Suggesion

Fourier: Theory + Math: 3, 4, 5, 6, 7, 9, 10 [2 set]

Laplace: Theory: 13, 14, 17, 19, 21, Laplace definition.

Math: 18, 20, 36, 48 [2 set] 22 (b)

Vector Dot and Cross: Theory + Math: 32, 37, 38, 47

Extra: 13, 12 [1 set]

Vector calculus: **Differential:** [1 or 2 set]

Theory: The Gradient, Divergence, curl

Math: 1, 3, 7, 10, 16, 17, 23, 24, 29, 32

Integral: Definition: Line Integrals

Math: 2

Green's theorem + Math: 2 and 9

[1 or 2 set]

The Laplace Transform

Definition: Let $F(t)$ be a function of t specified for $t > 0$. Then the laplace transform of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$ is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

where $F(t)$ is a f^n of time t,

s is a complex number

$f(s)$ is the laplace ~~fra~~ transform of $F(t)$.

~~Laplace transform of some elementary f^n:~~

~~#~~ Laplace transform is a mathematical technique that converts a time domain function $F(t)$ into a complex frequency domain function $f(s)$. It simplifies differential eq^n by turning them ~~#~~ into algebraic eq^n. Formula: $\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$.

$$\mathcal{L}\{F(t)\} = f(s)$$

$F(t) = 1 \quad \Rightarrow \quad \mathcal{L}\{1\} = \frac{1}{s}, \; s > 0$

$F(t) = t \quad \Rightarrow \quad \mathcal{L}\{t\} = \frac{1}{s^2}, \; s > 0$

$F(t) = t^n \quad \Rightarrow \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \; s > 0$

$, n = 0, 1, 2, \ldots$

$F(t) = e^{at} \quad \Rightarrow \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \; s > a$

$F(t) = \sin at \quad \Rightarrow \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \; s > 0$

$F(t) = \cos at \quad \Rightarrow \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \; s > 0$

$F(t) = \sinh at \quad \Rightarrow \quad \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \; s > |a|$

$F(t) = \cosh at \quad \Rightarrow \quad \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \; s > |a|.$

Laplace Derivatives

13: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

By the definition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= \left[e^{-st} \cdot F(t)\right]_0^\infty - \int_0^\infty -s \cdot e^{-st} F(t)\, dt$$

$$= -F(0) + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= s\,\mathcal{L}\{F(t)\} - F(0)$$

$$\therefore \mathcal{L}\{F'(t)\} = sf(s) - F(0)$$

(Proved)

14: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

By deffinition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

From 1st derivatives of laplace transform we get $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$.

$$\mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} \cdot F''(t)\,dt$$

$$= \left[e^{-st} \cdot F'(t)\right]_0^\infty - \int_0^\infty \left(\int_0^\infty \frac{d}{dt} e^{-st} \cdot F''(t)\,dt\right)dt$$

$$= -F'(0) - \int_0^\infty -s \cdot e^{-st} F'(t)\,dt$$

$$= -F'(0) + s\int_0^\infty e^{-st} F'(t)\,dt$$

$$= -F'(0) + s\left[sf(s) - F(0)\right]$$

$$\therefore \mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

☐ If $\mathcal{L}\{F(t)\} = f(s)$ then show that

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - \sum_{k=0}^{n-1} F^{(k)}(0)\, s^{n-k-1}.$$

প্রথমে $\mathcal{L}\{F'(t)\}$ and $\mathcal{L}\{F''(t)\}$ বের করব। তারপর, ...

Similarly $\mathcal{L}\{F'''(t)\} = \cancel{s^3 f(s) - s^2 F'(0) - sF''(0)} - F''(0)$

$$= s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$$

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \cdots - F^{n-1}(0)$$

$$\therefore \mathcal{L}\{F^n(t)\} = s^n f(s) - \sum_{k=0}^{n-1} F^k(0)\, s^{n-k-1} \quad \text{(showed)}$$

Example: show that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

or

$F(t) = \sin at$

so find $\mathcal{L}\{F''(t)\} = ?$

Let, $F(t) = \sin at$ $\qquad F(0) = 0$

$F'(t) = a\cos at$

$F''(t) = -a^2 \sin at$ $\qquad F'(0) = a$

From the 2nd derivative of laplace transform.

