

1. a) Define complex numbers. Is complex number algebra? Explain why or why not. For any complex number $z_1, z_2, \dots, z_n$ prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Answer: A complex number is a number that can be expressed in the form $a+bi$ where "a" and "b" are real numbers, and "i" is the imaginary unit, defined as the square root of -1.

A complex variable z is an ordered pair (x, y) where x & y are real numbers.

It is written as $z = x + iy$; $i = \sqrt{-1}$

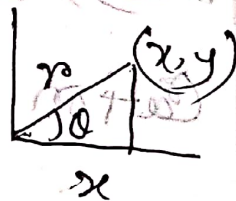
x is called real part and y is called imaginary part.

In polar coordinates,

$$x = r \cos \theta, y = r \sin \theta$$

$$x^2 + y^2 = r^2 \therefore r = \sqrt{x^2 + y^2} \therefore |z| = \sqrt{x^2 + y^2}$$

Called modulus of z .



$\tan \theta = \frac{y}{x} \dots \theta = \tan^{-1}(\frac{y}{x})$; θ is called argument of z .

Yes, complex numbers are part of algebra. Algebra is a branch of mathematics that deals with symbols and the rules for manipulating those symbols. It encompasses various topics such as equations, expressions, functions, and mathematical structures.

Complex numbers are an extension of the real numbers and form a mathematical structure called a field. The field of complex numbers, denoted by $\mathbb{C}$, includes both the real numbers ($\mathbb{R}$) and the imaginary unit (i). Complex numbers obey the same algebraic rules as real numbers, such as addition, subtraction, multiplication and division.

Complex numbers also have algebraic properties such as commutativity, associativity and distributivity. These properties allow for the manipulation and simplification of complex expressions, equations and inequalities using algebraic techniques.

Complex numbers are an integral part of algebra, and their properties and operations are studied within the framework of algebraic systems.

The conjugate of a complex number $a+bi$ is obtained by changing the sign of the imaginary part resulting in $a-bi$.

Proof:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \quad (\because |z|^2 = z \cdot \overline{z})$$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_1} + z_2 \overline{z_2}$$

$$= |z_1|^2 + |z_2|^2 + z_1 \overline{z_2} + z_2 \overline{z_1}$$

$$= |z_1|^2 + |z_2|^2 + z_1 \overline{z_2} + \overline{z_1} z_2$$

$$= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1 \overline{z_2}) \quad \left(\because \operatorname{Re}(z) = \frac{z + \overline{z}}{2} \right)$$

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

$$\operatorname{Re} z \leq |z|$$

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad x \leq \sqrt{x^2 + y^2}$$

$$|z_1 + z_2| \leq (|z_1| + |z_2|)^2 \quad |z| = \sqrt{x^2 + y^2}$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (\text{Proved})$$

b) Find all values of z for which $z^5 = -32$, and locate these values in the complex plane.

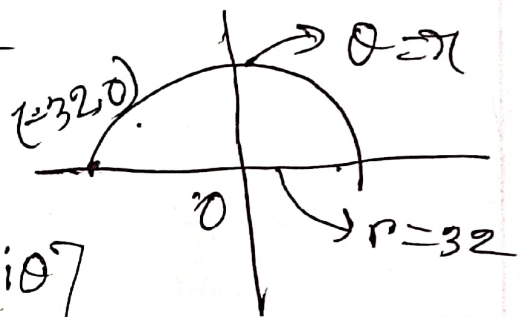
Solution:

$$z^5 = -32$$

$$\text{Let } z^5 = w$$

$$w = -32 + 0i \therefore x = r \cos \theta, y = r \sin \theta$$

$$r = \sqrt{(-32)^2 + 0^2} = 32$$



$$-32 = 32e^{i\pi} [\because re^{i\theta}]$$

$$= 32e^{i(\pi + 2k\pi)}$$

$$(-32)^{1/5} = 32^{1/5} e^{i(\frac{\pi + 2k\pi}{5})} = 2 e^{i(\frac{\pi + 2k\pi}{5})}$$

