* Metric space :

A set X be a non-empty set whose elements we shall call points. Then, A mapping $d: X \times X \rightarrow \mathbb{R}$ is called a metric or distance function on X if it satisfy the following properti

(i) $d(x,y) > 0$ for all $x, y \in X$.

(ii) $d(x,y) = 0$ iff $x = y$.

(iii) $d(x,y) = d(y,x)$ for all $(x,y) \in X$. (symmetry)

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality).

Here, $d(x,y)$ is ~~the~~ a real number called the distance from x to y. A metric space is denoted by (X, d).

***:

* $d(x,y)$ is the distance function defined the distance from x to y, that is

$$d(x,y) = |x-y|$$

* If $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on $\mathbb{R}^2$ then the distance ~~function~~ of the points defined by.

$$\text{distance, } AB = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

* Example-I:

The function $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $d(x,y) = |x-y| \ \forall \ x, y \in \mathbb{R}$. is called the usual metric ~~space~~ on the real line $\mathbb{R}$ and $(\mathbb{R}, d)$ is a metric space.

Example - II

Let X be a ~~met~~ non-empty set and let d be the function defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then d is a metric ~~space~~ on X and d is usually called trivial metric on X. Then/Thus (X,d) is a metric space.

Example - III:

Taxicab Metric:

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|, \text{ where, } x = (x_1, x_2),\ y = (y_1, y_2).$$

* Problem-I:

Define the function $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by $d(x,y) = |x-y|$. Then prove that d is a ~~metric space~~ usual metric on $\mathbb{R}$.

Solution:

usual metric is defined as $d(x,y) = |x-y|$.

(i) $d(x,y) = |x-y| \geq 0$. ~~and~~,

(ii) $d(x,y) = |x-y| = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$.

$\therefore d(x,y) = 0$ iff $x = y$.

(iii) $d(x.y) = |x-y|$

$= |-(y-x)|$

$= |y-x|$

$= d(y,x)$

(iv) Let z is any function on $\mathbb{R}$.

$\therefore d(x,y) = |x-y|$

and, $d(y,z) = |y-z|$

$\therefore d(x,y) + d(y,z) = |x-y| + |y-z|$

$\geq |x-y+y-z|$

$= |x-z|$

$= d(x,z)$

$\therefore d(x,y) + d(y,z) \geq d(x,z)$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$.

H.W.

Problem-II:

Let (X, d) be a metric space. For any two points $x, y \in X$, Prove that (X, d_1) is a metric space where

$d_1(x,y) = \min\{1, d(x,y)\}$

Solution:

(i) Since d is a metric. so,

$d(x,y) \geq 0, \forall x,y \in X$

$\therefore d_1(x,y) = \min\{1, d(x,y)\} \geq 0$.

(ii) Let $d_1(x,y) = 0$

$\Rightarrow \min\{1, d(x,y)\} = 0$

$\Rightarrow d(x,y) = 0$

But d is a metric. so,

$d(x,y) = 0$ iff $x = y$

$\therefore d_1(x,y) = 0$

$\Rightarrow x = y$.

Conversely, let $x = y$. Then $d(x,y) = 0$

$\therefore d_1(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, 0\}$

$= 0$.

$\Rightarrow d_1(x,y) = 0$

(iii) $d_1(x,y) = \min\{1, d(x,y)\}$

$= \min\{1, d(y,x)\}$ $\quad [\because d(x,y) = d(y,x)]$

$= d_1(y,x)$

(iv) For triangular inequality, we have to prove that,

$d_1(x,y) \le d_1(x,z) + d_1(z,y)$

Observe that, $d_1(x,y) = \min\{1, d(x,y)\} \le 1$

Hence, if $d_1(x,z) = 1$ or $d_1(z,y) = 1$, then the triangular inequality holds. But if both

$d_1(x,z) < 1$ and $d_1(z,y) < 1$,

then, $d_1(x,z) = \min\{1, d(x,z)\} = d(x,z)$
and, $d_1(z,y) = \min\{1, d(z,y)\} = d(z,y)$ ——— ①

Also since d is a metric. so,

$$d(x,y) \le d(x,z) + d(z,y) \quad \text{——— ②}$$

$\therefore$ $d_1(x,y) = \min\{1, d(x,y)\} = d(x,y)$ by ①

Now, from ② $\Rightarrow$

$\therefore$ $d_1(x,y) \le d(x,z) + d(z,y)$

$\Rightarrow$ $d_1(x,y) \le d_1(x,z) + d_1(z,y)$ [From 1]

~~The~~ Thus (x,d) is a metric space. (Ans).

H.W.

Problem-III:

Show that the mapping $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$$

where, $x=(x_1,x_2)$ and $y=(y_1,y_2)$ on $\mathbb{R}^2$ is a metric on $\mathbb{R}^2$.

Solution:

Let x,y,z be any arbitrary elements of $\mathbb{R}^2$, where $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1,z_2)$.

Given that, $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

① We have $|x_1-y_1| \ge 0$ and $|x_2-y_2| \ge 0$

$\therefore$ $\max\{|x_1-y_1|, |x_2-y_2|\} \ge 0$

$\therefore$ $d(x,y) \ge 0$

(ii) If $d(x,y)=0$.

Then, $\max\{|x_1-y_1|, |x_2-y_2|\}=0$

$\Rightarrow |x_1-y_1| = 0 = |x_2-y_2|$

$\Rightarrow \cancel{|x_1-y_1|}\; x_1-y_1=0 = x_2-y_2$

$\Rightarrow x_1=y_1, \; x_2=y_2$

since, $x=(x_1,x_2)$ and $y=(y_1,y_2)$

$\therefore x=y$.

Conversely, let $x=y$

$\Rightarrow (x_1,x_2) = (y_1,y_2)$

$\Rightarrow x_1=y_1$ and $x_2=y_2$.

$\Rightarrow x_1-y_1=0$ and $x_2-y_2=0$

$\therefore d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$= \max\{0,0\}$

$= 0$.

$\therefore d(x,y)=0$ iff $x=y$.

(iii) $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$= \max\{|y_1-x_1|, |y_2-x_2|\}$

$= d(y,x)$

(iv) $d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\}$

$\Rightarrow d(x,y) = \max\{|x_1-z_1+z_1-y_1|, |x_2-z_2+z_2-y_2|\}$

$\Rightarrow d(x,y) \leq \max\{|x_1-z_1|+|z_1-y_1|, |x_2-z_2|+|z_2-y_2|\}$

$\Rightarrow d(x,y) \leq \cancel{\max\ d(x,z)+d(}$

$\max\{|x_1-z_1|, |x_2-z_2|\} + \max\{|z_1-y_1|, |z_2-y_2|\}$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$\therefore d(x,y) \leq d(x,z) + d(z,y)$

Thus, d is a metric on $\mathbb{R}^2$. (Ans)

H.W.

