Suggesion

Fourier: Theory + Math: 3, 4, 5, 6, 7, 9, 10 [2 set]

Laplace: Theory: 13, 14, 17, 19, 21, Laplace definition.

Math: 18, 20, 36, 48 [2 set] 22 (b)

Vector Dot and Cross: ~~Ma~~ Theory + Math: 32, 37, 38, 47

Extra: 13, 12 [1 set]

Vector calculus: **Differential:** [1 or 2 set]

Theory: The Gradient, Divergence, curl

Math: 1, 3, 7, 10, 16, 17, 23, 24, 29, 32

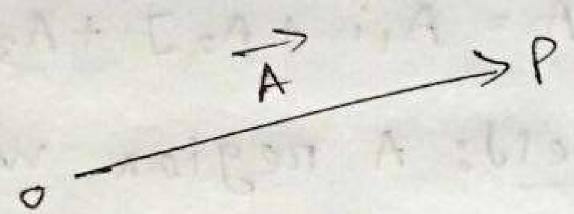
Integral: Definition: Line Integrals

Math: 2

Green's theorem + Math: 2 and 9

[1 or 2 set]

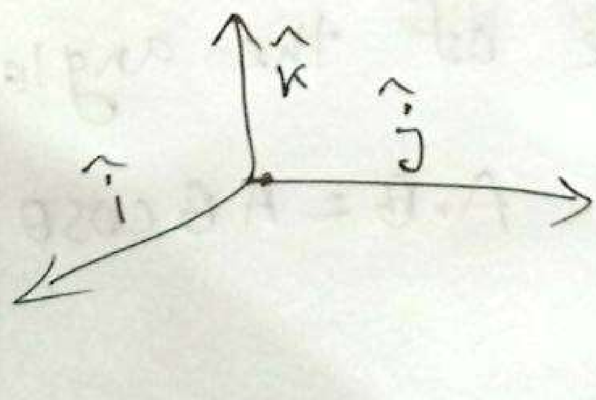
* A vector is a quantity that has both magnitude and direction.



* A scalar is a quantity that has only magnitude but no direction.

* <u>Unit vector:</u> A vector whose magnitude is 1.

* <u>Rectangular Unit vector:</u> Unit vectors along the x, y, and z axes, usually denoted as $\hat{i}$, $\hat{j}$ and $\hat{k}$.



Components of a vector: The parts of a vector along the coordinate axes. $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

Scalar field: A region where each point has a scalar value.
$\phi(x,y,z) = x^3y - z^2$

Vector field: A region where each point has a vector value.
$V(x,y,z) = xy^2\hat{i} - 2yz^3\hat{j} + x^2z\hat{k}$

Dot product: The product of two vectors that gives a scalar, found by multiplying their magnitudes and the cosine of the angle between them.

$$A \cdot B = AB\cos\theta$$

Line Integral: It gives the total effect of a vector field along a path. The integral of a function along a curve C:

$$\int_C F.dr$$

Surface Integral: It measures the total flow of a vector field through a surface.

$$\iint_S F.nds$$

Volume Integral: It gives the total value of a quantity within a 3D region.

$$\iiint_V f(x,y,z)dV$$

Vector Dot and Cross Product

The Dot or scalar Product:

If two vectors $\vec{A}$ and $\vec{B}$ denoted by $\bar{A}.\bar{B}$, defined as the magnitude of $\bar{A}$ and $\bar{B}$ and the ~~angle~~ the cosine of the angle θ between them, written as,

$$\bar{A}\cdot\bar{B} = AB\cos\theta \qquad 0 \le \theta \le \pi$$

The Cross or vector Product:

If two vectors $\vec{A}$ and $\vec{B}$ denoted by $\bar{A}\times\bar{B}$, defined as the magnitude of $\bar{A}\times\bar{B}$ and the sine ~~and~~ of the angle θ between them, written as,

$$\bar{A}\times\bar{B} = AB\sin\theta\,\hat{u} \qquad 0 \le \theta \le \pi$$

where $\hat{u}$ is the direction of $\bar{A}\times\bar{B}$.

☐ If $\bar{A}\cdot\bar{B} = 0$, then $\vec{A}$ and $\vec{B}$ are perpendicular

☐ If $\bar{A}\times\bar{B} = 0$, then $\bar{A}$ and $\bar{b}$ are parallel.

Triple Product:

Dot and cross multiplication of three vectors $\bar{A}$, $\bar{B}$ and $\bar{C}$ may produced meaningful product of the form $(A.B).C$, $A.(B\times C)$, and $A\times(B\times C)$.

- $A.(B\times C) = B.(A\times C) = C.(A\times B)$ = volume of parallelepiped.

- Scalar triple ~~vector~~ product $= A.(B\times C)$ or $[ABC]$

- Vector triple product $= A\times(B\times C)$

- Scalar triple product also known as box product

- Triple Product "Don't follow Associative Law".

- Imp: $A\times(B\times C) = B(A.C) - C(A.B)$

$$(A\times B)\times C = B(A.C) - A(B.C)$$

** সহজে মনে রাখার এই ভাবে যে, cross product এর সময় যেই vector গুলো ( ) এর ভিতরে থাকবে তারা scalar product এর সময় বাইরে থাকবে।
একবার করে

Exersice Problem:

12: Find angle of coordinate axes.

Let α, β, γ are the angles, which are made by $\vec{A}$ with the axes x, y, z respectively.

