

→ A proposition is a declarative statement that is either true (T) or false (F), but not both.

True = T = 1
False = F = 0

$2+2=5 \rightarrow$

Pigs can Fly $\rightarrow$ not possible

① What time is it? → Questions
② Read this carefully. → Commands / Imperative

④ $x+1=2 \rightarrow$ Non constant value/value is not defined

⑥ Shahin is the best lecturer. → Opinion

⑦ He is a college student. → Person isn't defined

Propositional variables / statement variables

■ Today is Friday. $\rightarrow p$ = Today is Friday.
↑
[propositional variables]

■ It is raining. $\rightarrow q$ = It is raining.

$\rightarrow$ The truth value of the propositional variable can be True or False.

Compound Proposition

Compound proposition is a proposition formed by combining two or more simple proposition.

The logical operators that are used to form compound proposition is called connectives.

Symbol	Math name	English name
$\neg$	Negation	NOT
$\vee$	Disjunction	OR
$\wedge$	Conjunction	AND
$\oplus$	Ex OR	"OR... but not both"
$\rightarrow$	Implication	"If... then"
$\Leftrightarrow$	Equivalence/biconditional	"If and only if"

$\neg$ (NOT) $\wedge$ (AND) $\vee$ (OR) $\oplus$ EXOR

$\rightarrow$ (Implication) $\Leftrightarrow$ (Bi-conditional)

NOT / Negation ($\neg$)

Today is Friday. (Negation) $\rightarrow$ Today is not Friday.

It is not the case that today is Friday.

(¬p)

P	$\neg P$
F	T
T	F

Example:

Example:
Find the negation of each of these proposition.

① 6 is negative.

① 6 is negative.
 $\rightarrow$ 6 is non negative / It is not the case that 6 is negative.

② $2+1=3 \rightarrow 2+1 \neq 3$

③ There is no pollution in New Jersey.

→ There is pollution in New Jersey.

④ The summer in marine is hot and sunny.

→ It is not the case that the summer is marine is hot and sunny.

⑤ Today is Thursday.

→ Today is not Thursday.

AND/Conjunction ($\wedge$)

Let p and q be propositions the compound proposition 'p and q'.

Today is Friday $\wedge$ It is raining.
 p q

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

OR/Disjunction ($\vee$)

Today is Friday $\vee$ It is raining.
 p q

Let p and q be propositions the compound proposition 'p or q'.

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Ex OR / Exclusive OR ($\oplus$)

"Students who have taken calculus $\overset{P}{\text{or}}$ computer science $\underset{Q}{\text{can take this class.}}$ "

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

"Students who have taken calculus or computer science but not both can take the class."

↕

P	Q	$P \oplus Q$
F	F	F
F	T	T
T	F	T
T	T	F

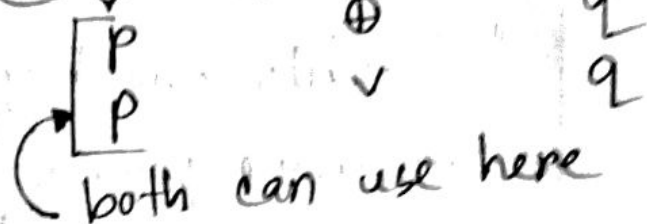
For each of these sentences, determine whether an exclusive OR or OR is intended.

① Coffee $\overset{P}{\text{or}} \oplus \underset{Q}{\text{tea}}$ comes with dinner. $\rightarrow$ because not both

② Experience with C++ $\overset{P}{\text{or}} \vee \underset{Q}{\text{java}}$ is required.

③ Lunch includes soup $\overset{P}{\text{or}} \oplus \underset{Q}{\text{salad}}$.

① You can pay using U.S. dollars or eq euros.



Conditional statement/Implication ($\rightarrow$)

p : Today is holy day.
 q : The store is closed.

Let p and q propositions
 the compound proposition
 "if p , then q "

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

You can make
 $\rightarrow$ the compound proposition false only when you satisfy the first

condition itself i.e. if that itself not satisfied, then we can't make compound proposition false (F).

(Not false mean true)

☐ Determine whether each of these conditional statements is true or false.

② If $\underbrace{1+1=2}_T$, then $\underbrace{2+2=5}_F \rightarrow (T \rightarrow F = F)$

⑥ If $1+1=3$, then $2+2=4 \rightarrow (F \rightarrow T = T)$

© If $1+1=3$, then dogs can fly $\rightarrow (F \rightarrow F=T)$

② If $1+1=2$, then dogs can fly $\rightarrow (T \rightarrow F = F)$

① "If p , then q ."