Problem-4:

If (X,d) is a metric space, then show that (X,d_1) is a metric space where d_1 is defined by,

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Solution:

(1) Since, d is a metric. so,

$d(x,y) \geq 0, \forall x,y \in X.$

$$\therefore d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0.$$

(ii) Let $d(x,y)=0$

$$\therefore d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{0}{1+0} = 0$$

Since, d is a metric. so,

$d(x,y)=0$ iff $x=y$

$\therefore d_1(x,y)=0$ iff $x=y$

(iii)
$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)} \quad [\because d(x,y) = d(y,x)]$$

$$= d_1(y,x).$$

(iv) Since, d is a metric. so,

$d(x,y) \leq d(x,z) + d(z,y)$

$$\therefore \cancel{\frac{d(x,y)}{1+d(x,y)}} \quad d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$\leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

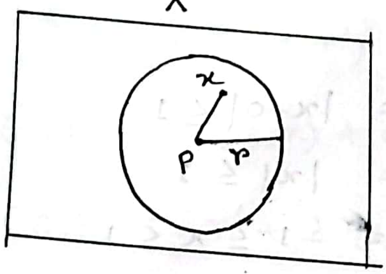
$$\leq d_1(x,z) + d_1(z,y)$$

$$\therefore d_1(x,y) \leq d_1(x,z) + d_1(z,y)$$

Thus, d is a metric space. (Ans)

* Open sphere:

Let (X, d) be a metric space. Then for any point $p \in X$ and ~~for~~ any real number $r>0$, the set $S(p,r) = \{x \mid d(x,p) < r\}$ is called an open sphere or open ball of center p and radius r.



$\boxed{\text{here, } d(x,p) < r}$

Example:

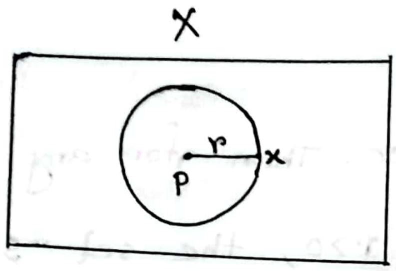
In the metric space (X,d) with the usual metric d, for $p=0$ and $r=1$

$$S(0,1) = \{x \mid d(x,0) < 1\} = |x-0| < 1$$
$$= |x| < 1$$
$$= -1 < x < 1$$
$$= (-1, 1).$$

Thus for any point on the interval $(-1,1)$ the sphere is open.

* Closed sphere:

Let (X,d) be a metric space. Then for any point $p \in X$ and any real number $r>0$, the set $S[p,r] = \{x \mid d(x,p) \le r\}$ is called an closed sphere or closed ball of center p and radius r.



$\boxed{\text{Here, } d(x,p) \le r}$

Example:

In the metric space (X, d) with the usual metric d, for $p=0$ and $r=1$

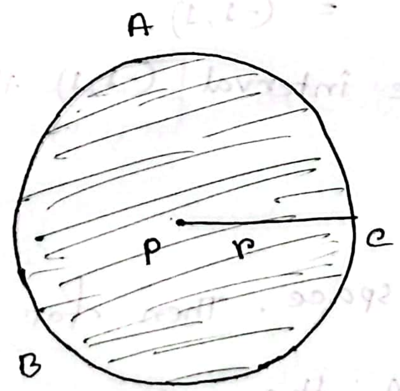
$$S[0,1] = \{x \mid d(x,0) \le 1\} = |x-0| \le 1$$
$$= |x| \le 1$$
$$= -1 \le x \le 1$$
$$= [-1, 1]$$

Thus, for any point ~~be~~ on the interval $[-1,1]$ the sphere is closed.

***

open and closed sphere (ball):

Concept



Let, ABC is a sphere with center p and radius r.

* Every point except the boundary point of the sphere is open. i.e, (a,b).
* ~~With th~~ Every point together with boundary point is closed. i.e, $[a,b]$

* **Theorem:**

Every open sphere in a metric space is an open set.

Proof:

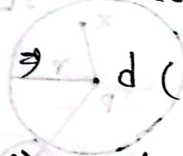
Let $x \in S(P,r)$, so that $d(x,P) < r$.

Putting $r' = r - d(x,P) > 0$, we have $S(x,r') \subset S(P,r)$

Now, for $y \in S(x,r')$ implies

$d(y,P) \leq d(y,x) + d(x,P)$

$\Rightarrow d(y,P) < r' + d(x,P)$ $\quad [\because d(y,x) < r']$



$\Rightarrow d(y,P) < r - d(x,P) + d(x,P)$

$\Rightarrow d(y,P) < r$

$\therefore y \in S(P,r)$

Then for each point x in $S(P,r)$, there is a sphere $S(x,r')$ contained in $S(P,r)$.

Hence, $S(P,r)$ is open.

~~Alternate proof:~~

Theorem:

Every closed sphere in a metric space is a closed set.

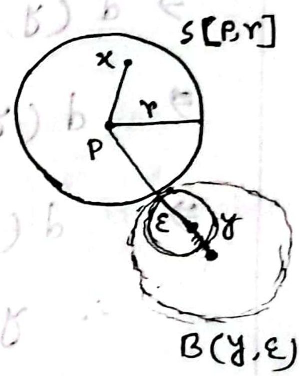
Proof:

To prove that, every closed sphere in a metric space is a closed set, we have to shown that, its complement is open.

Let (X,d) be a metric space, and consider a closed sphere $S[P,r]$ centered at P with radius r. so that,

$$S[P,r] = \{x \mid d(x,P) \leq r\}.$$

Now, let us consider the complement of $S[P,r]$, denoted as $S[P,r]^c$, which consists of all points outside of the closed sphere. We want to show that $S[P,r]^c$ is open.



Take any point $y \in S[P,r]^c$, this means $d(y,P) > r$.

Let $\varepsilon = d(y,P) - r$

consider the open sphere $B(y,\varepsilon)$ centered at y with radius ε. We want to show that $B(y,\varepsilon)$ is entirely contained in $S[P,r]^c$.

Take any point z in $B(y,\varepsilon)$, by triangular inequality

$\cancel{d(x,y)}$

$$d(P,y) \leq d(P,z) + d(z,y)$$

$$\Rightarrow d(P,z) \geq d(P,y) - d(z,y) > r - \varepsilon$$

$$\Rightarrow d(P,z) > r - d(y,P) + r = 2r - d(y,P)$$

$$\Rightarrow d(P,z) > r.$$

This implies that, z is outside of the closed sphere $S[P,r]$

So, $B(y,\varepsilon)$ is completely contained in $S[P,r]^c$. Since,

y is chosen arbitrarily, every point in $S[p,r]^c$ has an open ~~spel~~ sphere contained in $S[p,r]^c$. Therefore $S[p,r]^c$ is open. and hence $S[p,r]$ is closed set. [proved]

* Neighbourhood in Metric space :

Let S be a metric space with metric d, let $p \in S$. A set $N \subset S$ is called a nbd of p if it contains some open ball with center p. symbolically,

~~In other word~~, $S(p,r) \subseteq N_p$.

* Proposition:

Every open sphere is a nbd of each of its points.

Proof:

Let $S(p,r)$ be an open sphere and $x \in S(p,r)$.

If $x = p$, then $p \in N_p \subset S(p,r)$.

Therefore, suppose that $x \neq p$.

In order to show that $S(p,r)$ is a nbd of x, we must show that there exists $r_1 > 0$ such that $S(x,r_1) \subseteq S(p,r)$.

Now, $x \in S(p,r) \Rightarrow d(x,p) < r$

Take, $r_1 = r - d(x,p)$. Then $y \in S(x,r_1)$ implies by using triangular inequality,

$d(y,p) \leq d(y,x) + d(x,p) < r_1 + d(x,p) = r$

$\Rightarrow d(y,p) < r$

Therefore, $y \in S(p,r)$. $\therefore S(x,r_1) \subseteq S(p,r)$

* Open set in Metric space :

A subset U is open if every point in U has a nbd contained in U.

A subset U of a metric space (X,d) is called open if, for any point $x \in U$, there exists a real number $\varepsilon > 0$ such that any point satisfying $d(x,y) < \varepsilon$ belongs to U.

* Theorem:

In any metric space (X,d).

① The union of an arbitrary family of open set is open.