~~$\vec{A} = 3i - 6j + 2k$~~

$\vec{A} = 3i - 6j + 2k$

~~$|\vec{A}| = \sqrt{3^2 + (-6)^2 + 2^2} = 7$~~

$\vec{A}\cdot\hat{i} = A\cdot i \cos\alpha = \sqrt{3^2+(-6)^2+2^2}\cdot 1^2 \cos\alpha$

$= 7\cos\alpha$

$\vec{A}\cdot\hat{i} = (3i - 6j + 2k)\cdot i = 3$

$\vec{A}\cdot\hat{j} = (3i - 6j + 2k)\cdot j = -6$

$\vec{A}\cdot\hat{k} = (3i - 6j + 2k)\cdot k = 2$

$\therefore$ angles,

$\vec{A}\cdot i = 7\cos\alpha$

$\Rightarrow \cos\alpha = \frac{3}{7}$

$\therefore \alpha = 64.6^\circ$

Similarly,

$\beta = \cos^{-1}\left(\frac{-6}{7}\right) = 149^\circ$

$\gamma = \cos^{-1}\left(\frac{2}{7}\right) = 73.4^\circ$

13:

Projection of the vector $\vec{A}$ on the vector $\vec{B}$

Sol^n:

$\vec{A} = i - 2j + k$

$\vec{B} = 4i - 4j + 7k$

on the vector এর পর যে ভেক্টর তার unit vector বের করব। এর পর of the vector(-) এর সাথে Dot হবে।

unit vector of $\vec{B}$ is $b = \dfrac{4i - 4j + 7k}{\sqrt{4^2 + (-4)^2 + 7^2}}$

$= \frac{4}{9}i - \frac{4}{9}j + \frac{7}{9}k$

$\therefore$ The projection of the $\vec{A}$ on the $\vec{B}$ is,

$\vec{A} \cdot b = (i - 2j + k) \cdot \left(\frac{4}{9}i - \frac{4}{9}j + \frac{7}{9}k\right)$

$= \frac{4}{9} + \frac{8}{9} + \frac{7}{9} = \frac{19}{9}$ Ans

22:

Find a unit vector which is perpendicular to the plane of $\vec{A}$ and $\vec{B}$?

Given, $\vec{A} = 2i - 6j - 3k$

$\vec{B} = 4i + 3j - k$

unit vector আছে বললে $\vec{A} \times \vec{B}$ করতে হবে।

Bcz. $\vec{A} \times \vec{B} = AB \sin\theta \, \hat{u}$

$$\bar{A} \times \bar{B} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = i(6+9) - j(-2+12) + k(6+24)$$

$$= 15i - 10j + 30k$$

$\therefore$ the unit vector $\hat{u} = \frac{\bar{A} \times \bar{B}}{|A \times B|} = \frac{15i - 10j + 30k}{\sqrt{15^2 + (-10)^2 + 30^2}}$

$$= \frac{3}{7} i - \frac{2}{7} j + \frac{6}{7} k$$

(Ans)

Another unit vector opposite in direction,

$$-\frac{3}{7} i + \frac{2}{7} j - \frac{6}{7} k.$$

37. Show that $A \cdot (B \times C)$ is in absolute value equal to the volume of a parallelepiped with sides A, B, C.

Let n be a unit normal to parallelogram l, having direction of $\bar{B} \times \bar{C}$. and let h be the height of the end point of $\bar{A}$.

volume of ~~pall~~ parallelepiped = (height h) (area of parallel- l)

$= (A \cdot \hat{n}) \{ |B \times C| \}$

$= \bar{A} \cdot \{ |\bar{B} \times \bar{C}| \cdot \hat{n} \}$

$= A \cdot (B \times C)$

If $\bar{A}, \bar{B}$ and $\bar{C}$ don't form a right-handed system $\bar{A} \cdot \hat{n} < 0$ then the volume will take $|A \cdot (B \times C)|$.

(38) Show that $A \cdot (B \times C) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Soln:

Let, $\bar{A} = A_1 i + A_2 j + A_3 k$

$\bar{B} = B_1 i + B_2 j + B_3 k$

$\bar{C} = C_1 i + C_2 j + C_3 k$

$$\bar{B} \times \bar{C} = \begin{vmatrix} i & j & k \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = (B_2 C_3 - B_3 C_2) i - j (B_1 C_3 - B_3 C_1) + k (B_1 C_2 - B_2 C_1)$$

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = (A_1 i + A_2 j + A_3 k)\cdot\{(B_2C_3 - B_3C_2)i - j(B_1C_3 - B_3C_1) + k(B_1C_2 - B_2C_1)\}$$

$$= A_1(B_2C_3 - B_3C_2) - A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad \text{(showed)}$$

47: (a) $\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$
(b) $(\bar{A}\times\bar{B})\times\bar{C} = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{A}(\bar{B}\cdot\bar{C})$

Prove.