→ "If I am elected, then I will lower Taxes!"

• if p, q

→ "If it is below freezing, it is also snowing."
 $p \rightarrow q$

• "p is sufficient for q"

→ "Driving over 65 miles per hour is sufficient for getting a speeding ticket."

"q if p"

→ "Maria will get a good job if she learns Discrete mathematics."

$$p \rightarrow q$$

① "q whenever p"

② "q is necessary for p"

③ "q follows from p"

④ "a sufficient condition for q is p"

⑤ "q when p"

→ "Maria will find a good job when she learns discrete mathematics."

$$p \rightarrow q$$

⑥ "q unless $\neg p$ "

→ "Maria will find a good job unless she doesn't learn discrete mathematics."

⑦ "p only if"



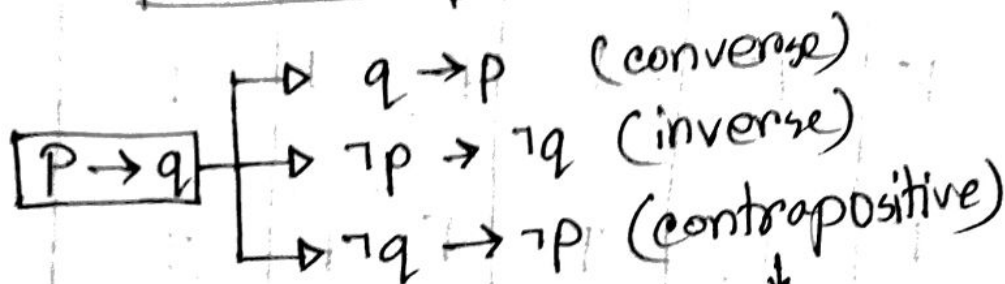
अरे 11 डि उनासु

conditional statement उनासु

असम दस-रस

Conditional Statement to Now conditional statement

Converse Inverse Contrapositive



converse + inverse

Example:

$P \rightarrow$ I am in Barishal.

$Q \rightarrow$ I am in Bangladesh.

Conditional statement: If I am in Barishal then I am in Bangladesh.

Converse: $Q \rightarrow P$

"If I am in Bangladesh then I am in Barishal."

Inverse: $\neg P \rightarrow \neg Q$

"If I am not in Barishal then I am not in Bangladesh."

Contrapositive: $\neg Q \rightarrow \neg P$

"If I am not in Bangladesh then I am not in Barishal."

P	q	$P \rightarrow q$	$q \rightarrow P$	$\neg P$	$\neg q$	$\neg P \rightarrow \neg q$	$\neg q \rightarrow \neg P$
F	F	T	T	T	T	T	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F
T	T	T	T	F	F	T	T

main conditional
statement

converse

Inverse Contrapositive

here main conditional statement and contrapositive
always totally same.

☐ State the converse, inverse and contrapositive
of each of these following statements —

@ If it snows today, I will ski tomorrow.
 $P \rightarrow Q$

Converse: $Q \rightarrow P$

→ I will ski tomorrow only if it snows today.

Inverse: $\neg p \rightarrow \neg q$

→ If it does not snow today, then I will not ski tomorrow.

Contrapositive: $q \rightarrow p$

→ If I don't ski tomorrow then I will not have the snowed today.

⑥ I come to class whenever there is going to be a quiz.

$p \rightarrow q$

Converse: $q \rightarrow p$

→ There is going to be a quiz whenever I come to class.

Inverse: $\neg p \rightarrow \neg q$

→ I don't come to class whenever there is not going to be a quiz.

Contrapositive: $\neg q \rightarrow \neg p$

→ There is not going to be a quiz whenever I don't come to class.

Biconditional statements/simplifications ($\leftrightarrow$)

You can take the flight, if and only if, you buy a ticket.

$$p \leftrightarrow q$$

- ⊙ "p is necessary and sufficient for q"
- ⊙ "p iff q"
- ⊙ "if p then q, and conversely."

P	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Example:

"That it is below freezing is necessary and sufficiency for it to be snowing."

$$(p \leftrightarrow q)$$

☐ Determine whether these biconditionals are true or false.