② The intersection of a finite number of open set is open.

Proof:

① Let A be a family of open subsets of X. and let $x \in \cup A$

Then $x \in A_0$ for some $A_0 \in A$ and therefore there exists an $r > 0$ such that,

$S(x,r) \subseteq A_0$

But $A_0 \in A$, so $S(x,r) \subseteq \cup A$.

Which shows that, $\cup A$ is open.

② Let $G = G_1 \cap G_2 \cap G_3 \cdots \cap G_n$ where $i = 1, 2, \ldots n$ are open sets.

To prove G is open. Let $p \in G$, Then $p \in G_i$, for all $i = 1, 2, \ldots n$.

So, therefore open sphere $S(p, r_1), S(p, r_2) \ldots S(p, r_n)$ such that $S(p, r_i) \subseteq G_i$.

Let $\delta = \min(r_1, r_2 \ldots r_n)$, then $\delta > 0$ and constract $S(p, \delta)$.

It is clear that,

$S(p, \delta) \subseteq S(p, r_i)$ for every $i = 1, 2, \ldots n$.

[proved].

* Closed set in Metric space:

Let (X, d) be a metric space. Let $U \in X$. Then U is said to be closed set if every limit point of U is a point of U.

* Theorem:

① The intersection of an arbitrary family of closed set is closed.

② The ~~intersection~~ union of a finite number of closed set is closed.

Proof:

① Let $\{A_i\}$ ~~$A_1, A_2 ---- A_n$~~ be an arbitrary family of closed sets. We want to show that,

$\cap A_i$ ~~$= A_1 \cap A_2 \cap ---- \cap A_n$~~ is closed.

Consider the complement of $\cap A_i$ is $(\cap A_i)^c$. We aim to show that $(\cap A_i)^c$ is open.

Since, A_i is closed for each i, $(A_i)^c$ is open for each i.

Now, we know that,

$(\cap A_i)^c = \cup A_i^c$ [De-Morgan's law]

This union of open sets is itself open. Therefore, $\cap A_i$ is closed.

[proved]

⑪ Proof:

Let $\{A_1, A_2, ---- A_n\}$ be a finite collection of closed sets. We want to show that, $\bigcup_{i=1}^{n} A_i$ is closed.

Consider the complement of $\bigcup_{i=1}^{n} A_i$, denoted by $(\bigcup_{i=1}^{n} A_i)^c$. We aim to show that $(\bigcup_{i=1}^{n} A_i)^c$ is open.

Now, by De-Morgan's Law

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c.$$

Since, each A_i is closed, A_i^c is ~~closed.~~ open for each i. Now, the Intersection of open sets is itself open. Therefore, $(\bigcup_{i=1}^{n} A_i)^c$ is open, which implies that $\bigcup_{i=1}^{n} A_i$ is closed.

[proved]

Lecture-3:

* Interior point in metric space:

In a metric space (X,d), a point p is an interior point of a set A if there exists a positive real number $r>0$ such that the open sphere $S(p,r) = \{x \mid d(x,p) < r\}$ is entirely contained within A. (i.e, $S(p,r) \subseteq A$).

* Exterior point in metric space:

In a metric space (X,d), a point p is an exterior point of a set A if there exists a positive real number $r>0$ such that the open sphere $S(p,r) = \{x \mid d(x,p) < r\}$ is entirely disjoint from A. That is $S(p,r) \cap A = \varphi$.

* Boundary point in metric space:

In a metric space (X,d), a point p is an boundary point of a set A if p is neither the interior point nor the exterior point of A.

* Dense set:

A set D is dense in a metric space (X,d) if the closure of D is the whole space X. That is, $\bar{D} = X$. Where $\bar{D}$ denotes the closure of D.

Example:

consider the rational number $(\mathbb{Q})$ as the subset of the real numbers $(\mathbb{R})$. In the metric space of real numbers, the set of rational number is dense. (i.e, $\bar{\mathbb{Q}} = \mathbb{R}$).

* A set is dense in a topological space if its closure is the entire space.

* **Nowhere dense set:**

A Nowhere dense set in a topological space is a set whose closure has an empty interior.

In metric space, denotes the set which does not contain any open spheres/open balls.

Ex: The set of irrational numbers between 1 and 2 on the real number line.

* **Dense in itself:**

A set is dense in itself if every point in the set is a limit point of the set.

For example, the set of all real numbers is dense in itself.

* **Perfect set:**

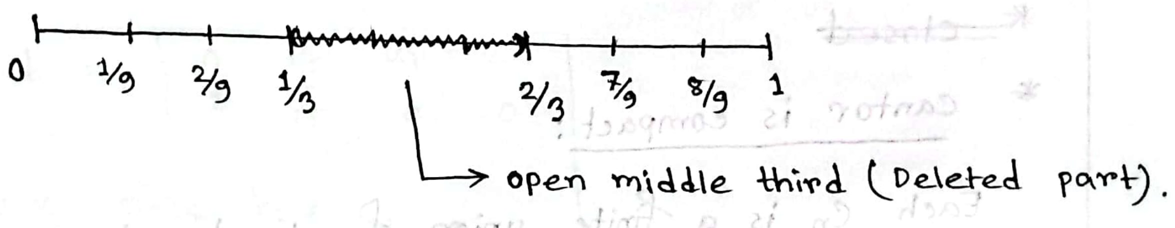
A perfect set in topology is a closed set with no isolated points, meaning every point is a limit of the set.

For example, Cantor set is a perfect set.

Cantor set

* cantor set :

The cantor set is a set of points lying on a single line segment that has a number of weired and surprising properties.



Remove the open middle third segment $(\frac{1}{3}, \frac{2}{3})$ and let

$$C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$

Now removing the open third segment in each part, let

$$C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$

Continuing in this way,

- - - - - - - -
- - - - - - - -

We get, $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots\cdots \supseteq C_n$; for each $n \in \mathbb{N}$.

C_k is the union of 2^k closed intervals and each length is 3^{-k} or $1/3^k$.

$\therefore$ $C = \bigcap_{i=1}^{\infty} C_i$, then C is called the cantor set.

* Properties

Cantor set is a ① perfect set.
② compact set
③ Uncountable set
④ closed.

* ~~closed~~

* Cantor is compact:

Each C_n is a finite union of closed sets. so, C_n is closed for all n.

Then $C = \cap C_n$ is also closed. Also C is bounded, since $C \subseteq [0,1]$. so by Heine Borel theorem C is compact.

* Cantor is perfect ~~and uncountable~~:

~~We have~~ To prove that cantor is perfect, we have to show that it is dense-in itself. Let $x \in X$, then

$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, where a_n is easier 0 or 2 be the ternary any expression of x. ~~যেন কোন মান আছে~~ we shall show that x is a limit point of C. choose the sequence $\langle x_n \rangle$ in C such that

$x_1 = a_1', a_2, a_3, \ldots, a_n$

$x_2 = a_1, a_2', a_3, \ldots, a_n$

$\vdots$

$x_n = a_1, a_2, a_3, \ldots, a_n'$

where $a_n' = 0$ if $a_n = 2$ and $a_n' \leq 2$ if $a_n = 0$.

Therefore, $\lim_{n \to \infty} x_n = x$, then every point of the cantor set is a limit point of the set and so it is dense-in itself.

Since, cantor set is closed and dense in itself, so it is perfect set.

Where, $b_i = \begin{cases} 0 & \text{if } a_{ii} = 2 \\ 2 & \text{if } a_{ii} = 0 \end{cases}$

Thus $y \neq x_1$, since $b_1 \neq a_{11}$,

$y \neq x_2$, since $b_2 \neq a_{22}$. and so on.