$$k=0, z_1 = 2e^{i\pi/5}$$

$$k=1, z_2 = 2e^{i3\pi/5}$$

$$k=2, z_3 = 2e^{i5\pi/5} = 2e^{i\pi} = 2(-1) = -2$$

$$k=3, z_4 = 2e^{i7\pi/5}$$

$$k=4, z_5 = 2e^{i9\pi/5}$$

Q) Represent graphically the set of values of z for which -

$$\left| \frac{z-3}{z+3} \right| = 2$$

Answer:

$$\left| \frac{z-3}{z+3} \right| = 2 \quad |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow \left| \frac{x-3+iy}{x+3+iy} \right| = 2 \quad z = x+iy$$

$$\Rightarrow \frac{\sqrt{(x-3)^2 + y^2}}{\sqrt{(x+3)^2 + y^2}} = 2$$

$$\Rightarrow \frac{(x-3)^2 + y^2}{(x+3)^2 + y^2} = 4 \quad \Rightarrow \frac{x^2 - 6x + 9 + y^2}{x^2 + 6x + 9 + y^2} = \frac{4}{1}$$

$$\Rightarrow (x^2 - 6x + 9 + y^2) = 4(x^2 + 6x + 9 + y^2)$$

$$\Rightarrow 9 - 36 = 4x^2 + 4y^2 + 24x - x^2 - y^2 + 6x$$

$$\Rightarrow -27 = 3x^2 + 3y^2 + 30x$$

$$\Rightarrow 3x^2 + 3y^2 + 30x + 27 = 0$$

$$\Rightarrow x^2 + y^2 + 10x + 9 = 0$$

$$\Rightarrow x^2 + 2x \cdot 5 + 5^2 + y^2 - 25 = 0$$

$$\Rightarrow (x+5)^2 + y^2 = 16$$

$$\Rightarrow (x+5)^2 + y^2 = 4^2 \quad \text{--- (1)}$$

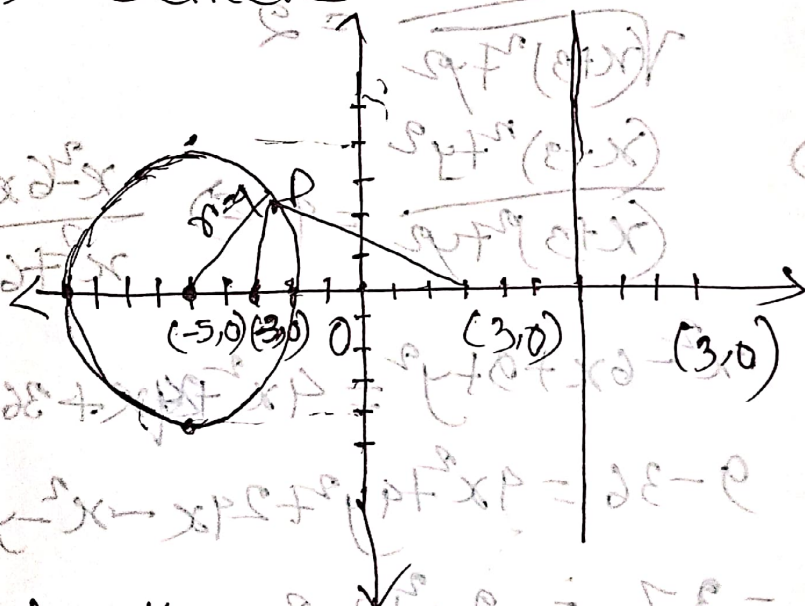
Compare with $(x-a)^2 + (y-b)^2 = r^2$ (circle)

$(a, b) \rightarrow$ centre

$r \rightarrow$ radius

So, centre $(-5, 0)$ and radius 4 unit.

this is a circle



Geometrically any point P on this circle is such that the distance from P to $(-3, 0)$ is twice the distance P to $(3, 0)$

Q. Define (limit and continuity of complex function). Let $w = f(z) = (z^2 + 1)^{1/2}$. Then show that (i) $z = \pm i$ are branch points of $f(z)$ and (ii) a circuit around both branch points produces no change in the branches of $f(z)$.