(a) Let $\bar{A} = A_1 i + A_2 j + A_3 k$

$\bar{B} = B_1 i + B_2 j + B_3 k$

$\bar{C} = C_1 i + C_2 j + C_3 k$

$$\bar{B}\times\bar{C} = \begin{vmatrix} i & j & k \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = (B_2C_3 - B_3C_2)i - (B_1C_3 - B_3C_1)j + (B_1C_2 - B_2C_1)k$$

$$\bar{A}\times(\bar{B}\times\bar{C}) = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_3 - B_3C_1 & B_1C_2 - B_2C_1 \end{vmatrix}$$

$$= \cancel{(A_1B_2C_3 - A_1B_3C_1)\,i - j}$$

$$= i\{(A_2B_1C_2 - A_2B_2C_1 - A_3B_1C_3 + A_3B_3C_1)\}$$

$$+ j\{(-A_1B_1C_2 + A_1B_2C_1 + A_3B_2C_3 - A_3B_3C_2)\}$$ [∵ [illegible]

$$+ k\{(A_1B_1C_3 - A_1B_3C_1 - A_2B_2C_3 + A_2B_3C_2)\}$$

Now, $\bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$

$$\bar{B}\cdot(\bar{A}\cdot\bar{C}) = (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})\cdot\{(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(C_1\hat{i} + C_2\hat{j} + C_3\hat{k})\}$$

$$= (B_1 i + B_2 j + B_3 k)\cdot(A_1C_1 + A_2C_2 + A_3C_3)$$

$$= (A_1B_1C_1 + A_2B_1C_2 + A_3B_1C_3)\,i + j(A_1B_2C_1 + A_2B_2C_2 + A_3B_2C_3)$$

$$+ k(A_1B_3C_1 + A_2B_3C_2 + A_3B_3C_3)$$

similarly,

$$\bar{C}(\bar{A}\cdot\bar{B}) = (A_1B_1C_1 + A_2B_2C_1 + A_3B_3C_1)i + j(A_1B_1C_2 + A_2B_2C_2 + A_3B_3C_2)$$
$$+ k(A_1B_1C_3 + A_2B_2C_3 + A_3B_3C_3)$$

Now:

$$B(A.C) - C(A.B) = \cancel{(A_2B_2C_1 + A_3B_3C_1}$$

$$= (A_2B_1C_2 + A_3B_1C_3 - A_2B_2C_1 - A_3B_3C_1)i +$$

$$= (A_1B_2C_1 + A_3B_2C_3 - A_1B_1C_2 - A_3B_3C_2)j +$$

$$= (A_1B_3C_1 + A_2B_3C_2 - A_1B_1C_3 - A_2B_2C_3)k$$

$$\therefore A\times(B\times C) = B(A.C) - C(A.B)$$ (showed)

same ~~ভাবে~~ প্রমাণেতে $(A\times B)\times C = B(A.C) - A(B.C)$

চেক করবা।

Vector Differential

The vector differential Operator Del (written as

defined by,

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

~~The Gradient~~

The Gradient: (grad ϕ)

Let $\phi(x, y, z)$ ~~is~~ be defined and differentiable at each point (x, y, z) in a region of a space, then the gradient of ϕ is denoted as $\nabla\phi$ and defined by

$$\nabla.\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \phi$$

$$\therefore \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

The Divergence: ~~let ϕ f.~~ ~~$\bar{\nabla} f$~~ ($\text{div.} \bar{\nabla}$)

Let $\bar{v} = v_1 i + v_2 j + v_3 k$ be defined and differentiable at each point (x, y, z) in a region of a space, then the divergence of $\bar{v}$ is denoted as $\nabla \cdot \bar{v}$ and defined by

$$\nabla \cdot \bar{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}\right)$$

$$\therefore \nabla \cdot \bar{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

The Curl: (curl $\bar{v}$ or rot $\bar{v}$)

Let $\bar{v} = v_1 i + v_2 j + v_3 k$ be defined and differentiable at each point (x, y, z) in a region of a space, then the curl of $\bar{v}$ is denoted as $\nabla \times \bar{v}$ and defined by

$$\nabla \times \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

2:

$\phi = (x,y,z) = 3x^2y - y^3z^2$ at point $(1,-2,-1)$

Soln:

The gradient is

$$\nabla\phi = i\frac{\partial}{\partial x}(3x^2y - y^3z^2) + j\frac{\partial}{\partial y}(3x^2y - y^3z^2) + k\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= i(6xy) + j(3x^2 - 3y^2z^2) + k(-2y^3z)$$

at point $(1,-2,-1) \Rightarrow$

$$= -12i - 9j - 16k \quad \text{Ans.}$$

3: (a) $\phi = \ln|r|$ (b) $\phi = \frac{1}{|r|}$

Find $\nabla\phi$:

Soln: (a) $\phi = \ln|r|$

Let, $\bar{r} = xi + y\hat{j} + zk$

$$|r| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\therefore \phi = \ln(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

Now,

$$\nabla\phi = \frac{1}{2}\nabla\{\ln(x^2+y^2+z^2)\}$$

$$= \cancel{\frac{1}{2}\{\frac{\partial}{\partial x} f\}}$$

$$= \frac{1}{2}\left[\hat{i}\frac{\partial}{\partial x}\{\ln(x^2+y^2+z^2)\} + \hat{j}\frac{\partial}{\partial y}\{\ln(x^2+y^2+z^2)\} + \hat{k}\frac{\partial}{\partial z}\{\ln(x^2+y^2+z^2)\}\right]$$

$$= \frac{1}{2}\left(\hat{i}\cdot\frac{2x}{x^2+y^2+z^2} + \hat{j}\cdot\frac{2y}{x^2+y^2+z^2} + \hat{k}\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{xi+yj+zk}{x^2+y^2+z^2}$$

$$\therefore \nabla\phi = \frac{\vec{r}}{|r^2|}$$

(b) $\phi = \frac{1}{|\vec{r}|}$

Let, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2} \qquad \therefore \frac{1}{|r|} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\therefore \nabla\phi = \{i\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} + j\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}} + k\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= -\frac{1}{\cancel{2}}\cdot i(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot\cancel{2}x - \frac{1}{\cancel{2}}\cdot j(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot\cancel{2}y$$