Since $b_i \in \{0, 2\}$ for each i, and $y \in C$.

Therefore C is uncountable.

* cantor is closed :

The cantor set C is the intersection of the family of sets $\{C_i : i \in \mathbb{N}\}$ where,

$C_1 = [0, 1/3] \cup [\frac{2}{3}, 1]$

$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

Each C_n is the union of 2^n closed intervals. we know that the union of a finite number of closed set is closed.

But $C = \bigcap_{i=1}^{\infty} C_i$

We know that, the intersection of closed set is closed.

Hence, cantor is closed.

* Cantor is uncountable :

Suppose C is countable and list its elements as

$C = \{x_1, x_2 ---- x_n\}$

Now we can write,

$x_1 = 0, a_{11}, a_{12}$ -------

$x_2 = 0, a_{21}, a_{22}$ -----

$\vdots$

$x_k = 0, a_{k1}, a_{k2}$ -----.

Where $a_{ij} = 0$ or 2 $\forall$ i, j

let, $y = 0, b_1, b_2$ -----

Where, $b_i = \begin{cases} 0 & \text{if } a_{ii} = 2 \\ 2 & \text{if } a_{ii} = 0 \end{cases}$

Thus $y \neq x_1$, since $b_1 \neq a_{11}$

$y \neq x_2$, since $b_2 \neq a_{22}$ and so on.

Since, $b_i = \{0, 2\}$ for each i, and $y \in C$.

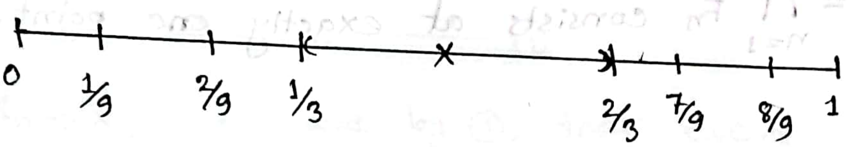
Therefore C is uncountable.

[proved]

* Length of cantor set is zero.

<u>proof:</u>

Let's consider the construction of the cantor set on the interval $[0,1]$.



Remove the open third segment $(1/3, 2/3)$ and let

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

Now, Remove the open third segment in each part, let

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

Continue in this way, In each iteration, remove the middle third of the remaining intervals.

The length of the cantor set is the sum of the lengths of the intervals that are ~~removed~~ not(?) at each step.

The length of the intervals removed in each step is $1/3^n$, Where n is the iteration number.

So, the length of the cantor set L is given by the infinite series,

$$L = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.$$

This is a geometric series with a common ratio less than 1. Thus the length of the cantor set is zero. [Proved]

*** Cantor intersection theorem :

Statement:

Let (X,d) is a complete metric space and let (F_n) be a nested sequence of non-empty closed subsets of X such that $diam(F_n) \to 0$ or $d(F_n) \to 0$ as $n \to \infty$.

Then, $F = \bigcap_{n=1}^{\infty} F_n$ consists at exactly one point.

Proof:

Since, $F_n \neq \Phi$, for each $n \in \mathbb{N}$

we can choose a sequence of points (x_n) such that, $x_n \in F_n$ [for $n = 1, 2 \ldots n$].

We need to show that (x_n) is a cauchy sequence in X.

As (F_n) is a decreasing sequence.

So, $F_{n+1} \subseteq F_n \; \forall n$.

Therefore x_n, x_{n+1} all lie in F_n.

Since $d(F_n) \to 0$ as $n \to \infty$, for every $\varepsilon > 0$, there exists a positive n_0 such that $d(F_{n_0}) < \varepsilon \; \forall n \geq n_0$.

Again, since (F_n) is nested (decreasing), we have

$m, n \geq n_0 \Rightarrow F_m, F_n \subset F_{n_0}$

$\Rightarrow x_m, x_n \in F_{n_0}$

$\Rightarrow d(x_m, x_n) \in \varepsilon$.

Thus (x_n) is a cauchy sequence.

As X is complete, so there exists a point $x \in X$, such that, $\lim_{n \to \infty} x_n = x$. $\Rightarrow$ $x_n \to x$ as $n \to \infty$.

Assume that, $x \in \bigcap_{n=1}^{\infty} F_n$.

To prove this, let m be any positive integer such that,

$n > m \Rightarrow x_n \in F_n$ ——— (1)

Since $x_n \to x$, we have by (1), that every nbd of x is contain an infinite number of points of F_m. Thus x has a limit point of F_m.

But we have, F_m is closed. so, $x \in F_m$. Since m is arbitrary, we have $x \in \bigcap_{n=1}^{\infty} F_n$.

Now we want to show that, $x \in \bigcap_{n=1}^{\infty} F_n$ is unique.

If possible let, there is another point $y \in \bigcap_{n=1}^{\infty} F_n$.

Then $d(x,y) \subseteq d(F_n)$, for every n. (by defn of diameter).

$\Rightarrow$ $d(x,y) \geq 0$ is always true

Hence $d(x,y) = 0$.

$\Rightarrow |x-y| = 0$

$\Rightarrow x = y$.

and so $\bigcap_{n=1}^{\infty} F_n = \{0\}$. Thus $\bigcap_{n=1}^{\infty} F_n$ consists at exactly one point.

[proved].

Lecture-04:

* power series:

Given a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, the series $\sum_{n=0}^{\infty} a_n x^n$ is called the power series.

Given any sequence (a_n), one of the following holds for its power series:

1. The series converges for all $x \in \mathbb{R}$.
2. The series converges for $x = 0$.
3. The series converges for all x in some bounded interval, centered at series the interval may be ~~interval~~. open or half open as closed.

* pointwise converges:

Let (f_n) be a sequence of real valued function, defined on a set $S \subseteq \mathbb{R}$, The sequences (f_n) converges pointwise [i.e, at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\ x \in S.$$

We often write [redacted] $\lim f_n = f_n$ pointwise on S as $f_n \to f$ pointwise on S.

Symobolically, $f_n \to f$ pointwise on S means exactly the following:

for $\varepsilon > 0$ and $x \in S$ $\exists$ N such that $|f_n(x) - f(x)| < \varepsilon$ for $n \in \mathbf{N}$.

Note that the value of N depends on both $\varepsilon > 0$ and x in S.

* uniformly converges:

Let (f_n) be a sequence of real valued function defined on a set $S \subseteq \mathbb{R}$.

The sequence (f_n) converges uniformly on S to a function f is defined on S if,

for each $\varepsilon > 0$ $\exists$ an number N such that,

$|f_n(x) - f(x)| < \varepsilon$ $\forall$ $x \in S$ and all $n > N$.

pointwise converges:

pointwise convergence for a sequence $f_n(x)$ to a function $f(x)$ on a set S is defined as:

The sequence of real numbers $f_n(x)$ converges to $f(x)$ as n approaches infinity for all $x \in S$.

symobilically,

for each x in S,

$$\lim_{n \to \infty} f_n(x) = f(x).$$

uniformly convergence :

A sequence of functions $f_n(x)$ defined on a set S converges uniformly to a function f on S if, for every $\varepsilon > 0$, there exists a ~~set~~ positive integer N such that for all $n \geq N$ and for all $x \in S$,

$$|f_n(x) - f(x)| < \varepsilon.$$

* Difference between P.C. and U.C.