Ⓐ $\frac{2+2=4}{T} \leftrightarrow \frac{1+1=2}{T}$ (True)

Ⓑ $\frac{1+1=2}{T} \leftrightarrow \frac{2+2=4}{T}$ (False)

① $1+1=3$ if and only if monkey can fly. (True)

② $0 > 1$ if and only if $2 > 1$. (False)

→ This is a Biconditional Statement.

If we think that, $P = 0 > 1$, $Q = 2 > 1$ then P is false because 0 is not greater than 1 but Q is true because 2 is greater than 1 . According to biconditional if one value is false and another value is true then it must return false.

So, it returns False (F).

Aspect	Implication ($\rightarrow$)	Biconditional ($\leftrightarrow$)
Meaning	If P then Q	P if and only if Q or, P is equivalent to Q
Truth Conditions	- True if P is false or both P and Q are true - False if P is true and Q is false	- True if both P and Q have the same truth value (either both true or both false) - False if they are different truth values.
Directionality	Unidirectional: from P to Q .	Bidirectional: Between P and Q
Dependency	P depends on Q	Both P and Q depend on each other
Causality	Implies a causal relationship from P to Q	Indicates an equivalence between P and Q
Usage Example	If it's raining (P), then the ground is wet (Q)	A shape is a triangle (P) if and only if it has three sides (Q)..

Truth table of Compound Propositions

* precedence of logical operators

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Number of Rows: 2^n

(where n is number of variables)

① $(p \vee \neg q) \rightarrow (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
F	F	T	T	F	F
F	T	F	F	F	T
T	F	T	T	F	F
T	T	F	T	T	T

② $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
F	F	T	T	F	T
F	T	F	F	T	T
T	F	T	F	T	T
T	T	F	T	F	T

$$(p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow \neg p)$$

p	q	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow \neg p$	$(p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow \neg p)$
F	F	T	T	T	T	T
F	F	T	T	T	F	F
F	T	T	F	F	F	T
F	T	T	F	F	T	F
T	F	F	T	F	T	F
T	F	F	T	F	F	T
T	T	F	F	T	F	F
T	T	F	F	T	T	T

① $\neg$ (NOT) $\rightarrow$ উল্টো হবে বলে

② $\wedge$ (AND) $\rightarrow$ সবগুলো True হলে True (T)

③ $\vee$ (OR) $\rightarrow$ যেকোনো একটি True হলে True (T)

④ $\rightarrow$ (Conditional) $\rightarrow$ T, F থাকলে False (F)

⑤ $\leftrightarrow$ (Biconditional) $\rightarrow$ সবগুলো True অথবা সবগুলো False হলে True (T)

⑥ $\oplus$ (ExOR) $\rightarrow$ সবগুলো True অথবা সবগুলো False হলে False (F)

অর্থাৎ Biconditional এর উল্টো ExOR

English sentence into Proposition

Ⓐ It is below freezing and snowing. ($p \wedge q$)

Ⓑ It is below freezing but not snowing. ($p \wedge \neg q$)

Ⓒ It is either snowing or freezing but not both. ($p \oplus q$)

Ⓓ If it is below freezing, it also snowing. ($p \rightarrow q$)

Ⓔ It is either below freezing or it is snowing, but it is not snowing if it is below freezing.

$$(p \vee q) \wedge (p \rightarrow \neg q)$$

Ⓕ That is below freezing is necessary and sufficient for it to be snowing. ($p \leftrightarrow q$)

Ⓖ You can access the internet from campus only if you are a computer science major or you are not a freshman.

$$p \rightarrow q \vee \neg r$$

Proposition to English Sentence

Let p and q be the propositions "Swimming at the New Jersey is allowed" and "sharks have been spotted near the shore", respectively.

Express each of these compound propositions as an English sentence.

- a) $\neg p \rightarrow$ Swimming at the New Jersey is not allowed and sharks have been spotted near the shore.
- b) $p \vee q \rightarrow$ Swimming at the New Jersey is allowed or sharks have been spotted near the shore.
- c) $p \rightarrow q \rightarrow$ If swimming at the New Jersey is allowed then sharks have been spotted near the shore.
- d) $p \leftrightarrow q \rightarrow$ Swimming at the New Jersey is allowed if and only if sharks have been spotted near the shore.
- e) $\neg p \leftrightarrow \neg q \rightarrow$ Swimming at the New Jersey is not allowed if and only if sharks have not been spotted near the shore.