we get

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$\mathcal{L}\{-a^2\sin at\} = s^2\mathcal{L}\{F(t)\} - s.0 - a$$

$$\Rightarrow \mathcal{L}\{-a^2\sin at\} = s^2\mathcal{L}\{\sin at\} - a$$

$$\Rightarrow s^2\mathcal{L}\{\sin at\} + a^2\mathcal{L}\{\sin at\} = a$$

$$\Rightarrow \mathcal{L}\{\sin at\} = \frac{a}{(s^2+a^2)} \quad \text{(showed)}$$

17: Laplace Integral

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$

Let, $\phi(t) = \int_0^t F(u)\,du$

$\phi'(u) = F(u)$

$\phi(0) = 0$

Now, $\cancel{\mathcal{L}\{\phi' u\} = \mathcal{L}\{F(u)\}}\ \cancel{s f(s) - F(0)}$

$\cancel{\mathcal{L}\{\phi'(u)\} = s\mathcal{L}\{\phi(u)\} - F(0)}$

$\mathcal{L}\{\phi'(u)\} = \cancel{sf}\ s\phi(u) - \phi(0)$

$\mathcal{L}\{\phi'(u)\} = s\,\mathcal{L}\{\phi(t)\} - 0$

$\Rightarrow \mathcal{L}\{\phi(u)\} = \frac{1}{s}\mathcal{L}\{\phi'(u)\}$

$\Rightarrow \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{F(u)\}$

$\therefore \mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} \cdot f(s)$ (proved)

18:

Evaluate or Find $\int_0^t \frac{\sin t}{t}\, dt$

We know,

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Prove: $\mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$.

Soln: ~~usin~~ using infinite series

$$\cancel{\int_0^t \frac{\sin u}{u}\, du = \int \frac{1}{u} - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots}$$

$$\int_0^t \frac{\sin u}{u}\, du = \int_0^t \frac{1}{u}\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\cdots\right)$$

$$= \int_0^t \left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \cdots\cdots\right)$$

$$= \left[u - \frac{u^3}{3.3!} + \frac{u^5}{5.5!} - \cdots\cdots\right]_0^t$$

$$= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \cdots\cdots$$

then $\mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\, dt\right\} = \mathcal{L}\left\{t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \cdots\cdots\right\}$

$$= \frac{1}{s^2} - \frac{1}{3 \cdot 3!} \cdot \frac{3!}{s^4} + \frac{1}{5 \cdot 5!} \cdot \frac{5!}{s^6} - \cdots$$

$$= \frac{1}{s}\left(\frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \cdots\right)$$

$$= \frac{1}{s}\left(\frac{\frac{1}{s}}{1} - \frac{\left(\frac{1}{s}\right)^3}{3} + \frac{\left(\frac{1}{s}\right)^5}{5} - \cdots\right)$$

$\mathcal{L}\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s} \tan^{-1} \frac{1}{s}$ (Proved)

Multiplication by Power of t

19: If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{t^n \{F(t)\}\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^n(s)$$

Proof:

We have,

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \quad \text{--- (1)}$$

differentiating (1) w.r.t s we get,

$$\frac{d}{ds} \mathcal{L}\{F(t)\} = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt$$

এখানে, $\mathcal{L}\{F(t)\}$ বদলে $f(s)$ হবে,

$$\Rightarrow \frac{d}{ds} f(s) = \int_0^\infty -t. e^{-st} F(t)\, dt$$

$$\Rightarrow (-1) \frac{d}{ds} f(s) = \int_0^\infty t. \, e^{-st} F(t)\, dt$$

$$\Rightarrow (-1) \frac{d}{ds} f(s) = \mathcal{L}\{t F(t)\}$$

$$\Rightarrow \mathcal{L}\{t F(t)\} = (-1) \frac{d}{ds} f(s)$$

similarly, $\mathcal{L}\{t^2 F(t)\} = (-1)^2 \frac{d^2}{ds^2} f(s)$

$$\mathcal{L}\{t^3 F(t)\} = (-1)^3 \frac{d^3}{ds^3} f(s)$$

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$$\therefore \mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$\text{or } (-1)^n f^n(s).$$

$$\text{or } \mathcal{L}\left\{\int_0^\infty e^{-st}. t^n F(t)\, dt\right\} = (-1)^n f^n(s)$$

$$\frac{d}{ds} \int_0^\infty e^{-st}. t^n F(t)\, dt = \frac{d}{ds} (-1)^n f^n(s)$$