① f_n converges pointwise to f on S if for all $x \in S$ and $\varepsilon > 0$, $\exists$ N such that for all $n \geq N$, we have

$|f_n(x) - f(x)| < \varepsilon$. (N depends on both x and ε)

f_n converges uniformly to f on S if for all $\varepsilon > 0$, $\exists$ N such that for all $n \geq N$, we have

$|f_n(x) - f(x)| < \varepsilon$. (N depends only from ε).

② Example:

Consider the sequence of functions $\{f_n\}$ defined on the interval $[0,1]$ by $f_n(x) = x^n$.

pointwise:

For each fixed x in $[0,1)$.

if $x = 0$ $f_n(x) = f_n(0) = 0^n = 0$

if $x = 1$, $f_n(x) = f_n(1) = 1^n = 1$

if $0 < x < 1$, $f_n(x) \to 0$ as $n \to \infty$.

Thus, $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$.

So, f_n converges pointwise to f.

Uniformly:

$f_n(x)$ does not converge uniformly to f on $[0,1]$.

because the convergence is not uniform as x approaches to 1. For any $\varepsilon < 1$, no matter how Large N is chosen, there will be an x close to 1, where $|f_n(x) - f(x)| > \varepsilon$.

Thus, $f_n(x)$ does not converge uniformly.

Example:

If $f_n(x) = \dfrac{x}{1+nx^2}$ for $x \in \mathbb{R}$.

Then the sequence (f_n) converges to '0' uniformly on $\mathbb{R}$.

Solution:

Here, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x}{1+nx^2} = 0$.

So, $f_n \to 0$ pointwise on $\mathbb{R}$.

Now, given $f_n(x) = \dfrac{x}{1+nx^2}$

$$\Rightarrow f_n'(x) = \frac{1-2nx^2+nx^2}{(1+nx^2)^2}$$

Now, $f_n'(x) = 0$

$$\Rightarrow 1-2nx^2+nx^2 = 0$$

$$\Rightarrow 1-nx^2 = 0$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{n}}$$

$$\therefore x = \pm \frac{1}{\sqrt{n}},$$

$\Rightarrow x = \dfrac{1}{\sqrt{n}} \rightarrow$ ~~supremum~~ maximum

$\Rightarrow x = -\dfrac{1}{\sqrt{n}} \rightarrow$ minimum.

Now, f_n is taken by its max value at $x = \dfrac{1}{\sqrt{n}}$ and min value at $x = -\dfrac{1}{\sqrt{n}}$.

Since, $f_n\left(\dfrac{1}{\sqrt{n}}\right) = \dfrac{1}{2\sqrt{n}}$, we conclude that

$$\lim_{n\to\infty} \sup\{ |f_n(x) - 0| : x \in \mathbb{R} \}$$

$$= \lim_{n\to\infty} \frac{1}{2\sqrt{n}}$$

$$= 0.$$

$\therefore f_n$ converges to '0' uniformly on $\mathbb{R}$.

Example:

~~Let (f_n) be a sequence of continious function on $[a,b]$ and suppose that $f_n \to f$ uniformly on $[a,b]$~~. Then

$\lim_{n\to\infty}$

Example:

Show that, the sequence (f_n) where $f_n(x) = \tan^{-1} nx$; $x \geq 0$ is uniformly convergent in any interval $[a,b]$, $a>0$ but only pointwise convergent in $[0,b]$.

Solution:

Here, $f_n(x) = \tan^{-1} nx \; ; \; x \geq 0$

$$\therefore f(x) = \lim_{n\to\infty} \tan^{-1} nx \; ; \; x > 0$$

$$= \tan^{-1}(\infty)$$

$$= \pi/2$$

and $f(x) = \cancel{\tan^{-1}} \lim_{n\to\infty} \tan^{-1} nx \; ; \; x = 0$

$$= \tan^{-1}(0)$$

$$= 0.$$

$$\therefore f(x) = \begin{cases} \pi/2 & ; \text{if } x > 0 \\ 0 & ; \text{if } x = 0 \end{cases}$$

For $x > 0$ $\; |f_n(x) - f(x)| = |\tan^{-1} nx - \pi/2|$

$$= |\tan^{-1} nx - \tan^{-1} nx - \cot^{-1} nx|$$

$$= \cot^{-1} nx < \varepsilon.$$

$\therefore \cot^{-1} nx < \varepsilon$

if, $nx > \cot \varepsilon$

$\Rightarrow n > \frac{\cot \varepsilon}{x}$

Now choose a positive integer $N > \frac{\cot \varepsilon}{x}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \ \forall \ n \geq N$$

Thus, $< f_n(x) >$ is uniformly converges on $[a,b]$, $a>0$

But, $x \to 0$, $\frac{\cot \varepsilon}{x} \to \infty$, so that it is not possible to choose a positive integer N such that,

$$|f_n(x) - f(x)| < \varepsilon \ \forall \ n \geq N$$

Hence, $< f_n(x) >$ is pointwise converges on $[0,b]$

[Showed]

Theorem:

Let (f_n) be a sequence of continious function on $[a,b]$ and suppose that $f_n \to f$ uniformly on $[a,b]$. Then,

$$\lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

proof:

Since, (f_n) is a sequence of continious function converges to f, f is also continious and so the function

$(f_n - f)$ are integrable on $[a,b]$.

Let $\varepsilon > 0$ be given, and since $f_n \to f$ uniformly ~~continious~~ on $[a,b]$ $\exists$ an integer N such that,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall\ x \in [a,b] \text{ and } \forall\ n \geq N.$$

consequently, $n \geq N$ implies,

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$= \left| \int_a^b [f_n(x) - f(x)]\,dx \right|$$

$$\leq \int_a^b [f_n(x) - f(x)]\,dx$$

$$< \int_a^b \frac{\varepsilon}{b-a}\,dx$$

$$= \varepsilon.$$

Thus for $\varepsilon > 0$ $\exists$ an integer N such that

$$\boxed{\cancel{\int_a^b f_n(x) - f(x)}}$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon \text{ and}$$

hence, $\displaystyle\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

[proved]

Lecture-05:

* Series of function:

Definition:

A series of functions is an expression $\sum_{k=0}^{\infty} g_k$ which makes sense provided the sequence of partial sum $\sum_{k=0}^{\infty}$ $\sum_{k=0}^{n} g_k(x)$ converges or diverges in $-\infty$ to ∞ pointwise.

Example:

[illegible]

Example:

① $\sum \frac{x^k}{1+x^k}$ is a series of functions but not power series.

② $\sum a^n x^n$ is a power series and also a series of functions.

* Differentiation and integration of power series:

Theorem:

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$. If $0<R_1<R$ then the power series converges uniformly on $[-R_1, R_1]$ to a continious function.

Proof:

consider $0 < R_1 < R$,
The series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of converges. Since $|R_1| < R$, we have $\sum |a_n| R_1^n < \infty$ and so $|a_n x^n| \leq |a_n| R_1^n \;\forall\; x \in [-R_1, R_1]$

By ~~Wer~~ Weirstrass M-test, the series $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$.

Now each partial sum,

$f_n = \sum_{k=0}^{n} a_k x^k$ is continious on $[-R_1, R_1]$ and $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$ and the limit function is continious on $[-R_1, R_1]$.

* Weirstrass M-test:

Let (M_k) be a sequence of non-negative real number such that, $\sum M_k < \infty$. If $|g_k(x)| \leq M_k \;\forall\; x \in \Delta g_k$ (Domain of g_k). Then $\sum g_k$ converges uniformly on Δg_k.

* Lemma: (অনুসিদ্ধান্ত)

If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergen R, then the power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ also have radius of convergence R.

Proof:

The series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} n a_n x^n$ have the same radius of convergence, because they converge for exactly the same values of x.