Proposition to English Sentence

Let p and q be the propositions "swimming at the New Jersey is allowed" and "sharks have been spotted near the shore", respectively.

Express each of these compound propositions as an English sentence.

① $\neg p \rightarrow q$ → Swimming at the New Jersey is not allowed and sharks have been spotted near the shore.

② $p \vee q$ → Swimming at the New Jersey is allowed or sharks have been spotted near the shore.

③ $p \rightarrow q$ → If swimming at the New Jersey is allowed then sharks have been spotted near the shore.

④ $p \leftrightarrow q$ → Swimming at the New Jersey is allowed if and only if sharks have been spotted near the shore.

⑤ $\neg p \leftrightarrow \neg q$ → Swimming at the New Jersey is not allowed if and only if sharks have not been spotted near the shore.

④ $\neg p \vee (p \wedge q) \rightarrow$ Swimming at the New Jersey is not allowed or swimming at the New Jersey is allowed and sharks have been spotted near the shore

⑤ $\neg p \wedge (p \vee \neg q) \rightarrow$ Swimming at the New Jersey is not allowed, and swimming at the New Jersey is allowed or sharks have been spotted near the shore.

Propositional Equivalences

$\neg(p \wedge q)$	$\equiv$	$\neg p \vee \neg q$
F	↑ logically equivalent means logically same	F
F		F
F		F
T		F
		T

Compound Proposition

Tautology Contradiction Contingency

Example:

- ① $p \vee \neg p$ ② $q \wedge \neg q$ ③ $p \rightarrow q$

p	q	$\neg p$	$\neg q$	$p \vee \neg p$	$q \wedge \neg q$	$p \rightarrow q$
F	F	T	T	T	F	T
F	T	T	F	T	F	T
T	F	F	T	T	F	F
T	T	F	F	T	F	T

Tautology
(Always fully true)

Contradiction
(Always fully false)

Contingency
(mixed of tautology & contradiction)

□ Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology using truth table.

P	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	T	F	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

From this truth table, we can say that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

☐ Determine whether $(\neg p \wedge (p \rightarrow q)) \rightarrow p$ is a tautology.

P	q	$\neg p$	$p \rightarrow q$	$\neg p \wedge (p \rightarrow q)$	$(\neg p \wedge (p \rightarrow q)) \rightarrow p$
F	F	T	T	T	F
F	T	T	T	T	F
T	F	F	F	F	T
T	T	F	T	F	T

→ This is not a tautology

Q1 $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology or not.

p	q	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
F	F	F	T
F	T	F	T
T	F	F	T
T	T	T	T

Not
tautology
A

Logical Equivalence using truth table

Q2 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

Show that $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.

P	q	r	$p \rightarrow r$	$q \rightarrow r$	$p \wedge q$	$(p \rightarrow r) \vee (q \rightarrow r)$	$(p \wedge q) \rightarrow r$
F	F	F	T	T	F	T	T
F	F	T	T	T	F	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	F	T	T
T	F	F	F	T	F	T	T
T	F	T	T	T	F	T	T
T	T	F	F	F	T	F	F
T	T	T	T	T	T	T	T

So, $(p \rightarrow r) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$ are logically equivalent.

Logic Laws

① Double negation law: $\neg(\neg P) \equiv P$

② Idempotent law: $P \vee P \equiv P$
 $P \wedge P \equiv P$

③ Identity law: $P \wedge T \equiv P$
 $P \vee F \equiv P$

→ P true (T) or false (F)
if P is true (T) then $P \wedge T$ is true (T)
if P is false (F) then $P \wedge T$ is false (F)

④ Domination law: $P \vee T \equiv T$
 $P \wedge F \equiv F$

⑤ Commutative law: $P \vee Q \equiv Q \vee P$

$(P \wedge Q) \equiv Q \wedge P$

⑥ Associative law: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

⑦ Inverse law: $P \vee \neg P \equiv T$
 $P \wedge \neg P \equiv F$

⑧ Absorption law: $P \vee (P \wedge Q) \equiv P$
 $P \wedge (P \vee Q) \equiv P$

⑨ Distributive law: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

⑩ Conditional law: $p \rightarrow q \equiv \neg p \vee q$

⑪ Biconditional law: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

⑫ De Morgan's law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(\neg R \wedge \neg S \wedge \neg T) \equiv \neg \neg R \vee \neg \neg S \vee \neg \neg T$$