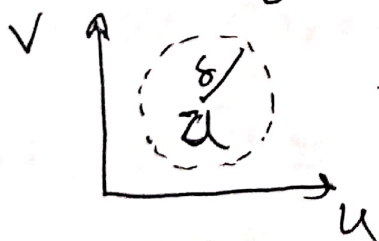
Ans:

limit: Let $f(z)$ be any function of the complex variable z defined in a bounded and closed domain D . Then ℓ is said to be the limit of $f(z)$ as if for any $\epsilon > 0 \exists \delta > 0$ such that

$$|f(z) - \ell| < \epsilon \text{ for all values of } z$$

$$\text{i.e. } 0 < |z - a| < \delta$$

In symbols $\lim_{z \rightarrow a} f(z) = \ell$.



Complex continuity: A function $f(z)$ of a complex variable z defined in the closed and bounded domain D is said to be continuous at $a \in D$ if and only if for $\epsilon > 0$ $\exists \delta > 0$ such that

$$|f(z) - f(a)| < \epsilon \quad \forall |z - a| < \delta$$

$z \rightarrow a$ if $\lim_{z \rightarrow a} f(z) = f(a)$

Let $f(z)$ be a function defined in a bounded and closed domain D . Then $f(z)$ is said to be continuous at $a \in D$ if for all $\epsilon > 0$ $\exists \delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for all $z \in D$ with $|z - a| < \delta$.

$\lim_{z \rightarrow a} f(z) = f(a)$



$$i. w = f(z) = (z^2 + 1)^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2} = f_1(z)$$

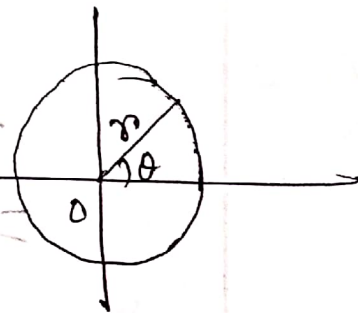
Let, $z = 0$

$$f(z) = z^{1/2} = (re^{i(\theta+2\pi)})^{1/2} = (re^{i\theta})^{1/2} e^{i\pi/2}$$

$$= r^{1/2} e^{i\theta/2} e^{i\pi/2}$$

$$= -r^{1/2} e^{i\theta/2}$$

$$= f_2(z)$$



$$i. f(z) = (z^2 + 1)^{1/2} = (z^2 - i^2)^{1/2}$$

$$= \{(z+i)(z-i)\}^{1/2} = \sqrt{z+i} \cdot \sqrt{z-i}$$

Let, $z+i = r_1 e^{i\theta_1}$ & $z-i = r_2 e^{i\theta_2}$

$$\sqrt{z+i} = \sqrt{r_1} e^{i\theta_1/2} \text{ and } \sqrt{z-i} = \sqrt{r_2} e^{i\theta_2/2}$$

Encircle $z=i$, not $z=-i$

Encircle

$$\sqrt{r_1} e^{i(\theta_1+2\pi)/2}$$

$z=-i$, not $z=i$

$$\sqrt{r_2} e^{i(\theta_2+2\pi)/2}$$

$$= \sqrt{r_1} e^{i\theta_1/2} e^{i\pi}$$

$$= \sqrt{r_2} e^{i\theta_2/2} e^{i\pi}$$

$$= \sqrt{r_1} e^{i\theta_1/2} (\cos \pi + i \sin \pi)$$

$$= \sqrt{r_2} e^{i\theta_2/2} (\cos \pi + i \sin \pi)$$

$$z = \sqrt{r_1} e^{i\theta_1/2}$$

$$z = \sqrt{r_2} e^{i\theta_2/2}$$

the function is changed.

$$z = f(a) \quad [0]_0 = 2\pi + f(a)$$

AVAILABLE AT:

Onebyzero Edu - Organized Learning, Smooth Career

The Comprehensive Academic Study Platform for University Students in Bangladesh (www.onebyzeroedu.com)

so, $z = -i$ is a branch point.

$$(b) f(z) = (z^2 + 1)^{1/2}$$

$$= \sqrt{(z+i)(z-i)}$$

$$= \sqrt{r_1 r_2 e^{i\theta_1} e^{i\theta_2}}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$$

Let, encircle both $z = i$ & $z = -i$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + 2\pi + \theta_2 + 2\pi)/2}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} e^{2\pi i}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \cdot 1$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$$

$$\begin{aligned} &= \cos 2\pi + i \sin 2\pi \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