$$\Rightarrow -\int_0^\infty e^{-st} . t . t^n F(t)\, dt = (-1)^n f^{n+1}(s)$$

$$\therefore \int_0^\infty e^{-st}\, t^{n+1} F(t)\, dt = (-1)(-1)^n f^{n+1}(s)$$

$$\Rightarrow \int_0^\infty e^{-st}\, t^{n+1} F(t)\, dt = (-1)^{n+1} f^{n+1}(s).$$

$$\Rightarrow \mathcal{L}\{t^{n+1} F(t)\} = (-1)^{n+1} f^{n+1}(s).$$

(20) (a) $\mathcal{L}\{t \sin at\}$ (b) $\mathcal{L}\{t^2 \cos at\}$

We know,

$$\mathcal{L}\{t F(t)\} = (-1)\frac{d}{ds} f(s)$$

Let, $F(t) = \sin at$

$$\therefore f(s) = \frac{a}{s^2+a^2}$$

$$\therefore \mathcal{L}\{t \sin at\} = (-1)\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \cancel{-1 \cdot \frac{a \cdot 2s - (s^2+a^2)\cdot 0}{(s^2+a^2)^2}}$$

$$= -1\, \frac{(s^2+a^2)\cdot 0 - a \cdot 2s}{(s^2+a^2)^2}$$

$$\therefore \mathcal{L}\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$$

(b) we know,

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

$$= \mathcal{L}\{t^2 F(t)\} = \frac{d^2}{ds^2} f(s)$$

Let $F(t) = \cos at$

$$\therefore f(s) = \frac{s}{s^2+a^2}$$

$$\therefore \mathcal{L}\{t^2 \cos at\} = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

Now,

$$\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2}$$

then,

$$\frac{d^2}{ds^2}\left(\frac{a^2-s^2}{(s^2+a^2)^2}\right) = \frac{(s^2+a^2)^2 . -2s - (a^2-s^2).2(s^2+a^2).2s}{(s^2+a^2)^4}$$

$$= \frac{s\cancel{(s^2+a^2)}\{-2(s^2+a^2) - 4(a^2-s^2)\}}{(s^2+a^2)^{\cancel{4}}}$$

$$= \frac{-2s^3 - 2sa^2 - 4sa^2 + 4s^3}{(s^2+a^2)^3}$$

$$\therefore \mathcal{L}\{t^2 \cos at\} = \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}$$

21) Division by t

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$

Proof:

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt \quad \text{--- (1)}$$

integrating both side of (1) =>

$$\int_s^\infty f(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} F(t)\,dt\right] ds$$

$$= \cancel{\int_0^\infty \left[\frac{e^{-st}}{-t}\right] dt}$$

$$= \int_0^\infty F(t)\left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_0^\infty F(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty F(t)\left\{\frac{e^{-st}}{t}\right\} dt$$

$$\Rightarrow \int_s^\infty f(s)\,ds = \mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

$$\therefore \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

Example:

22(b)

show that $\int_0^\infty \frac{\sin at}{t}\,dt = \frac{\pi}{2}$

L.H.S

we know,

$$\int_0^\infty e^{-st}\,\frac{F(t)}{t}\,dt = \int_s^\infty f(s)\,ds$$

Let $F(t) = \sin at = \frac{a}{s^2+a^2}$

$\therefore F(t) = \sin t = \frac{1}{s^2+1} = f(s)$

taking $s \to 0+$ so, we get,

$$\int_0^\infty 1.\,\frac{\sin t}{t}\,dt = \int_0^\infty 1\,\frac{1}{s^2+1}\,ds$$

$$= \tan^{-1} s\Big|_0^\infty$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} = R.H.S$$

48:

Find $\mathcal{L}\{\sin\sqrt{t}\}$

Soln:

~~un~~ using series -

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + - -$$

$$\therefore \sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - - - -$$

using gamma function and ~~we get~~ -

laplace ~~fr~~ transform -

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}\,1!} - \frac{\Gamma(5/2)}{3!\,s^{5/2}} + \frac{\Gamma(7/2)}{5!\,s^{7/2}} - -$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{3\sqrt{\pi}}{4s^{5/2}\cdot 3!} + \frac{15\sqrt{\pi}}{5!\,8s^{7/2}} - -$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{1}{8s^{5/2}} + \frac{1}{64s^{7/2}} - - -$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left(1 - \frac{1}{4s\,1!} + \frac{1}{(4s)^2\cdot 2!} - -\right)$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s}$$