Similarly, the series $\sum \frac{a_n}{n+1} x^{n+1}$ and $\sum \frac{a^n}{n+1} x^n$ have the same radius of convergence.

We have, $R = \frac{1}{\beta}$ where

$\beta = \lim \sup (a_n)^{1/n}$

Now for the series $\sum n a_n x^n$,

$$\lim \sup (n a_n)^{1/n} = \lim \sup (n)^{1/n} . (a_n)^{1/n}$$

$$= \lim \sup (a_n)^{1/n} \quad [\text{ since, } \lim n^{1/n} = 1]$$

$$= \beta.$$

Hence the radius of convergence of these series is R.

Similarly, for the series $\sum \frac{a_n}{n+1} x^n$

Now, $\lim \sup \left(\frac{a_n}{n+1}\right)^{1/n} = \lim \sup \left(\frac{1}{n+1}\right)^{1/n} . (a_n)^{1/n}$

$$= \lim \sup (a_n)^{1/n} \quad \left[\because \lim \left(\frac{1}{n+1}\right)^{1/n} = 1\right]$$

$$= \beta$$

Hence the radius of convergence of $\sum \frac{a_n}{n+1} x^n$ is R.

* **Theorem:**

Suppose that, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R>0$, then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ for } |x|<R.$$

proof:

Fix x and assume $x<0$, on the interval $[x,0]$. the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ converges uniformly to $f(t)$. consequently we have,

$$\int_x^0 f(t)\,dt = \lim_{n\to\infty} \int_x^0 a_k t^k\,dt$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \int_x^0 a_k t^k\,dt$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left[\frac{t^{k+1}}{k+1}\right]_x^0$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \left(-\frac{x^{k+1}}{k+1}\right)$$

$$\Rightarrow -\int_x^0 f(t)\,dt = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}$$

$$\therefore \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

(proved).

Theorem:

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R>0$, then f is differentiable on $(-R,R)$ and

$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$.

proof:

Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

The series $\sum_{n=0}^{\infty} a_n x^n$ converges ~~to R~~ for $|x|<R$ and so the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges for $|x|<R$.

Since, we can integrate a series term-by-term, we have

$$\int_0^x g(t)\,dt = \sum_{n=1}^{\infty} a_n x^n$$

$$\Rightarrow \int_0^x g(t)\,dt = \sum_{n=0}^{\infty} a_n x^n - a_0$$

$$\Rightarrow \int_0^x g(t)\,dt = f(x) - a_0$$

$$\Rightarrow f(x) = \int_0^x g(t)\,dt + a_0 \quad \text{for } |x|<R$$

Thus if $0<R_1<R$, then

$$f(x) = \int_0^{-R_1} g(t)\,dt + \int_{-R_1}^{x} g(t)\,dt + a_0, \quad \text{for } |x|<R_1$$

$$= \int_{-R_1}^{x} g(t)\,dt + k \quad ; \left[\text{where } k = -\int_{-R_1}^{0} g(t)\,dt + a_0\right]$$

Since, g is continious by fundamental theorem of calculus (II), shows that f is differentiable and, $f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x|<R$.

Sequence

* Sequence in metric space:

A sequence in a metric space (X, d) is a function $f: \mathbb{N} \to X$, denoted by (x_n), where x_n represent the nth term of the sequence.

Convergence of the sequence (x_n) to a limit L means that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t. for all $n \geq N$, $d(x_n, L) < \varepsilon$.

* Cauchy sequence:

A sequence (x_n) ~~is~~ in a metric space (X, d) is called ~~an~~ a cauchy-sequence if it satisfy the following properties:

For $\varepsilon > 0$, there is an integer N such that $d(x_m, x_n) < \varepsilon$ whenever $n \geq N$ and $m \geq N$.

* Complete metric space:

A complete metric space is a metric space in which every cauchy sequence is convergent. That is a metric space (X, d) is said to be complete if every cauchy sequence is in X.

A subset S of X is called complete if the metric sub space (S, d) is complete.

Theorem:

A convergent sequence in metric space has a unique limit.

proof:

If possible, let $\lim_{n\to\infty} x_n = x$

and $\lim_{n\to\infty} x_n = x'$

Then for $\varepsilon>0$ $\exists$ $m_1, m_2 \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon/2 \quad \forall \quad n \geq m_1 \text{ ——— (i)}$$

$$d(x_n, x') < \varepsilon/2 \quad \forall \quad n \geq m_2 \text{ ——— (ii)}$$

Let $m = \max\{m_1, m_2\}$

$\therefore$ $d(x_n, x) < \varepsilon/2 \ \forall \ n \geq m$

$d(x_n, x') < \varepsilon/2 \ \forall \ n \geq m$

Now

$$\begin{aligned} d(x, x') &\leq d(x, x_n) + d(x_n, x') \\ &= d(x_n, x) + d(x_n, x') \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Since, $\varepsilon>0$ is an arbitrary small.

Then, ~~If possible~~ It is possible iff $d(x, x') = 0$

$\therefore$ $x = x'$

$\therefore$ The limit is unique (proved).

Theorem:

Every convergence sequence is a cauchy sequence but the ~~converg~~ converse of this statement need not to be true.

proof:

Let (x, d) be a metric space and (x_n) be a convergent sequence and it is converges to a point $x_0 \in X$, if for any $\varepsilon > 0$, $\exists$ a natural number $n_0 > 0$ such that,

$$d(x_n, x_0) < \varepsilon/2 \quad \forall \; n \geq n_0 \longrightarrow (i)$$

Again if $m \geq n_0$, then

$$d(x_m, x_0) < \varepsilon/2 \quad \forall \; m \geq n_0 \longrightarrow (ii)$$

By triangular inequalities; we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_0) + d(x_0, x_m) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

$\therefore \; d(x_n, x_m) < \varepsilon \quad \forall \; n, m \geq n_0.$

Hence, every convergence sequence is a cauchy-sequence.

Converse part :

Let (x, d) be a metric space where $X = (0, 1]$ and d is a usual metric space.

$\therefore d(x,y) = |x-y|$

Let $\{x_n\} = \langle \frac{1}{n} \rangle$ be a sequence in a metric space (x, d)

$\therefore \frac{1}{n} \to 0$ as $n \to \infty$.

But $0 \notin X$.

$\therefore$ The sequence $\langle \frac{1}{n} \rangle$ is not convergent.

Now, $d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right|$

$$\leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$< \varepsilon, \ \forall \ n, m \geq \frac{2}{\varepsilon}$$

$\therefore d(x_n, x_m) < \varepsilon, \ \forall \ n, m \geq \frac{2}{\varepsilon}$

$\therefore$ The sequence $\frac{1}{n}$ is a Cauchy sequence.

[proved]

~~Hence~~

* Differentiation in $\mathbb{R}^n$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, if there is a linear transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^m$, such that,

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

or, $$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0$$

* Theorem:

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, there is a unique linear transformation $\lambda: \mathbb{R}^n \to \mathbb{R}^m$ such that,

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0$$

proof:

Assume that, $\mu: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation satisfies

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{h} = 0$$

If $d(h) = f(a+h) - f(a)$

then,

$$\lim_{h\to 0} \frac{\|\lambda(h) - \mu(h)\|}{\|h\|}$$

$$\leq \lim_{h\to 0} \frac{\|\lambda(h) - d(h) + d(h) - \mu(h)\|}{\|h\|}$$

$$\leq \lim_{h\to 0} \frac{\|\lambda(h) - d(h)\|}{\|h\|} + \lim_{h\to 0} \frac{\|d(h) - \mu(h)\|}{\|h\|}$$

$= 0$.