The function remain unchanged

B) Solution: Define harmonic function with example. Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

Soln: If the 2nd partial derivatives of u and v with respect to x & y exists and continuous in $\mathbb{R}^2$, then we find -

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

* If G is an open subset of $\mathbb{C}$ then function $u: G \rightarrow \mathbb{R}$ is said to be harmonic if u has continuous second order partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

This equation is called Laplace's equation

$$z = 1 + i(a) \quad \angle \theta \Big|_0 = 2\pi + \theta(a)$$

i. A function F on a region G is analytic iff $\Re f = u$ and $\Im f = v$ are harmonic functions which satisfy the Cauchy Riemann eq.

ii. A region G is simply connected iff each harmonic function u on G such that $F = u + iv$ is analytic in G .

$$* u = e^{-x} (x \sin y - y \cos y)$$

$$* \frac{\partial u}{\partial x} = e^{-x} (\sin y) - (e^{-x}) (x \sin y - y \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y - e^{-x} \sin y + x e^{-x} \sin y + y e^{-x} \cos y$$

$$= -2e^{-x} \sin y + x e^{-x} \sin y + y e^{-x} \cos y$$

Again, $\frac{\partial u}{\partial y} = e^{-x} (x \cos y + y \sin y - \cos y)$

$$= x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -x e^{-x} \sin y + e^{-x} \sin y$$

$$+ y e^{-x} \cos y + e^{-x} \sin y \quad \textcircled{1} \text{ po} + \textcircled{2} \text{ po}$$

$$2 + x^2 e^{-x} \sin y + 2 e^{-x} \sin y + y e^{-x} \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= -2 e^{-x} \sin y + x e^{-x} \sin y - y e^{-x} \cos y$$

$$- x e^{-x} \sin y + 2 e^{-x} \sin y + y e^{-x} \cos y$$

$\therefore u$ is harmonic

*

Example:

$$f(z) = u + iv$$

$$\text{let } u = x^2 - y^2 \quad \textcircled{1}$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \textcircled{1}$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \quad \textcircled{II}$$

eq ① + eq ②

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \quad (\text{harmonic or Laplace's eqn})$$

$$f(z) = u + iv$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x, y} \quad \text{--- ①}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x, y} \quad \text{--- ②}$$

by ① and ②

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x, y} - \frac{\partial^2 v}{\partial x, y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (5)}$$

Example if $v(x, y) = 3x^2y - y^3$

3. a) Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region.
 R. Prove Cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz$$

Soln:

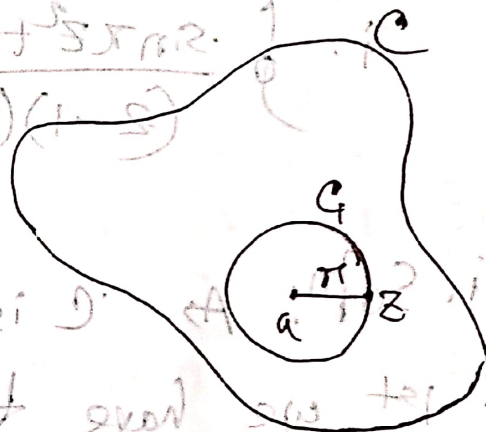
eqn of circle (C_1)

$$|z-a| = r$$

$$\Rightarrow z-a = r e^{i\theta}$$

$$\Rightarrow z = a + r e^{i\theta} \quad (1)$$

$$dz = r i e^{i\theta} d\theta \quad (2)$$



when $r \rightarrow 0$
 then $f(z) \rightarrow f(a)$

$$\oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz$$

$$= \int_0^{2\pi} \frac{f(a + r e^{i\theta})}{r e^{i\theta}} \cdot r i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} f(a + r e^{i\theta}) d\theta = i \int_0^{2\pi} f(a) d\theta$$

$$= i f(a) \int_0^{2\pi} d\theta = i f(a) [\theta]_0^{2\pi} = 2\pi i f(a)$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \quad (\text{Proved})$$

3. Evaluate:

i. $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

ii. $\oint_C \frac{e^{2z}}{(z+1)^4} dz$

Soln. As C is the circle $|z| = 3$,
1st we have to do partial fraction of

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

AVAILABLE AT:

By Cauchy's Integral formula with $a=2$ & $a=1$, we get —

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz = 2\pi i (\sin \pi 2^2 + \cos \pi 2^2) \quad [a=2]$$

$$= 2\pi i (\sin 4\pi + \cos 4\pi)$$

$$= 2\pi i (0 + 1)$$

$$= 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz = 2\pi i (\sin \pi 1^2 + \cos \pi 1^2) \quad [a=1]$$

$$= 2\pi i (0 - 1)$$

$$[a=1]$$

$$\text{The required integral} = 2\pi i - (-2\pi i) = 4\pi i$$

ii. Solⁿ: $\oint \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z|=3$.

Let, $f(z) = e^{2z}$ and $a = -1$ in Cauchy integral formula,

Comparing with,

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (1)$$

$$n+1=4 \quad \therefore n=3$$

$$\therefore f'''(z) = 8e^{2z}$$

$$\therefore f'''(1-i) = 8e^{-2}$$

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8e^{-2} \cdot 2\pi i}{3} = \frac{8\pi i e^{-2}}{3}$$

(Ans.)

4. a) Find Laurent series about the indicated singularity for each of the following functions

i. $\frac{e^{2z}}{(z-1)^3}$; $z=1$ ii. $\frac{z}{(z+1)(z+2)}$; $z=-2$

Soln:

Let $z-1=u$ $\therefore z=1+u$

$$\therefore \frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2 \cdot 2^{2u}}{u^3} = \frac{e^2}{u^3} \cdot e^{2u}$$

$$= \frac{e^2}{u^3} \left\{ 1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \dots \right\}$$

$$= \frac{e^2}{u^3} + \frac{2e^2}{u^2} + \frac{2e^2}{u} + \frac{4e^2}{3} + \dots$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{2(z-1)} + \frac{4e^2}{3} + \dots$$

$z=1$ is a pole of order 3

$\therefore$ The series converges for all value of $z \neq 1$ without 1.

for $\frac{z}{(z+1)(z+2)}$; $z = -2$ — trivial bit (a) $\therefore$
 let, $z+2 = u \therefore z = u-2$

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-2+1)u} = \frac{u-2}{(u-1)u} = \frac{2-u}{(1-u)u}$$

$$\Rightarrow \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u} (1+u+u^2+u^3+\dots)$$

$$= \frac{2-u}{u} + \frac{2u-u^2}{u} + \frac{2u^2-u^3}{u} + \dots$$

$$= \frac{2}{u} - 1 + 2 - u + 2u - u^2 + 2u^2 - u^3 + \dots$$

$$= \frac{2}{u} + 1 + u + u^2 + u^3 + \dots$$

$$= \frac{2}{(z+2)} + 1 + (z+2) + (z+2)^2 + \dots$$

for $\frac{z}{(z+1)(z+2)}$; $z = -2$ is a pole
 $0 < |z+2| < 1$

D) Evaluate $\int \frac{e^{zt}}{z^2(z^2+2)} dz$ around the circle C
with equation $|z|=3$.

Soln! $|z|=3 \therefore \text{Centre} \equiv (0,0) \text{ radius}=3$.

there is a simple pole at $z=2$ inside C

5. a) i. Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$

ii. show that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$

i. $e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \text{Real part}(e^{i\theta})$

$\sin\theta = \text{Im part}(e^{i\theta})$

$I = \text{Real part} \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta} d\theta$

Putting

$\cos\theta = \frac{z+z^{-1}}{2}$

$d\theta = \frac{dz}{zi}$

$I = \int_C \frac{z^3}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{zi}$

$= \int_C \frac{z^2}{5-2z^2-2} \frac{dz}{i} = \frac{1}{i} \int_C \frac{z^3}{5z-2z^2-2} dz$

$= -\frac{1}{i} \int_C \frac{z^3}{2z^2-5z+2} dz = -\frac{1}{i} \int_C \frac{z^3}{2z^2-4z-z+2} dz$