$$\neg(R \vee S \vee T) \equiv \neg R \wedge \neg S \wedge \neg T$$

~~with~~ $\neg(R \wedge S \wedge T) \wedge \neg(R \vee S \vee T)$

$$\equiv (\neg R \vee \neg S \vee \neg T) \wedge (\neg R \wedge \neg S \wedge \neg T)$$

$$\equiv (\bar{R} + \bar{S} + \bar{T}) \wedge (\bar{R} \cdot \bar{S} \cdot \bar{T})$$

$$\equiv \bar{R} \bar{R} \bar{S} \bar{T} + \bar{R} \bar{S} \bar{T} \bar{S} + \bar{R} \bar{S} \bar{T} \bar{T}$$

$$\equiv \bar{R} \bar{S} \bar{T} + \bar{R} \bar{S} \bar{T} + \bar{R} \bar{S} \bar{T}$$

$$\equiv \bar{R} \bar{S} \bar{T}$$

$$\equiv \neg R \wedge \neg S \wedge \neg T$$

De Morgan's law to find the negation:

Question: Use De Morgan's law to find the negation of each of the following statement.

① John is rich and happy.

$$\rightarrow \neg(p \wedge q) \equiv \neg p \vee \neg q$$

John is not rich or not happy.

② Carlos will bicycle or run tomorrow.

$$\rightarrow \neg(p \vee q) \equiv \neg p \wedge \neg q$$

Carlos will not bicycle and not run tomorrow.

Question: Show that each of these conditional statement is a tautology without using truth table.

① $(p \wedge q) \rightarrow p$

$$\equiv \neg(p \wedge q) \vee p$$

$$\equiv \neg p \vee \neg q \vee p$$

$$\equiv \neg p \vee p \vee \neg q$$

$$\equiv T \vee \neg q$$

$$\equiv T$$

→ Tautology

② $(p \wedge q) \rightarrow (p \vee q)$

$$\equiv \neg(p \wedge q) \vee (p \vee q)$$

$$\equiv \neg p \vee \neg q \vee p \vee q$$

$$\equiv \neg p \vee p \vee \neg q \vee q$$

$$\equiv T \vee T$$

$$\equiv T$$

→ Tautology

Questions Show that each of statement are logically equivalent without using truth table

(a) $\neg(p \rightarrow q) \equiv p \wedge \neg q$ (b) $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

L.H.S. = $\neg(p \rightarrow q)$

= $\neg(\neg p \vee q)$

= $\neg(\neg p) \wedge \neg q$

= $p \wedge \neg q$

= R.H.S.

L.H.S. = $\neg(p \vee (\neg p \wedge q))$

= $\neg p \wedge \neg(\neg p \wedge q)$

= $\neg p \wedge [\neg(\neg p) \vee \neg q]$

= $\neg p \wedge (p \vee \neg q)$

= $(\neg p \wedge p) \vee (\neg p \wedge \neg q)$

= $F \vee (\neg p \wedge \neg q)$

= $\neg p \wedge \neg q$

= R.H.S.

Predicates and Quantifiers

Predicates:

A predicate is a statement that contains variables and becomes a proposition when specific values are substituted for those variables. It can be thought of as a function that returns a true or false value depending on the inputs. For example,

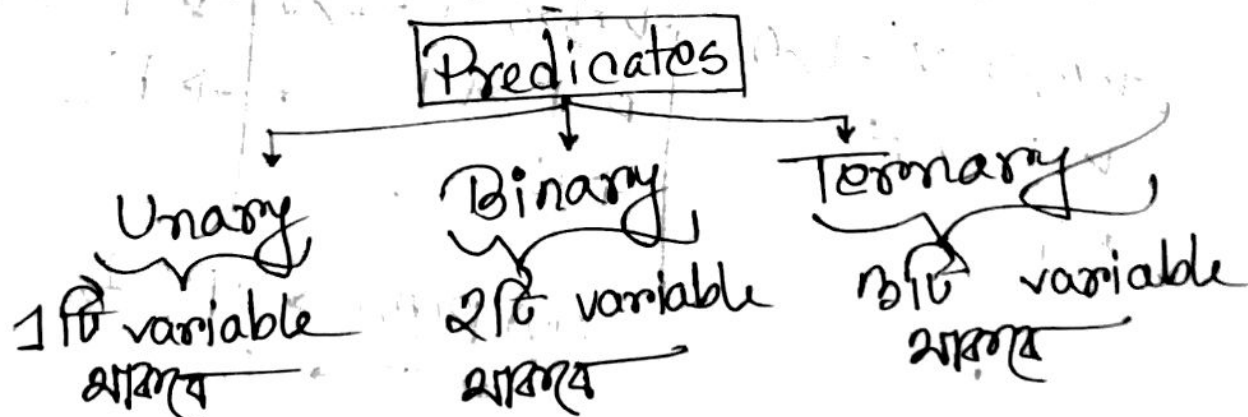
$P(x)$ could be " x is greater than 5."