$$- \frac{1}{\cancel{2}}\cdot k(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot\cancel{2}z$$

$$= -\left(x\hat{i} \cdot \frac{1}{\sqrt[3]{x^2+y^2+z^2}} + y\hat{j} \cdot \frac{1}{\sqrt[3]{x^2+y^2+z^2}} + z\hat{k} \cdot \frac{1}{\sqrt[3]{x^2+y^2+z^2}}\right)$$

$$= -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt[3]{x^2+y^2+z^2}}$$

$$= -\frac{\bar{r}}{r^3} \quad \text{(Ans)}$$

$$\boxed{\begin{aligned} \therefore\ r &= \sqrt[3]{x^2+y^2+z^2} \\ \therefore\ r^3 &= x^2+y^2+z^2 \end{aligned}}$$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

(7.)

Soln:

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(2xz^2 - 3xy - 4x) + \hat{j}\frac{\partial}{\partial y}(2xz^2 - 3xy - 4x) + \hat{k}\frac{\partial}{\partial z}(2xz^2 - 3xy + 4x)$$

$$= i(2z^2 - 3y - 4) + j(-3x) + k(4xz)$$

at point $(1,-1,2) \Rightarrow$

$$= 7i - 3\hat{j} + 8k$$

Now, Let, the eqn passing through $\vec{r_0}$ and perpendicular to a normal vector $\bar{N}$. so $(\bar{r} - \bar{r_0}) \cdot \bar{N} = 0$

$$\left(x\hat{i}+y\hat{j}+z\hat{k} - \hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(7i-3j+8k\right)=0$$

$$\Rightarrow \left\{\hat{i}(x-1)+\hat{j}(y+1)+\hat{k}(z-2)\right\}\left(7\hat{i}-3\hat{j}+8\hat{k}\right)=0$$

$$\Rightarrow 7(x-1)-3(y+1)+8(z-2)=0$$

$$\Rightarrow 7x-7-3y-3+8z-16=0$$

$$\Rightarrow 7x-3y+8z=26$$ required eq^n.

10: $\phi = x^2yz+4xz^2$ point $(1,-2,-1)$

~~direct~~ $\bar{a} = 2i-j-2k$

$$\cancel{\nabla\times\phi} = \begin{vmatrix} \cancel{i} & \cancel{j} & \cancel{k} \\ \cancel{\frac{\partial}{\partial x}} & & \end{vmatrix}$$

$$\nabla\cdot\phi = i\left(2xyz+4z^2\right)+j\left(x^2z\right)+k\left(x^2y+8xz\right)$$

at point $(1,-2,-1) \Rightarrow = 8i-j-10k$

~~normal~~ unit vector of $\bar{a}$ is $\hat{a} = \dfrac{2i-j-2k}{\sqrt{2^2+(-1)^2+(-2)^2}}$

$$= \frac{2}{9}i-\frac{1}{9}j-\frac{2}{9}k$$

$\therefore$ the directional derivatives is,

$$\nabla\phi \cdot \hat{a} = (8i - j - 10k)\left(\frac{2}{9}i - \frac{1}{9}j - \frac{2}{9}k\right)$$

$$= \frac{16}{9} + \frac{1}{9} + \frac{20}{9} = \frac{37}{9} \text{ (Ans)}$$

(16) $\phi = 2x^3y^2z^4$ Find $\nabla \cdot \nabla\phi$?

Soln:

(a) $\nabla\phi = 6x^2y^2z^4\,\hat{i} + 4x^3yz^4\,\hat{j} + 8x^3y^2z^3\,\hat{k}$

$$\therefore \nabla \cdot \nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(6x^2y^2z^4\,\hat{i} + 4x^3yz^4\,\hat{j} + 8x^3y^2z^3\,\hat{k}\right)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

(b) Show that $\nabla \cdot \nabla\phi = \nabla^2\phi$ where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Soln:

$\nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ [from the definition of gradient]

$$\nabla \cdot \nabla\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \cdot \phi$$

$$= \nabla^2 \cdot \phi \quad \text{(showed)}$$

17: Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = \cancel{x^2+y^2+} (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\therefore \frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\nabla \cdot \frac{1}{r} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x\hat{i} - \frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2y\hat{j} - \frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2z\hat{k}$$

$$\nabla \cdot \frac{1}{r} = -\left\{x\hat{i} \cdot (x^2+y^2+z^2)^{-\frac{3}{2}} + y\hat{j} \cdot (x^2+y^2+z^2)^{-\frac{3}{2}} + z\hat{k} \cdot (x^2+y^2+z^2)^{-\frac{3}{2}}\right\}$$

$$\nabla^2 \frac{1}{r} = -\left\{-\frac{3}{2} \cdot x(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2x - \frac{3}{2} y (x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2y - \frac{3}{2} z (x^2+y^2+z^2)^{-\frac{5}{2}} \cdot 2z\right\}$$

$$\nabla^2\left(\frac{1}{r}\right) = 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= \frac{3x^2 + 3y^2 + 3z^2}{\sqrt[5]{x^2+y^2+z^2}}$$

$$= \underbrace{3(x^2+y^2+z^2)}\cdot(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= 3(x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$=$$

17: Prove $\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 0$

Given $\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 0$

where $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\therefore \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