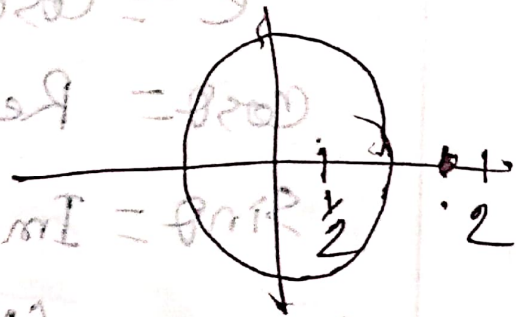
$$= \frac{2\pi i}{2\pi i} \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz$$

$$I = \oint_C \frac{1}{2i} \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz, |z|=1/2 \quad \text{--- (1)}$$

$$I_1 = \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz$$

Pole $z - \frac{1}{2} = 0 \Rightarrow z = \frac{1}{2}$

$z - 2 = 0 \Rightarrow z = 2$



$z = \frac{1}{2}$ lies inside

the circle

$$I_1 = \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz$$

$$= 2\pi i \cdot \frac{z^3}{z-2} \Big|_{z=\frac{1}{2}}$$

$$I_1 = 2\pi i \left(\frac{1}{2} \right)^3$$

From (1)

$$I = \frac{1}{2i} \times \frac{\pi i}{6} = \frac{\pi}{12}$$

AVAILABLE AT:

Onebyzero Edu - Organized Learning, Smooth Career

The Comprehensive Academic Study Platform for University Students in Bangladesh (www.onebyzeroedu.com)

$$\int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta} d\theta = \frac{\pi}{12}$$

Real part

$$\int_0^{2\pi} \frac{\cos 3\theta + i \sin 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$$

$$= \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$$

ii. Solⁿ: $f(a) = \int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx$ (where a is Parameter, $a \geq 0$)

$$f'(a) = \int_0^{\infty} \frac{\partial}{\partial a} \left(e^{-ax} \frac{\sin x}{x} \right) dx$$

$$= \int_0^{\infty} -x e^{-ax} \frac{\sin x}{x} dx = - \int_0^{\infty} e^{-ax} \sin x dx$$

$$I = \int_0^{\infty} e^{-ax} \sin x dx = e^{-ax} \int \sin x dx - \int a e^{-ax} \cos x dx$$

$$= -e^{-ax} \cos x - a \int e^{-ax} \cos x dx - \int a e^{-ax} \sin x dx$$

$$I = -e^{-ax} (\cos x + a \sin x) - a I$$

$$I = - \frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1} + C$$

$$- \int e^{-ax} \sin x dx = \frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1} + C$$

$$\therefore - \int_0^B e^{-ax} \sin x dx = \left[\frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1} \right]_0^B$$

When $B \rightarrow \infty$ then

$$- \int_0^{\infty} e^{-ax} \sin x = 0 - \frac{1}{a^2 + 1}$$

$$f'(a) = \frac{1}{a^2 + 1} \therefore f(a) = \tan^{-1} a + C$$

$$f(0) = \tan^{-1}(0) + C \Rightarrow f(a) = \int_0^{\infty} \frac{\sin x}{x} dx$$

$$\therefore f(a) = 0$$

$$f(a) = \lim_{a \rightarrow \infty} \int_0^{\infty} e^{-ax} \frac{\sin x}{x} dx = 0$$

Again, $f(a) = \tan^{-1}(a) + C$, $f(\infty) = \tan^{-1}(\infty) + C$
 $\Rightarrow C - \frac{\pi}{2} = 0 \Rightarrow C = \frac{\pi}{2}$

$$f(b) = 0 = \frac{\pi}{2} = \int_0^{\infty} \frac{\sin x}{x} dx$$

Other:-

* $\int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx$ where a and b are variable. at first treating a as a constant also diff w.r.t b .

$$I(b) = \int_0^{\infty} \frac{e^{-ax} (\cos bx)}{x} dx$$

$$I'(b) = \frac{e^{-ax}}{a^2 + b^2} [-a \cos bx + b \sin bx]_0^{\infty}$$

$$I'(b) = \frac{a}{a^2 + b^2}$$

$$I(b) = \tan^{-1}\left(\frac{b}{a}\right) + C$$

$$\text{Substitute } b=0$$

$$\Rightarrow 0 = 0 + C \text{ (equation)}$$

$$I(b) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\int_0^{\infty} \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1}\left(\frac{b}{a}\right)$$