If $x \in \mathbb{R}^n$, then $tx \to 0$ as $t \to 0$

Hence for $x \neq 0$, we have

$$\lim_{t\to 0} \frac{\lambda(tx) - \mu(tx)}{\|tx\|} = 0.$$

or, $$\frac{\|\lambda(tx) - \mu(tx)\|}{\|tx\|} = 0$$

Therefore, $\lambda(x) - \mu(x) = 0 \;\; \forall\, x \in \mathbb{R}^n$

$\Rightarrow \lambda(x) = \mu(x)$

$\therefore \lambda = \mu$. [proved]

* Mean value theorem ($F: \mathbb{R}^n \to \mathbb{R}$):

Statement:

Let f be defined on an open subset A of $\mathbb{R}^n$ and have values in $\mathbb{R}$. Suppose that the set A contain the point a, b and the line segment S joining and that f is differentiable at every point of this segment. Then, there exist a point c on S such that,

$$f(b)-f(a) = \Delta f(c)(b-a).$$

<u>proof</u>:

Let $\phi: \mathbb{R} \to \mathbb{R}^n$ be defined by $\cancel{\phi(t) = (1-t)+tb}$

$$\phi(t) = (1-t)a + tb \quad \text{for } t \in [0,1].$$

So that, $\phi(0) = a$ and $\phi(1) = b$ and $\phi(t) \in S \subseteq A$; for $t\,(0,1)$.

Since, A is open and ϕ is continious, there is a number >0, such that ϕ maps the interval $(-\gamma, 1+\gamma)$ into A.

Now, let $F: (-\gamma, 1+\gamma) \to \mathbb{R}$ be defined by

$$F(t) = (f \circ \phi)(t) = f(\phi(t)) = f\{(1-t)a + tb\}$$

By ~~claim~~ chain rule, it follows that,

$$F'(t) = \Delta f(\phi(t))(b-a).$$

Now, apply mean value theorem, there exist $\phi: c\,(0,1)$ such that,

$$F(1) - F(0) = F'(t_0)$$

let $c = \phi(t_0) \in S$, we obtain

$$f(b) - f(a) = F'(t_0) = \Delta f(\phi(t_0))(b-a)$$

$$\therefore f(b) - f(a) = \Delta f(c)(b-a)$$

[Proved]

* Implicit function theorem:

Statement:

Suppose $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continious ~~containing~~ differentiable in an open set containing (a,b) and $f(a,b)=0$.

let M be the $m \times n$ matrix $(D_{n+j}\, f_i(a,b))$, $1 \le (i,j) \le m$

If $\det M \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing 'a' and an open set $B \subset \mathbb{R}^m$ containing 'b' with the following property:

For each $x \in A$, there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$

the function g is differentiable.

Proof:

Let us define,

$F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$F(x,y) = (x, f(x,y))$.

Then $\det F'(a,b) = \det M \neq 0$

By inverse function theorem, there is an open set $W \in \mathbb{R}^n \times \mathbb{R}^n$ containing $F(a,b) = (a,0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^n$ containing (a,b) which we may ~~have~~ take to be of the for $A \times B$ such that $F: A \times B \to W$ has a differentiable inverse $h: W \to A \times B$.

clearly h is of the form,

$h(x,y) = (x, K(x,y))$ for some dif. function K.

let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be defined by,

$\pi(x,y) = y$, then

$$(\pi \circ F)(x,y) = \pi(F(x,y))$$
$$= \pi(x, f(x,y))$$
$$= f(x,y)$$

$\therefore \pi \circ F = f$.

Therefore, $f(x, K(x,y)) = f(h, (x,y))$

$$= ((\pi \circ F) \circ h)(x,y)$$
$$= (\pi \circ (F \circ h))(x,y)$$
$$= \pi(x,y)$$

$\therefore f(x, K(x,y)) = y$

Thus, $f(x, K(x,y)) = 0$, that is the differentiable function g can be defined as

$g(x) = K(x,0)$ ~~such~~ so that

$$(f(x), g(x)) = 0$$

[proved]

* problem / Exercise :

* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \left\{ xy \frac{x^2-y^2}{x^2+y^2} \right. , \quad (x,y) \neq 0 .$$

Show that,

~~(1) $D_2 (f(x,y)$~~

(1) $D_2 f(x,0) = x \quad \forall x$.

(2) $D_1 f(0,y) = -y \quad \forall y$.

(3) $D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$

Solution

(1)
$$D_2 f(x,0) = \lim_{k \to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{k \to 0} \frac{1}{k} \left[xk \frac{x^2-k^2}{x^2+k^2} - 0 \right]$$

$$= \lim_{k \to 0} x \cdot \frac{x^2-k^2}{x^2+k^2}$$

$$= x \quad \forall x .$$

(2)
$$D_1 f(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[hy \cdot \frac{h^2-y^2}{h^2+y^2} - 0 \right]$$

$$= \lim_{h \to 0} y \cdot \frac{h^2-y^2}{h^2+y^2}$$

$$= -y \quad \forall y .$$

③ $D_{1,2} f(0,0) = D_2 \left(D_1 f(0,0) \right)$

$$= \lim_{k \to 0} \frac{D_1 f(0,k) - D_1 f(0,0)}{k}$$

$$= \lim_{k \to 0} \left(\frac{-k}{k} \right) \quad [\text{by } (ii)]$$

$$= -1$$

and $D_{2,1} f(0,0) = D_1 \left(D_2 f(0,0) \right)$

$$= \lim_{h \to 0} \frac{D_2 f(h,0) - D_2 f(0,0)}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h} \right) \quad [\text{by } (ii)]$$

$$= 1$$

$\therefore D_{1,2} f(0,0) \neq D_{2,1} f(0,0)$ [proved]

* Integration in $\mathbb{R}^n$:

* **Theorem:**

Let S be a rectangle in $\mathbb{R}^n$ and let $f: S \to \mathbb{R}$ be a continious function then f is integrable.

proof:

To prove that theorem, we claim that for any $\varepsilon > 0$, there is a partition P of S so that

$$\cancel{U(P,f)} \quad U(f,P) - L(f,P) < \varepsilon.$$

Suppose that, $\varepsilon > 0$, since S is compact and f is continious, it follows that f is uniformly continious.

Then, there is a $\delta > 0$ such that

$$|f(u) - f(w)| < \frac{\varepsilon}{\mu(S)} \quad \text{whenever } \|u - w\| < \delta$$

Let P be a partition of S with mesh $P < \delta$. we denote the sub rectangle of S by $S_1, S_2, \dots, S_K$ and define number M_i and m_i, $1 \le i \le K$ by

$$M_i = \text{lub}\{f(u) : u \in S_i\}$$

$$m_i = \text{glb}\{f(u) : v \in S_i\}$$

Then
$$U(f,P) = \sum_{i=1}^{K} M_i \mu(S_i)$$

$$L(f,P) = \sum_{i=1}^{K} m_i \mu(S_i)$$

Now, f is continious and each S_1 is compact. so, we can find points u_i, w_i with $M_i = f(u_i)$, and $m_i = f(w_i)$; $1 \le i \le k$.

Furthermore, since mesh $P < \delta$ we have, $\|u_i - w_i\| \le \delta$, $1 \le i \le k$

It follows that,

$$M_i - m_i = f(u_i) - f(w_i) < \frac{\varepsilon}{\mu(S)}$$

Then $U(f,P) - L(f,P)$

$$= \sum_{i=1}^{k} (M_i - m_i)\,\mu(S_i)$$

$$< \sum_{i=1}^{k} \frac{\varepsilon}{\mu(S)}\,\mu(S_i)$$

$$= \frac{\varepsilon}{\mu(S)} \sum_{i=1}^{k} \mu(S_i)$$

$$= \frac{\varepsilon}{\mu(S)} \cdot \mu(S)$$

$$= \varepsilon$$ [proved].