Let, $f(x,y,z) = \frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\therefore \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Now, $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

then,

$$\frac{\partial^2 f}{\partial x^2} = \frac{d}{\partial x}\left\{-x(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}$$

$$= -1\cdot(x^2+y^2+z^2)^{-\frac{3}{2}} - x\cdot -\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot\cancel{2}x$$

$$= -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} \text{ ---- (i)}$$

similarly,

$$\frac{\partial^2 f}{\partial y^2} = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} \text{ ---- (ii)}$$

$$\frac{\partial^2 f}{\partial z^2} = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} \text{ ---- (iii)}$$

Now, add (i), (ii) and (iii) =>

$$-(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$
$$- (x^2+y^2+z^2)^{-\frac{3}{2}} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\Rightarrow -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)^1(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\Rightarrow -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$1-\frac{5}{2} = \frac{-3}{2}$$

$$\Rightarrow 0$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = 0 \quad \text{(Proved)}$$

The curl:

23: $A = xz^3 i - 2x^2yz j + 2yz^4 k$

at point $(1,-1,1)$, Find $\nabla \times A$.

Soln:

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \cancel{i(-4xyz - 0)}$$

$$= i(2z^4 + 2x^2y) - j(0 - 3xz^2) + k(-4xyz - 0)$$

$$= 2\hat{i}(z^2 + x^2y) + 3xz^2\hat{j} - 4xyzk$$

at point $(1,-1,1)$ $\Rightarrow$

$$= 2i(1^2 + 1^2 \cdot -1) + 3.1.1^2\hat{j} - 4.1.-1.1\hat{k}$$

$$= 3\hat{j} + 4\hat{k} \quad \text{Ans}$$

24: $\bar{A} = x^2 y \hat{i} - 2xz \hat{j} + 2yz \hat{k}$

find curl curl $\bar{A}$ or $\nabla \times (\nabla \times \bar{A})$?

Soln:

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$$

$$= i(2z + 2x) - \hat{j}(0 - 0) + k(-2z - x^2)$$

$$= i(2z + 2x) - k(x^2 + 2z)$$

$$\nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} i & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 2x & 0 & -x^2 - 2z \end{vmatrix}$$

$$= i(0 - 0) - \hat{j}(-2x - 0 - 2) + k(0 - 0)$$

$$\therefore \nabla \times (\nabla \times \bar{A}) = \hat{j}(2x + 2) \text{ Ans}$$

(29) Prove $\nabla \times (\nabla \times \bar{A}) = -\nabla^2 \bar{A} + \nabla(\nabla \cdot \bar{A})$.

Sol^n:

Here

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

Now,

$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \times \bar{A} = i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + j\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

L.H.S

$$\nabla \times (\nabla \times \bar{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left\{ \frac{\partial}{\partial y}\cdot\frac{\partial A_2}{\partial x} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial}{\partial z}\cdot\frac{\partial A_3}{\partial x} \right\}$$

$$+ \hat{j}\left\{ -\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial}{\partial x}\cdot\frac{\partial A_1}{\partial y} + \frac{\partial}{\partial z}\cdot\frac{\partial A_3}{\partial y} - \frac{\partial^2 A_2}{\partial z^2} \right\}$$

$$+ \hat{k}\left\{ \frac{\partial}{\partial x}\cdot\frac{\partial A_1}{\partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial}{\partial y}\cdot\frac{\partial A_2}{\partial z} \right\}$$

Now, take elements of $\hat{i}$:

$$= \left\{ \left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right) \right\}$$

add and substruct $\frac{\partial^2 A_1}{\partial x^2}$ =>

$$= \left\{ \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \frac{\partial A_1}{\partial x} \right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_1}{\partial x^2} \right) \right\}$$

$$= \left[\frac{\partial}{\partial x}(\nabla \cdot \bar{A}) - \nabla^2 \hat{A}_1 \right]$$

Similarly, by taking elements of $\hat{j}$ an $\hat{k}$, we get,

$$\left[\frac{d}{dy}(\nabla \cdot \bar{A}) - \nabla^2 \hat{A}_2 \right] \text{ and } \left[\frac{\partial}{\partial z}(\nabla \cdot \bar{A}) - \nabla^2 \hat{A}_3 \right]$$

Now add all $\Rightarrow$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial x}\left\{(\nabla\cdot\bar{A})-\nabla^2\cdot A_1\right\}\right]+\hat{j}\left[\frac{\partial}{\partial y}\left\{(\nabla\cdot\bar{A})-\nabla^2 A_2\right\}\right]+k\left[\frac{\partial}{\partial z}\left\{(\nabla\cdot\bar{A})-\nabla^2 A_3\right\}\right]$$

$$= \cancel{\nabla(\nabla\cdot\bar{A})-\nabla^2\cdot\bar{A}}$$

$$= \hat{i}\frac{\partial}{\partial x}(\nabla\cdot\bar{A})-\nabla^2\cdot A_1\hat{i}+\hat{j}\frac{\partial}{\partial y}(\nabla\cdot\bar{A})-\nabla^2 A_2\hat{j}+k\frac{\partial}{\partial z}(\nabla\cdot\bar{A})-\nabla^2\cdot A_3\hat{k}$$

$$= \hat{i}\frac{\partial}{\partial x}(\nabla\cdot\bar{A})+\hat{j}\frac{\partial}{\partial y}(\nabla\cdot\bar{A})+\hat{k}\frac{\partial}{\partial z}(\nabla\cdot\bar{A})-(\nabla^2 A_1 i+\nabla^2 A_2 j+\nabla^2 A_3 k)$$

$$= \nabla(\nabla\cdot\bar{A})-\nabla^2\bar{A}$$

= R.H.S

(Proved)

☐ If, $\nabla \cdot \bar{v} = 0$ then it is solenoidal

☐ If $\nabla \times \bar{v} = 0$ then it is irrotational or conservative.

