi - \ldots\ldots\right]$$

Or, $\frac{\pi}{2} = 1 - 2\left[0 - \frac{1}{3} - 0 + \frac{1}{15} - 0 - \frac{1}{35} - 0 + \ldots\right]$

$$= \frac{2}{2} + \frac{2}{3} - \frac{2}{15} + \frac{2}{35} - \ldots$$

Or, $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \ldots$

Or, $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} - \ldots.$

Example 12. Find the Fourier series of the function f(*x*) of period 2π, where $f(x) = x\cos x$ for $-\pi < x \le \pi$. **[D. U. P 1986]**

Solution : By definition of the Fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$, $n \neq o$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$.

The given function is $f(x) = x \cos x$

$\therefore f(-x) = -x\cos(-x) = -x\cos x = -f(x)$. and so the function $f(x) = x \cos x$ is an odd function and in this case $a_n = o$ for $n = o, 1, 2, 3, \ldots$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx.$$

Thus equation no (1) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx \quad (2)$$

$$\text{where } b_1 = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin x\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos x \sin x\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \sin 2x\, dx$$

$$= -\frac{1}{2\pi}[x \cos 2x]_0^{\pi} + \frac{1}{2\pi}\int_{0}^{\pi} \cos 2x\, dx$$

$$= -\frac{1}{2\pi}(\pi \cos 2\pi - 0) + \frac{1}{4\pi}[\sin 2x]_0^{\pi}$$

$$= -\frac{1}{2} + o = -\frac{1}{2}.$$

$$\text{Also } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx \ (n \neq 1,\ n = 2, 3, 4, \ldots)$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos x \sin nx\, dx$$

$$=\frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x+\sin(n-1)x]\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx+\frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$=-\frac{1}{\pi(n+1)}\left[x\cos(n+1)x\right]_0^{\pi}+\frac{1}{\pi(n+1)}\int_0^{\pi}\cos(n+1)x\,dx$$

$$-\frac{1}{\pi(n-1)}\left[x\cos(n-1)x\right]_0^{\pi}+\frac{1}{\pi(n-1)}\int_0^{\pi}\cos(n-1)x\,dx$$

$$=-\frac{1}{(n+1)}.\pi\cos(n+1)\pi-\frac{1}{\pi(n-1)}.\pi\cos(n-1)\pi$$

$$+\frac{1}{\pi(n+1)^2}\left[\sin(n+1)x\right]_0^{\pi}+\frac{1}{\pi(n-1)^2}\left[\sin(n-1)x\right]_0^{\pi}$$

$$=-\frac{1}{(n+1)}.\cos(n\pi+\pi)-\frac{1}{(n-1)}\cos(n\pi-\pi)+o$$

$$=\frac{\cos n\pi}{n+1}+\frac{\cos n\pi}{n-1}$$

$$=\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\cos n\pi=\frac{2n}{n^2-1}.(-1)^n.\ (n\neq 1)$$

$$\therefore [\cos n\pi=(-1)^n]$$

Now putting the values of b_1 and b_n $(n\neq 1)$ in (2), we get

$$f(x)=-\frac{1}{2}\sin x+\sum_{n=2}^{\infty}\frac{2n}{n^2-1}(-1)^n.\sin nx$$

$$=-\frac{1}{2}\sin x+\frac{4}{3}\sin 2x-\frac{6}{8}\sin 3x+\frac{8}{15}\sin 4x-\dots$$

$$=-\frac{1}{2}\sin x+2\left[\frac{2}{1.3}\sin 2x-\frac{3}{2.4}\sin 3x+\frac{4}{3.5}\sin 4x-\dots\right].$$

Example 13. Find the series of sines and cosines of multiples of x which represents f(x) in the interval $-\pi<x<\pi$ where

$$f(x)=\begin{cases} o \text{ where } -\pi<x<0 \\ \frac{\pi x}{4} \text{ where } 0<x<\pi \end{cases}$$

and deduce that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\dots$ **[D.U.P. 1969]**

Solution : By definition of the Fourier series,

we have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (1)

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx$ $(n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nxdx$.

Now $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} 0.dx + \int_{0}^{\pi} \frac{\pi x}{4}\,dx\right]$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{8}.$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nxdx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{4} \cos nxdx$$

$$= 0 + \frac{1}{4}\int_{0}^{\pi} x\cos nxdx$$

$$= \frac{1}{4n}\left[x \sin nx\right]_0^{\pi} - \frac{1}{4n}\int_{0}^{\pi} \sin nx\,dx$$

$$= 0 + \frac{1}{4n^2}\left[\cos nx\right]_0^{\pi}$$

$$= \frac{1}{4n^2}\left[\cos n\pi - \cos 0\right] = \frac{1}{4n^2}\{(-1)^n - 1\}.$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{4}\sin nx dx$$

$$= 0 + \frac{1}{4}\int_{0}^{\pi} x \sin nx\, dx$$

$$= -\frac{1}{4n}[x\cos nx]_0^{\pi} + \frac{1}{4n}\int_0^{\pi} \cos nx dx$$

$$= -\frac{1}{4n}[\pi\cos n\pi - 0] + \frac{1}{4n^2}[\sin nx]_0^{\pi}$$

$$= -\frac{\pi}{4n}.(-1)^n + 0 = -\frac{\pi}{4n}.(-1)^n. \therefore [\cos n\pi = (-1)^n]$$