Substitute, $a=0$ and $b=1$ $\int_0^{\infty} \frac{\sin x}{x} dx = \tan^{-1} 1 = \frac{\pi}{2}$

b) Find a bilinear transformation that maps points $z = 0, -1, -i$ into $w = i, 1, 0$, respectively.

$$z_1 = 0$$

$$w_1 = i$$

$$z_2 = -1$$

$$w_2 = 1$$

$$z_3 = -i$$

$$w_3 = 0$$

We know that the bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i + 1)}{(0 + i)(-1 - z)}$$

$$\Rightarrow \frac{(w - i)}{w(1 - i)} = \frac{z(1 - i)}{-i(1 + z)}$$

$$\Rightarrow \frac{w(1 - i)}{(w - i)} = \frac{-i(1 + z)}{z(1 - i)}$$

$$\frac{z - i}{w - i} = \lim_{n \rightarrow \infty} \frac{i - n}{n}$$

6. a) Using Gauss-Jordan method solve the system:

$$x+y+2z=9, 2x+4y-3z=1, 3x+6y-5z=0$$

Solve:

Gauss Jordan elimination method

$$x+y+2z=9$$

$$2x+4y-3z=1$$

$$3x+6y-5z=0$$

'in matrix form'

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

Augmented

$$C = [A:B]$$

matrix is given by:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$$\times A:B = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$$C = [A:B]$$

$$C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$$[r'_1 = r_2, r'_2 = r_1]$$

Applying operations $R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - 3R_1$

$$C \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -3 & -5 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

$$C = [A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

Now, $R_3 \rightarrow R_3 - R_2$, $R_3 \leftrightarrow R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 1 & -9 & -10 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -9 & -10 \\ 0 & 2 & -7 & -17 \end{array} \right]$$

Now,

$$R_3 \rightarrow R_3 - 2R_2; C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -4 & : & -10 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 4R_3; C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 2R_3; C = \begin{bmatrix} 1 & 1 & 0 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2; C = \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

eqn: $x = 1$

Ans: $\begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

b) Using the result of part (a), determine whether the vector $w = (9, 1, 0)$ can be expressed as a linear combination of the vectors

$$v_1 = (1, 2, 3), \quad v_2 = (1, 4, 6), \quad v_3 = (2, -3, 5)$$

and, if so, find such a linear combination.

Soln: Suppose $w = (9, 1, 0)$ can be written as L.C. of v_1, v_2 & v_3

$$(9, 1, 0) = x(1, 2, 3) + y(1, 4, 6) + z(2, -3, 5)$$

$$= (x, 2x, 3x) + (y, 4y, 6y) + (2z, -3z, 5z)$$

$$= (x + y + 2z, 2x + 4y - 3z, 3x + 6y - 5z)$$

we get a system of equation

$$x + y + 2z = 9 \quad \text{--- (1)}$$

$$2x + 4y - 3z = 1 \quad \text{--- (2)}$$

$$3x + 6y - 5z = 0 \quad \text{--- (3)}$$

$$C = [A : B]$$

$$= \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & -4 & -7 & : & -17 \\ 0 & -6 & -11 & : & -27 \end{bmatrix}$$

$$(2x-5)y + (2y+1)z + (2x+1)z = (0 \quad 1 \quad 0)$$

$$R_3 \Rightarrow R_3 - R_2 \quad \& \quad R_3 \leftrightarrow R_2$$

$$C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & -4 & -7 & : & -17 \\ 0 & -2 & -4 & : & -10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & -2 & -4 & : & -10 \\ 0 & -4 & -7 & : & -17 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{-2} \quad C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & 1 & 2 & : & 5 \\ 0 & -4 & -7 & : & -17 \end{bmatrix}$$

(12)
(13)

$$0 = 5x - y + z$$

$$R_3 \rightarrow R_3 + R_2;$$

$$C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & 1 & 2 & : & 5 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Here, $z = 3$ [3rd equ]

$y + 2z = 5$ [2nd equ]