☐ gradient of a scalar function or scalar potential means find ϕ.

32: (a) $\bar{v}$ is <u>irrotational</u> (means $\nabla \times \bar{v} = 0$

Find the constant a, b, c,

where $\bar{v} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

Soln:

$$\nabla \times \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow i(c+1) - \hat{j}(4-a) + k(b-2) = 0$$

$$\Rightarrow i(c+1) + \hat{j}(a-4) + k(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

comparing $\hat{i}, \hat{j}, \hat{k}$ we ~~find~~ get,

$a = 4, b = 2, c = -1$.

Now put a, b, c value into vector $\bar{v} \Rightarrow$

$$\bar{v} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

(b) show that $\bar{v}$ can be expressed as the gradient of a scalar ~~fn~~ fn or scalar potential.

Soln:

From (a) we get,

$$\bar{v} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

~~Let~~, Assume, $\bar{v}_A = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

compare $\bar{v}$ and $\bar{v}_A$ we get $\Rightarrow$

~~Then~~, (i) $\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \partial\phi = (x+2y+4z)\partial x$

(ii) $\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \partial\phi = (2x-3y-z)\partial y$

(iii) $\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \partial\phi = (4x-y+2z)\partial z$

Now integrating (i), (ii) and (iii) we get,

(i) $\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz$ $\qquad \frac{x^2}{2} + \frac{3y^2}{2} + z^2$

(ii) $\Rightarrow \phi = x^2 - 3xy - xz$

(iii) $\Rightarrow \phi = 2x^2 - xy + 2xz$

[illegible] main নিয়ম

So, $\partial\phi = (x+2y+4z)dx + (2x-3y-z)dy + (4x-y+2z)\partial z$

Now, integrating both side,

$\int \partial\phi = \int (x+2y - \text{—})dx + (\quad)dy + (\quad)dz$

কিন্তু এ math হবে শুধু partial ~~de~~ integration করতে হবে।

Now, integrating (i) partially with respect to x when y and z constant.

$\int \partial\phi = \int (x+2y+4z)dx$

$\Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$ —— (4).

integrating (ii) partially w.r.t. y when x and z const

$$\int d\phi = \int (2x - 3y - z)\, dy$$

$$\Rightarrow \phi = 2xy - \frac{3y^2}{2} - yz + g(x, z) \quad \text{(5)}$$

integrating (iii) partially w.r.t z when x and y constant

$$\int d\phi = \int (4x - y + 2z)\, dz$$

$$\Rightarrow \phi = 4xz - yz + z^2 + h(x, y) \quad \text{(6)}$$

~~Now take all the common ~~val~~ element among~~

~~(4), (5) and (6).~~

~~So that $\phi =$~~

From (4), (5) and (6) we get the value of

ϕ is,

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz \quad \text{An}$$

Vector Integral

Line Integrals:

If $\vec{A}(x,y,z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ is a vector field and

$\vec{r}(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k}$ is a smooth curve

C, then the line integral of the tangential component of $\vec{A}$ from P_1 ~~point~~ to

P_2 along C is ~~denoted~~ defined by

$$\int_{P_1}^{P_2} \vec{A}\cdot d\vec{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

and if the curve C is closed then the line integral written as.

$$\oint \vec{A}\cdot d\vec{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

Theorem: If $\vec{A} = \nabla\phi$ everywhere in a region R of space, defined by $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ and $c_1 \le z \le c_2$, where $\phi(x,y,z)$ is single valued and has continuous derivatives in R, then.

1. $\int_{P_1}^{P_2} \bar{A} d\bar{r}$ is independent of the path C in R joining P_1 and P_2

2. $\oint_C \bar{A} d\bar{r} = 0$ around any closed curve C in R

In such case $\bar{A}$ is called conservative vector field and ϕ is scalar potential.

A vector field $\bar{v}$ is conservative if and only if $\nabla \times \bar{v} = 0$ or $\bar{v} = \nabla\phi$. In such case,

$\bar{v}\, d\bar{r} = \cancel{A_1 dx +} v_1 dx + v_2 dy + v_3 dz = d\phi$

an exact differential.

Math:

(2) The acceleration of a particle at any time $t \geq 0$ is giveb by

$$a = \frac{d\bar{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$$

If the ~~ve~~ velocity $\bar{v}$ and displacement $\bar{r}$ are zero at $t=0$ and find $\bar{v}$ and $\bar{r}$ at any time.

Soln:

Integrating $\frac{d\bar{v}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$

$$\int d\bar{v} = i\int 12\cos 2t\,dt + j\int -8\sin 2t\,dt + k\int 16t\,dt$$

$$\Rightarrow \bar{v} = 12 \cdot \frac{\sin 2t}{2}\,\hat{i} - 8\left(\frac{-\cos 2t}{2}\right)\hat{j} + 16 \cdot \frac{t^2}{2}\,\hat{k}$$

$$\therefore \bar{v} = 6\sin 2t\,i + 4\cos 2t\,\hat{j} + 8t^2\,\hat{k} + c_1$$

putting $\bar{v} = 0$ when $t = 0$ we get,

$$0 = 6\sin 2.0\,i + 4\cos 2.0\,\hat{j} + 8.0^2\,\hat{k} + c_1$$

$$\therefore c_1 = -4\hat{j}$$

So, $\bar{v} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$ Ans

Now, $\frac{d\bar{r}}{dt} = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$

integrating,

$\int d\bar{r} = i\int 6\sin 2t\,dt + j\int 4\cos 2t - 4\,dt + k\int 8t^2\,dt$

$\Rightarrow \bar{r} = i.\cancel{6}^{3}\frac{-\cos 2t}{\cancel{2}} + j.\left(\cancel{4}^{2}\frac{\sin 2t}{\cancel{2}} - 4t\right) + k.\frac{8t^3}{3}$