Thus substituting the values of a_0, a_n and b_n in (1) we get

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{1}{4n}\{(-1)^n - 1\}\cos nx + \left\{-\frac{\pi}{4n}(-1)^n\right\}\sin nx\right]$$

$$= \frac{\pi^2}{16} + \left[-\frac{2}{4.1^2}\cos x + 0 - \frac{2}{4.3^2}\cos 3x + 0 - \frac{2}{4.5^2}\cos 5x + \ldots\right]$$

$$-\frac{\pi}{4}\left[-\frac{\sin x}{1} + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x - \ldots\right]$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

$$+ \frac{\pi}{4}\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right] \quad (2)$$

Now $f(\pi) = \frac{1}{2}[f(-\pi) + f(\pi)] = \frac{1}{2}\left(0 + \frac{\pi^2}{4}\right) = \frac{\pi^2}{8}$.

Therefore, putting $x = \pi$ in (2) we get

$$f(\pi) = \frac{\pi^2}{8} = \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos\pi}{1^2} + \frac{\cos 3\pi}{3^2} + \frac{\cos 5\pi}{5^2} + \ldots\right]$$

$$+\frac{\pi}{4}\left[\frac{\sin\pi}{1} - \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} - \ldots\right]$$

$$\text{or, } \frac{\pi^2}{8} = \frac{\pi^2}{16} - \frac{1}{2}\left[-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \ldots\right] + 0$$

$$\text{or, } \frac{\pi^2}{16} = \frac{1}{2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\text{or, } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example 14. If f(x) is given by

$$f(x) = \begin{cases} 1 & \text{when } o \le x \le \pi \\ -1 & \text{when } -\pi \le x \angle 0 \end{cases}$$

expand f(x) in the Fourier series. Hence deduce

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$ **[C. U. P 1975, 1979]**

Solution : By definition of the Fourier series we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

$$\text{where } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \; (n \ne o)$$

$$\text{and } b_n = \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

$$\text{Now } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{o} f(x)\,dx + \frac{1}{\pi}\int_{o}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{o} (-1)\,dx + \frac{1}{\pi}\int_{o}^{\pi} 1\,dx$$

$$= -\frac{1}{\pi}[x]_{-\pi}^{o} + \frac{1}{\pi}[x]_{o}^{\pi}$$

$$= -\frac{1}{\pi}(o + \pi) + \frac{1}{\pi}(\pi - o) = -1 + 1 = 0.$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx dx$$

$$= -\frac{1}{\pi n}[\sin nx]_{-\pi}^{0} + \frac{1}{\pi n}[\sin nx]_{0}^{\pi}$$

$$= -\frac{1}{\pi n}(\sin 0 + \sin n\pi) + \frac{1}{\pi n}(\sin n\pi - \sin 0) = 0.$$

since $\sin 0 = 0$, $\sin n\pi = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} 1.\sin nx dx$$

$$= \frac{1}{\pi n}[\cos nx]_{-\pi}^{0} - \frac{1}{\pi n}[\cos nx.]_{0}^{\pi}$$

$$= \frac{1}{\pi n}(\cos 0 - \cos n\pi) - \frac{1}{\pi n}(\cos n\pi - \cos 0)$$

$$= \frac{2}{\pi n}(\cos 0 - \cos n\pi) = \frac{2}{\pi n}\{1 - (-1)^n\}.$$

Therefore, putting the values of a_o, a_n and b_n in (1), we get

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{2}{\pi n}\{1 - (-1)^n\}\sin nx.$$

$$= \frac{2}{\pi}\left[\frac{2\sin x}{1} + 0 + \frac{2\sin 3x}{3} + 0 + \frac{2\sin 5x}{5} + \dots\right]$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\dots\dots\right] \quad (2)$$

Now since $0 < x < \pi$

if $x = \pi/2$, $f(x) = 1$. Hence putting $x = \pi/2$ in (2), we get

$$1 = \frac{4}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \ldots \ldots\right]$$

$$\text{or, } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \ldots$$

Example 15. Find the Fourier series expansion of the function below, having period 2π :

$$f(x) = \begin{cases} -x \text{ if } -\pi < x < 0 \\ x \text{ if } 0 < x < \pi. \end{cases}$$ [D. U. P. 1985]