variable/
subject predicate

$P(x) = x \text{ is greater than } 5$

$x = 4$; $P(4) = 4 \text{ is } " " 5 \rightarrow \text{False}$

$x = 6$; $P(6) = 6 \text{ is } " " 5 \rightarrow \text{True}$



Question: Let $P(x)$ denote the statement " $x > 3$ ". what are the truth values of $P(4)$ and $P(2)$?

$$\rightarrow P(x) = x > 3$$

$$P(4) = 4 > 3 \rightarrow \text{True (T)}$$

$$P(2) = 2 > 3 \rightarrow \text{False (F)}$$

Question: Let $Q(x, y)$ denote the statement " $x = y + 3$ ". what are the truth values of the proposition $Q(1, 2)$ and $Q(3, 0)$?

$$\rightarrow Q(x, y) \Rightarrow x = y + 3$$

$$Q(1, 2) \Rightarrow 1 = 2 + 3 \rightarrow \text{False (F)}$$

$$Q(3, 0) \Rightarrow 3 = 0 + 3 \rightarrow \text{True (T)}$$

Compound Expression:

Question: If $P(x)$ denotes " $x > 0$ ", find these truth values.

$$\textcircled{I} P(3) \vee P(-1)$$

$$\rightarrow 3 > 0 \vee -1 > 0$$

$$\rightarrow T \vee F$$

$$\rightarrow T$$

$$\textcircled{II} P(3) \wedge P(-1)$$

$$\rightarrow 3 > 0 \wedge -1 > 0$$

$$\rightarrow T \wedge F$$

$$\rightarrow F$$

$$\textcircled{III} P(3) \rightarrow P(-1)$$

$$\rightarrow T \rightarrow F$$

$$\rightarrow F$$

$$\textcircled{IV} P(3) \rightarrow \neg P(-1)$$

$$\rightarrow T \rightarrow \neg F$$

$$\rightarrow T$$

Quantifiers: Quantifiers specify the extent to which a predicate applies to a set of elements. The two main types are:

Universal Quantifier ($\forall$): Indicates that a predicate is true for all elements in a domain.

for all, for every, for each, for any,
all of, given any.....

$\forall x P(x)$ means " $P(x)$ means is true for all x ."

Existential Quantifier ($\exists$): Indicates that there is at least one element in a domain for which the predicate is true. For example,
 $\exists x P(x)$ means "There exist at least one x such that $P(x)$ is true."

there exist, for some, for at least one,

Question: Let $Q(x)$ be the statement " $x < 2$ " what is the truth value of the quantification $\forall x Q(x)$ where the domain consists of all real numbers?

$$\rightarrow Q(x) = x < 2$$

$$\forall x Q(x) \rightarrow Q(x) = x < 2$$

$$\Rightarrow Q(1) = 1 < 2$$

$$\Rightarrow Q(2) = 2 < 2 \rightarrow \text{False}$$

So, the truth value is False.

Question: What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement " $x^2 < 10$ " and the domain consists of the positive integers not exceeding 4?

$$\rightarrow P(x) = x^2 < 10 \quad \text{domain } x = \{0, 1, 2, 3, 4\}$$

$$\forall x P(x) = x^2 < 10$$

$$P(0) = T$$

$$P(1) = T$$

$$P(2) = T$$

$$P(3) = T$$

$$P(4) = F$$

So, the truth value is False.