$\therefore U(f,P) - L(f,P) < \varepsilon$

*** Abel's theorem : (আবালী 😐).

Statement:

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of convergence R. If the series converges at $x=R$, then f is continious at $x=R$. If the series converges at $x=-R$, then f is continious at $x=-R$.

proof:

case-I

To prove the theorem first consider the case-I, that the radius of convergence of the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is 1 and the series converges at $x=1$. we need to prove that $f(x)$ is continious at $x=1$.

Let, $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and

$$s_n = \sum_{k=0}^{n} a_k = f_n(1), \quad \text{for } n = 0, 1, 2, \ldots$$

and let $s = \sum_{k=0}^{\infty} a_k = f(1)$, so that $\lim s_n = s$.

For $0<x<1$, we have

$$f_n(x) = \sum_{k=0}^{n} a_k x^k$$

$$= a_0 + \sum_{k=1}^{n} a_k x^k$$

$$= s_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k$$

$$= s_0 + \sum_{k=1}^{n} s_k x^k - x \sum_{k=1}^{n} s_{k-1} x^{k-1}$$

$$= s_0 + s_n x^n + \sum_{k=1}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k$$

$$= s_0 + s_n x^n + \sum_{k=1}^{n-1} s_k x^k - x s_0 - x \sum_{k=1}^{n-1} s_k x^k$$

$$= (1-x) s_0 + \sum_{k=1}^{n-1} (1-x) s_k x^k + s_n x^n$$

$$= \sum_{k=0}^{n-1} (1-x) s_k x^k + s_n x^n$$

Let us now take the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n-1} (1-x) s_k x^k + \lim_{n\to\infty} s_n x^n$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} (1-x) s_n x^n + 0$$

Thus, $f(x) = \sum_{n=0}^{\infty} (1-x) s_n x^n$ for $0<x<1$.

Again, since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$\Rightarrow \sum_{n=0}^{\infty} (1-x) x^n = 1$$

we also have, $f(1) = \cancel{f(x) =} \sum_{n=0}^{\infty} a_n = s = \sum_{n=0}^{\infty} s(1-x) x^n$

Hence, we have

$$f(1) - f(x) = \sum_{n=0}^{\infty} s(1-x) x^n - \sum_{n=0}^{\infty} (1-x) s_n x^n$$

$$= \sum_{n=0}^{\infty} (s - s_n)(1-x) x^n \qquad \text{——— (1)}$$

Now let $\varepsilon > 0$ be given,

Since, $\lim s_n = s$, there exists N such that

$$\Rightarrow |s - s_n| < \frac{\varepsilon}{2}$$

Let $g_N(x) = \sum_{n=0}^{N} |s - s_n| (1-x) x^n$

from (1), we have

$$|f(1) - f(x)| = \sum_{n=0}^{N} |s - s_n| (1-x) x^n + \sum_{n=N+1}^{\infty} |s - s_n| (1-x) x^n$$

$$= g_N(x) + \sum_{n=N+1}^{\infty} |s - s_n| (1-x) x^n$$

$$< g_N(x) + \sum_{n=N+1}^{\infty} \frac{\varepsilon}{2} (1-x) x^n$$

$$< g_N(x) + \frac{\varepsilon}{2} \qquad \text{——— (2)} \quad \text{for } 0 < x < 1.$$

The function g_N is continious and $g_N(1) = 0$. and hence for $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$1-\delta < x < 1 \Rightarrow g_N(x) < \varepsilon/2$$

Then from ②, we obtain

$$1-\delta < x < 1 \Rightarrow |f(1) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

which shows that f is continious at $x=1$.

Case-II

Suppose, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0<R<\infty$ and that the series converges at $x=R$.

Let $g(x) = f(Rx)$

$$\Rightarrow g(x) = \sum_{n=0}^{\infty} a_n R^n x^n \text{ for } |x|<1.$$

This series has radius of convergence 1 and it converges at $x=1$.

Hence, g is continious at $x=1$.

Since $f(x) = g\left(\frac{x}{R}\right)$ it follows that f is continious at $x=R$.

case-III:

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, $0 < R < \infty$ and that the series converges at $x = -R$.

~~The series for h converges at x=R~~

Let $h(x) = f(-x)$, then $h(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n$

The series for h converges at $x = R$ and so h is continious at $x = R$.

Hence it follows that,

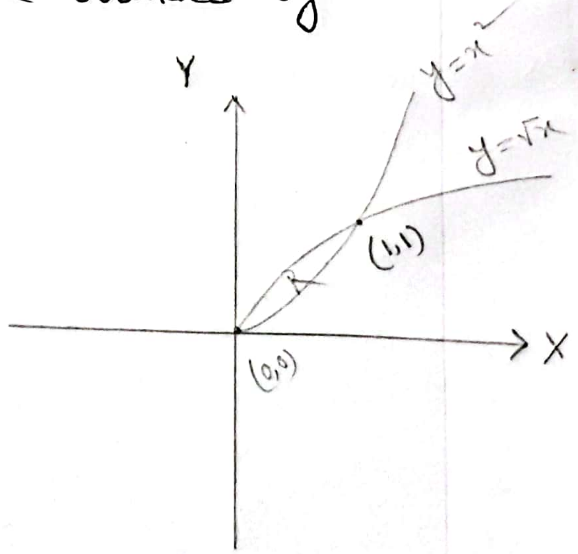
$f(x) = h(-x)$ is continious at $x = -R$.

[proved]

এতো বড় theorem তোরে কে করতে বলছে!!
Shala Abal

* Math problem:

* Find the volume of the solid under the surface $z = xy^2$ and above the region R bounded by $y = x^2$ and $x = y^2$.



Solution: $V = \iint_R xy^2 \, dA$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} xy^2 \, dy \, dx$$

$$= \int_0^1 \left[\frac{xy^3}{3} \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{x}{3}\left[(\sqrt{x})^3 - (x^2)^3\right] dx$$

$$= \int_0^1 \frac{1}{3}\left[x^{5/2} - x^7\right] dx$$

$$= \frac{1}{3}\left[\frac{x^{7/2}}{7/2} - \frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{2}{7} - \frac{1}{8}\right]$$

$$= \frac{1}{3} \times \frac{9}{56}$$

$$= \frac{3}{56} \text{ (Ans).}$$

*** এরকম type Volume/Area নির্ণয়ের Math গুলো দেখতে হবে। (previous question math + calculus).

End

[illegible].

Lecture-05:

* Series of function:

Definition:

A series of functions is an expression $\sum_{k=0}^{\infty} g_k$ which makes sense provided the sequence of partial sum $\sum_{k=0}^{\infty}$ $\sum_{k=0}^{n} g_k(x)$ converges or diverges in $-\infty$ to ∞ pointwise.

Example:

① Any power series is a series of functions $\sum_{n=0}^{\infty} a_n x^n$.

Note (viva)

Every series of function is not a power series, but any power series is a series of functions.

Example: ① $\sum \frac{x^k}{1+x^k}$ is a series of functions but not power series.

② $\sum a^n x^n$ is a power series and also a series of functions.

* Differentiation and integration of power series:

Theorem:

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$. If $0<R_1<R$ then the power series converges uniformly on $[-R_1, R_1]$ to a continious function.