Solution : By the definition of the Fourier series we have

$$f(x) = \frac{a_n}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

$$\text{where } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$\text{Now } a_n = \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= -\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}\left(0 - \frac{\pi^2}{2}\right) + \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos nx\, dx$$

$$= -\frac{1}{\pi n}\left[x \sin nx \right]_{-\pi}^{0} + \frac{1}{\pi n}\int_{-\pi}^{0} \sin nx\, dx$$

$$+ \frac{1}{\pi n}\left[x \sin nx \right]_{0}^{\pi} - \frac{1}{\pi n}\int_{0}^{\pi} \sin nx\, dx$$

$$= 0 - \frac{1}{\pi n^2}\left[\cos nx \right]_{-\pi}^{0} + 0 + \frac{1}{\pi n^2}\left[\cos nx \right]_{0}^{\pi}$$

$$= -\frac{1}{\pi n^2}\left[\cos 0 - \cos n\pi \right] + \frac{1}{\pi n^2}\left[\cos n\pi - \cos 0 \right]$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0 \right] = \frac{2}{\pi n^2}\left[(-1)^n - 1 \right]$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx dx$$

$$= \frac{1}{n\pi}\left[x \cos nx \right]_{-\pi}^{0} - \frac{1}{n\pi}\int_{-\pi}^{0} \cos nx dx$$

$$- \frac{1}{n\pi}\left[x \cos nx \right]_{0}^{\pi} + \frac{1}{\pi n}\int_{0}^{\pi} \cos nx dx$$

$$= \frac{1}{n\pi}\left[0 + \pi \cos n\pi \right] - \frac{1}{\pi n^2}\left[\sin nx \right]_{-\pi}^{0}$$

$$-\frac{1}{\pi n}[\pi \cos n\pi + 0] + \frac{1}{\pi n^2}[\sin nx]_0^{\pi}$$

$$= \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + 0 = 0.$$

Now putting the values of a_o, a_n and b_n in (1) we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\{(-1)^n - 1\} \cos nx + 0$$

$$= \frac{\pi}{2} + \left[-\frac{2.2}{\pi 1^2}\cos x + 0 - \frac{2.2}{\pi 3^2}\cos 3x + 0 - \frac{2.2}{\pi . 5^2}\cos 5x + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right].$$

Example 16. Find the Fourier series for the function f(x) the interval $(-\pi, \pi)$ where

$$f(x) = \begin{cases} \pi + x \text{ when } -\pi \angle x \angle 0 \\ \pi - x \text{ when } 0 \angle x \angle \pi. \end{cases}$$ **[C. U. P. 19**

Solution : By definition of the Fourier series, we have

$$f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nxdx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nxdx.$

$$\text{Now } a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (\pi + x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[0 + \pi^2 - \frac{\pi^2}{2} + 0 + \pi^2 - \frac{\pi^2}{2}\right] = \frac{1}{2\pi} \times \pi^2 = \frac{\pi}{2}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\pi\int_{-\pi}^{0}\cos nx\,dx + \int_{-\pi}^{0} x\cos nx\,dx + \pi\int_{0}^{\pi}\cos nx\,dx - \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\left[\sin nx\right]_{-\pi}^{0} + \frac{1}{n}\left[x\sin nx\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx + \frac{\pi}{n}\left[\sin nx\right]_{0}^{\pi} - \frac{1}{n}\left[x\sin nx\right]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\left[\cos nx\right]_{-\pi}^{0} + 0 - \frac{1}{n^2}\left[\cos nx\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[\cos 0 - \cos n\pi - \cos n\pi + \cos 0\right]$$

$$= \frac{2}{\pi n^2}\left[1 - \cos n\pi\right] = \frac{2}{\pi n^2}\left[1 - (-1)^n\right].$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x)\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx dx$$

$$= \frac{1}{\pi}\Big[\pi\int_{-\pi}^{0} \sin nx dx + \int_{-\pi}^{0} x\sin nx dx + \pi\int_{0}^{\pi} \sin nx dx - \int_{0}^{\pi} x\sin nx dx\Big]$$

$$= \frac{1}{\pi}\Big[-\frac{\pi}{n}\big[\cos nx\big]_{-\pi}^{0} + \frac{1}{n}\big[x\cos nx\big]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0} \cos nx dx - \frac{\pi}{n}\big[\cos nx\big]_{0}^{\pi} + \frac{1}{n}\big[x\cos nx\big]_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi} \cos nx dx\Big]$$

$$= \frac{1}{\pi}\Big[-\frac{\pi}{n} + \frac{\pi}{n}\cos n\pi + 0 - \frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\big[\sin nx\big]_{-\pi}^{0} - \frac{\pi}{n}\cos n\pi + \frac{\pi}{n} + \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}\big[\sin nx\big]_{0}^{\pi}\Big]$$

$= \frac{1}{\pi} \times 0 = 0 \qquad \therefore b_n = 0.$

Hence putting the values of a_o, a_n and b_n in (1), we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\{1 - (-1)^n\}\cos nx + 0$$

$$= \frac{\pi}{2} + \frac{2}{\pi.\,1^2}.\,2\cos x + 0 + \frac{2}{\pi.\,3^2}\,2\cos 3x + 0 + \frac{2}{\pi.5^2}.\,2\cos 5x + \ldots\ldots$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\Big[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\Big]$$