Question: What is the truth value of $\exists x P(x)$ where $P(x)$ is the statement " $x^2 > 10$ " and the domain consists of the positive integers not exceeding 4?

$$\rightarrow \exists x P(x) = x^2 > 10$$

$$= 0^2 > 10$$

$$1^2 > 10$$

$$2^2 > 10$$

$$3^2 > 10$$

$$4^2 > 10 \rightarrow \text{True}$$

For x from
Given the statement $P(x)$
defined as " $x^2 > 10$ " and considering
the domain of the positive integers not
exceeding 4, let's evaluate the expression.
The statement $P(x)$ is a false statement
because 4 can't satisfy the expression " $x^2 > 10$ ".
Since $P(x)$ is false for 4, the
expression $\exists x P(x)$ is also false.
Therefore, the truth value of $\exists x P(x)$
is false because there is no x that
can satisfy the equation.

So, the truth value is True.

☐ Translating from English into logical expression using predicate and quantifiers:

① Every student in the class has studied calculus.

$\rightarrow \forall x Q(x)$ = For every student x in the class,
 x has studied calculus.

② For every person x , if person x is a student in the class then x has studied calculus.

$\rightarrow P(x)$ = x is a student

$R(x)$ = x has studied calculus

$\therefore \forall x (P(x) \rightarrow R(x))$

• Universal Quantifier ($\forall x$) $\forall x (A(x) \rightarrow B(x))$

• Existential Quantifier ($\exists x$) $\exists x (A(x) \wedge B(x))$

⑤ Some student in this class has visited US.

→ $\exists x P(x)$ = There is a student x in the class having the properties that x has visited US.

④ There is a person x , having the properties that x is a student in the class and x has visited US.

→ $\exists x (S(x) \wedge M(x))$

⑥ For every person x , if x is a student in the class, then x has visited US or x has visited Mexico.

→ $\forall x (P(x) \rightarrow R(x) \vee S(x))$

Using Quantifier in system specification -

⑥ Every mail message larger than one megabyte will be compressed.

→ $S(m, y) = m$ message larger than y megabyte.
 $C(m) = m$ is a message that will be compressed

$$\forall m (S(m, y) \rightarrow C(m))$$

Question: Let $P(x)$ be the statement " x can speak English" and let $Q(x)$ be the statement " x knows the computer language C++".

Express each of these sentences in terms of $P(x)$, $Q(x)$ quantifiers and logical connectives.

Domain → all students at your school.

(a) There is a student at your school who can speak English and who knows C++.

$$\rightarrow \exists x (P(x) \wedge Q(x))$$

(b) There is a student at your school who can speak English but who doesn't know C++.

$$\rightarrow \exists x (P(x) \wedge \neg Q(x))$$

③ No student at your school can speak English or know C++.

$$\rightarrow \exists x (P(x) \vee Q(x))$$

④ Every student at your school either can speak English or knows C++.

$$\rightarrow \forall x (P(x) \vee Q(x))$$

□ Logical expression to English:

$P(x)$ is "x is a gammer"

$Q(x)$ is "x is funny"

Domain $\rightarrow$ all people

$$\textcircled{a} \forall x (P(x) \rightarrow Q(x))$$

$\rightarrow$ For all x, if x is a gammer then x is funny person.

or,

Every gammer is funny.

$$\textcircled{1} \forall x (P(x) \wedge Q(x))$$

→ For all x , x is a gammero and x is funny.

Or, Every person is a funny gammer.

□ Negating Quantified Expression:

$$\begin{aligned} \forall x P(x) &\equiv \neg (\exists x \neg P(x)) \\ \neg (\forall x P(x)) &\equiv \exists x \neg P(x) \\ \neg (\exists x P(x)) &\equiv \forall x \neg P(x) \end{aligned}$$

① Every student in your class has taken a course in calculus.

$$\begin{aligned} \rightarrow \forall x P(x) &\equiv \neg (\exists x \neg P(x)) \\ &\equiv \neg (\exists x \neg C(x)) \end{aligned}$$

There is a student in your class who has not taken a course in calculus.

② There is an honest man.

$$\rightarrow \exists x M(x) \equiv \neg (\forall x \neg M(x))$$

Not all man are honest.

③ Some old dogs can learn new tricks.

domain $\rightarrow$ all dogs.

$\rightarrow P(x) = x$ can learn new tricks

$$\exists x P(x) \equiv \neg(\forall x \neg P(x)) \equiv \forall x \neg \neg P(x)$$

$\rightarrow$ No old dogs can learn new tricks.

④ No Rabbit knows Calculus.

$\rightarrow P(x) = x$ knows calculus

$$\forall x \neg P(x) \equiv \neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

$\rightarrow$ There is at least one Rabbit that knows calculus.