$\therefore y = 5 - 6 = -1$

Q $x + 4y + 2z = 9$

$\Rightarrow x = 9 + 4 - 6 = 7$

So, $(9, -1, 0) = 7(1, 2, 3) - 1(1, 4, 6) + 3$

$(2, -3, -5)$

$= (7 + 14 + 21) - (1, 4, 6) + (6, -9, -15)$

$= (7 - 1 + 6, 14 - 4 - 9, 21 - 6 - 15)$

$= (12, 1, 0)$

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \end{bmatrix} = 0$$

$$[up\ down] \quad 2 = 3 \text{ (off)}$$

$$[up\ down] \quad 2 = 55 + 1$$

$$1 = 2 - 2 = 1 \therefore$$

c) Find the inverse of A (if possible) by elementary row operations and also state that rank of

A where - $A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$

Soln. Start with $[A | I]$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right] =$$

$$(from \ A A^{-1} = I)$$

Step 1: $R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ -1 & 12 & 5 & 0 & 10 & 1 \end{array} \right]$$

2: $R_3 \rightarrow R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

3: $R_3 \xrightarrow{\text{2nd}} R_3 + R_2$ & $R_2 \xrightarrow{\text{1st}} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array} \right]$$

4: $R_3 \rightarrow R_3 - 8R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array} \right]$$

5: $R_1 \rightarrow R_1 - 9R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array} \right]$$

6. $R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 4 & 0 \\ 0 & 1 & 0 & 17 & 9 & -1 \\ 0 & 10 & 1 & -15 & -8 & 1 \end{array} \right]$$

7. $R_1 \rightarrow R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -41 & -22 & 2 \\ 0 & 1 & 0 & 17 & 9 & -1 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array} \right]$$

$\therefore A^{-1} = \begin{bmatrix} 0 & -41 & -22 & 2 \\ 1 & 17 & 9 & -1 \\ 0 & -15 & -8 & 1 \end{bmatrix}$

Firstly we need to find the possibility of inverse: $|A| \neq 0$ [condition]

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 2 & 4 & -1 & 2 \\ -1 & 2 & 5 & -1 \end{array} \right]$$

$$1(20 + 2) - 6(10 - 1) + 4(9 + 1)$$

$$= 22 - 54 + 32 = 0$$

So we can determine A^{-1} .

Rank: As the value of $|A|$ is zero (6)

So the rank is not 3, it may be less than 3.

Now, we consider $|A|_{2 \times 2}$ form:

$$|A| = \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4 - 12 = -8 \neq 0$$

So Rank = 2.

$$2 = (1 - 8) = \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} \Rightarrow 1 - 8 = -7$$

$$2 = (1 - 8) = \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} \Rightarrow 1 - 8 = -7$$

$$5 = (8 - 1) = \begin{vmatrix} 8 & 1 \\ 1 & 8 \end{vmatrix} \Rightarrow 8 - 1 = 7$$

7. a) find inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ by finding its adjoint

Solve! Sol, Let, $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ is matrix

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + 1(4-3) = 6 - 6 + 2 = 2 \neq 0$$

So, the inverse of matrix is exist.

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 9 \\ 1 & 9 \end{vmatrix} = (18-12) = 6$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = (4-2) = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = - (9-4) = -5$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = (9-1) = 8$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = - (4-1) = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = (3-2) = 1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = - (3-1) = -2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (2-1) = 1$$

$$\text{So, } \text{Adj } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{2} \begin{pmatrix} 8 & -6 & 2 \\ -5 & 8 & -3 \\ -1 & 2 & 1 \end{pmatrix}$$

b) Using the result of part (c), solve the system of equations

$$x + y + z = 3, \quad x + 2y + 3z = 9, \quad x + 4y + 9z = 6$$

Soln:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\therefore X = A^{-1}B$$

A^{-1} exist if $|A| \neq 0$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} + (-1)^3 \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= (18 - 12) - (9 - 3) + (4 - 2)$$

$$= 6 - 6 + 2$$

$$= 2$$

We can take the result of part (a), where

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned} X &= A^{-1}B = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 9 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 9 - 12 + 6 \\ -\frac{15}{2} + 16 - 9 \\ -\frac{3}{2} - 9 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{-1}{2} \\ -\frac{2}{1} \end{pmatrix} \end{aligned}$$