$= -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8t^3}{3}\hat{k} + c_2$

putting $\bar{r} = 0$ when $t = 0$

$0 = -3\cos 0\,\hat{i} + (2\sin 0 - 4.0)\hat{j} + \frac{8.0^3}{3}\hat{k} + c_2$

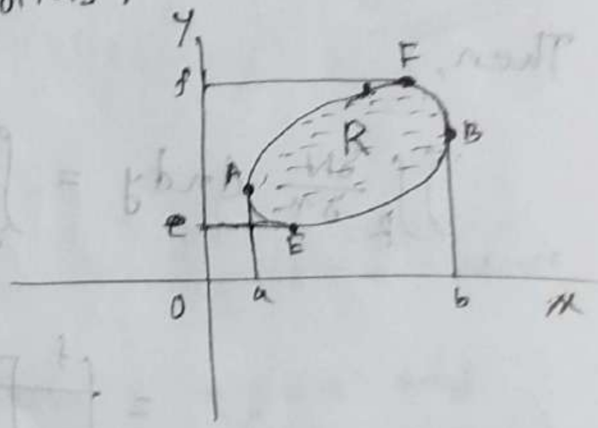
$\therefore c_2 = 3i$

So, $\bar{r} = 3(1 - \cos 2t)i + (2\sin 2t - 4t)\hat{j} + \frac{8t^3}{3}\hat{k}$

Green theorem:

1. Prove green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in the most two points.

C = AEBFA



Let the eq^n of curves AEB and AFB be $y = y_1(x)$ and $y = y_2(x)$ respectively.

If R region is bounded by curve C then we have

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_{x=a}^{x=b}\left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial M}{\partial y}\, dy\right] dx$$

$$= \int_a^b M(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)} dx$$

$$= \int_a^b \left[M(x,y_2) - M(x,y_1)\right] dx$$

$$= -\int_a^b M(x,y_1) - \int_b^a M(x,y_2) = -\oint_C M\, dx$$

$$\therefore \oint M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy \quad \text{———} \ (1)$$

Similarly ~~eq~~^{n}: let the eq^n of curve EAF and EBF be $x = x_1(y)$ and $x = x_2(y)$ respectively.

Then,

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{y=f}\left[\int_{x=x_1(y)}^{x=x_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy$$

$$= \int_e^f \Big[N(x,y)\Big]_{x_1(y)}^{x_2(y)}$$

$$= \int_e^f \left[N(x_2,y) - N(x_1,y)\right]dy$$

$$= \int_e^f N(x_2,y)\,dy + \int_f^e (x_1,y)\,dy$$

$$\therefore \oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy \quad \text{———} \ (2)$$

$$\cancel{= \iint_R \frac{\partial N}{\partial x}\,dx\,dy}$$

adding ① and ② $\Rightarrow$

$$\oint_c M\,dx + \oint_c N\,dy = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy + \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

$$\therefore \oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

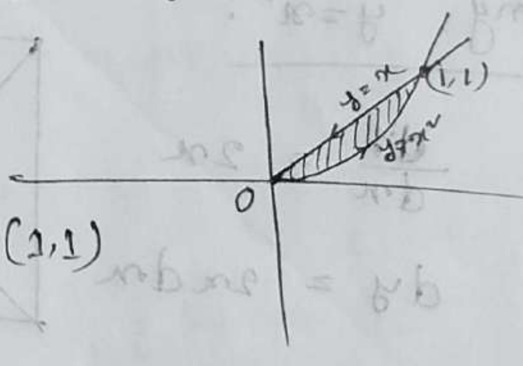
(Proved)

Math: 2

verify Green's theorem,

$\oint_c (xy+y^2)\,dx + x^2 dy$ where C is closed curve of the region bounded by $y=x$ and $y=x^2$.

$y=x$ and $y=x^2$ intersects at point $(0,0)$ and $(1,1)$



~~Line in~~ integral:

Along, $y=x^2$, the line integral equals

$\oint_c M\,dx$

Sol^n:

Here, $M = xy+y^2$

$N = x^2$

~~Along, $y=x^2$, the line integral,~~

$\oint_c M\,dx$

We know,

Green's theorem,

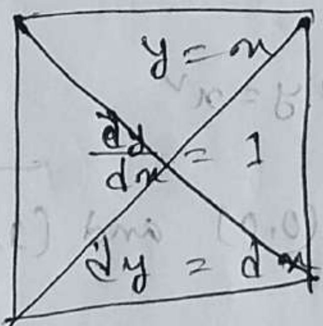
$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

L.H.S

Along, $y = x^2$:

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$



Now,

$$\oint_{y=x^2} M\,dx + N\,dy = \int_0^1 (xy+y^2)\,dx + x^2\,dy$$

$$= \int_0^1 \left(x.x^2 + (x^2)^2\right) dx + x^2.2x\,dx$$

$$= \int_0^1 \left(x^3 + x^4\right) dx + 2x^3\,dx$$

$$= \cancel{\left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^4}{4}\right]_0^1}$$

$$= \cancel{\left(\frac{1}{4} + \frac{1}{5} + \frac{2}{4_2} - 0 - 0 - 0\right)}$$

$$= \cancel{\frac{5+4+2}{20} = \frac{11}{20}}$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{19}{20}$$

Along $y=x$:

$$y = x$$

$$\frac{dy}{dx} = 1$$

$$\therefore dy = dx$$

Now, $\oint_{y=x} M\,dx + N\,dx = \int_1^0 (xy + y^2)\,dx + x^2\,dy$

$$= \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx$$

$$= \int_1^0 (3x^2)\,dx = \left[\frac{3x^3}{3}\right] = -1$$

Then,

$$\oint_C M\,dx + N\,dy = \frac{19}{20} - 1 = \frac{-1}{20}.$$

Here, $M = xy + y^2$

$N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y$$

$$\frac{\partial N}{\partial x} = 2x$$

<u>R.H.S</u>

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dx\, dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dx\, dy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[x.x - \cancel{x^2} - x.x^2 + x^4\right] dx$$

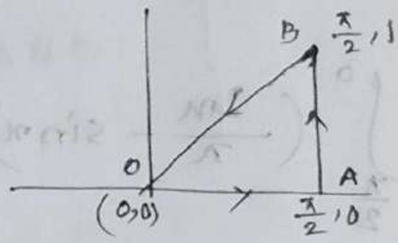
$$= \int_0^1 \left[-x^3 + x^4\right] dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \left[-\frac{1}{4} + \frac{1}{5}\right] = \frac{-1}{20}$$

$\therefore$ L.H.S = R.H.S

So that the theorem is verified.

Q: $\oint_C (y - \sin x)dx + \cos x\,dy$



(a) $\oint_C M dx + N dy.$

Along OA: $y = 0,\ dy = 0$

$\therefore \int_0^{\frac{\pi}{2}} (0 - \sin x)\,dx + \cos x \cdot 0$

$= \int_0^{\frac{\pi}{2}} (-\sin x)\,dx = \left[\cos x\right]_0^{\frac{\pi}{2}} = -1$

Along AB: $x = \frac{\pi}{2}$ $dx = 0$

$\therefore \int_0^1 \left(y - \sin\frac{\pi}{2}\right) \cdot 0 + \cos\frac{\pi}{2}\,dy$

$= \int_0^1 0 \cdot dy$

$= 0$

Along BO:

$$y = \frac{2x}{\pi} \qquad dy = \frac{2}{\pi}\,dx$$

$$\therefore \int_{\frac{\pi}{2}}^{0} \left(\frac{2x}{\pi} - \sin x\right) dx + \cos x \cdot \frac{2}{\pi}\,dx$$

$$= \left[\frac{x^2}{\pi} + \cos x + \sin x \cdot \frac{2}{\pi}\right]_{\frac{\pi}{2}}^{0}$$

$$= \left(0 + 1 + 0 - \frac{\frac{\pi^2}{4}}{\pi} - 0 - 1\cdot\frac{2}{\pi}\right)$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi} \cancel{= \frac{4\pi - \pi - 8}{4\pi} = \frac{3\pi - 8}{4\pi}}$$

Now,

$$\oint_C M\,dx + N\,dy = \cancel{-1 + 0 + \frac{3\pi - 8}{4\pi} = \frac{-\pi - 8}{4\pi}}$$

$$\cancel{-1} + 0 \cancel{+1} - \frac{\pi}{4} - \frac{2}{\pi}$$

$$\therefore -\frac{\pi}{4} - \frac{2}{\pi}$$

(b) Using green's theorem,

Here, $M = y - \sin x$

$$\frac{\partial M}{\partial y} = 1 - 0 = 1$$

$N = \cos x$

$$\frac{\partial N}{\partial x} = -\sin x$$

Now,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dx\,dy$$

$$= \int_{x=0}^{\frac{\pi}{2}} \left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1)\,dy\right] dx$$

$$= \int_{x=0}^{\frac{\pi}{2}} \left[-y\sin x - y\right]_0^{\frac{2x}{\pi}} dx$$

$$= \int_0^{\frac{\pi}{2}} \left(-\frac{2x}{\pi}\sin x - \frac{2x}{\pi} - 0 - 0\right) dx$$

$$= -\int_0^{\frac{\pi}{2}} \left(\frac{2x}{\pi}\sin x + \frac{2x}{\pi}\right) dx$$

$$= \left[\frac{2x}{\pi}\sin x + \cos x + \frac{x^2}{\pi}\right]$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\underset{u}{x}\,\underset{v}{\sin x} + x \right) dx$$

$$= -\frac{2}{\pi} \left[-x.\cos x + \sin x + \frac{x^2}{2} \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{2}{\pi} \left[-\frac{\pi}{2}\cdot 0 + 1 + \frac{\pi^2}{8} - 0 - 0 + 0 \right]$$

$$= -\frac{2}{\pi} - \frac{2}{\pi}\cdot\frac{\pi^2}{8}$$

$$= -\frac{2}{\pi} - \frac{\pi}{4}$$

L.H.S. = R.H.S.

$u = x$

$\frac{du}{dx} = 1$, $du = dx$

$\frac{dv}{dx} = \sin x$

$\Rightarrow dv = \sin x\, dx$

$\int dv = \int \sin x\, dx$

$\Rightarrow v = -\cos x$

Formula,

$= uv - \int v\, du$

$\int -\cos x\, dx$

$= -\sin x$