⑤ Every bird can fly.

$\rightarrow P(x) = x$ can fly

$$\forall x P(x) \equiv \neg(\exists x \neg P(x)) \equiv \forall x P(x)$$

$\rightarrow$ There exist a bird that can't fly

Nested Quantifier:

$$\forall x \exists y P(x, y) \leftarrow \text{Nested Quantifier}$$

Question: Translate the following statement into English.

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

Domain: Real number

→ For every real number x and y , if x is positive and y is negative then xy is negative.

Question: Statement to logical expression.

The sum of two positive integers is always positive.

$$\rightarrow \forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

Question: Translate nested quantifiers into English

$$\forall x (C(x) \vee \exists y (C(y) \wedge F(x, y)))$$

$C(x)$ = x has a computer

$C(y)$ = y " " " "

$F(x, y)$ = x and y are friends

Domain = All students

→ For every student in your school, has a computer or there is a student that has a computer and they are friends.

Question: Use quantifiers to express each of these statements.

Let $L(x, y) = "x \text{ loves } y"$

Domain = All people

(a) Everybody loves Jerry.

→ $\forall x L(x, \text{Jerry})$

(b) Everybody loves somebody.

→ $\forall x \exists y L(x, y)$

(c) There is somebody whom everybody loves.

→ $\exists y \forall x L(x, y)$

Important

① $\forall x \exists y (x = y^2)$

② $\exists y \forall x (x = y^2)$

False

For Exam Question: Express the statements -

"Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

→ Let:

$S(x)$ represent " x is a student in this class"

$C(x)$ represent " x has visited Canada"

$M(x)$ represent " x has visited Mexico."

"Some student in this class has visited Mexico":

$$\exists x (S(x) \wedge M(x))$$

This means "There exists at least one student x such that x is a student in this class and x has visited Mexico."

"Every student in this class has visited either Canada or Mexico": $\forall x (S(x) \rightarrow (C(x) \vee M(x)))$

This means "For every student x , if x is a student in this class, then x has visited either Canada or Mexico."

Free variable:

A free variable in logic is, a variable that appears in a logical expression but is not bound by any quantifiers (such as $\forall$ for "for all" or $\exists$ for "there exists").

This means its value is not fixed by the expression and can vary within the domain.

Example

Consider the predicate $P(x)$ where $P(x)$ represents " $x > 5$ ".

→ In the expression $P(x)$, x is a free variable. The expression $x > 5$ is not a complete logical statement on its own because it does not specify what x is.

→ To make the complete statement, you might use a quantifier.

$$\begin{aligned} \text{even} &= 2k \\ \text{odd} &= 2k+1 \end{aligned}$$

Basic Proof Techniques:

① Direct Proof: (P true $\Rightarrow$ q true)
 $P \rightarrow Q$

Example:

① If n is odd, then n^2 is odd.

$\rightarrow p: n$ is odd.

$q: n^2$ is odd

Let's say,

$n = 2k+1$; where k is an integer

$$\therefore n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2m + 1 \quad [m = 2k^2 + 2k \text{ let}]$$

$$\therefore n^2 = 2m + 1$$

Since n^2 can be written in this form, it is odd.

② Give direct proof of the statement

"the product of two odd integers is odd"

$\rightarrow$ Let's say, $a = 2m+1$; where a, b are two odd integers.
 $b = 2n+1$

$$\begin{aligned}
 & (2m+1)(2n+1) \\
 &= 4mn + 2m + 2n + 1 \\
 &= 2(2mn + m + n) + 1 \\
 &= 2k+1 \quad [\text{let } k = 2mn + m + n]
 \end{aligned}$$

$\therefore 2k+1$ is odd. for this reason, the product of two odd integers is odd.

② Indirect Proof (Contraposition/Contrapositive):
 $\neg q$ - true $\Rightarrow$ original statement is true.

$$p \rightarrow q$$

Contrapositive: $\neg q \rightarrow \neg p$

Example:

Give an indirect proof of the theorem
 "If $3n+2$ is odd, then n is odd".

$\rightarrow$ p : $3n+2$ is odd logical expression: $p \rightarrow q$
 q : n is odd

contrapositive: $\neg q \rightarrow \neg p$

$\neg q = n$ is even $\neg p = 3n+2$ is even

$$\therefore n = 2k$$

$$\therefore 3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1) \\ \equiv 2m \quad [m=3k+1]$$

So, $3n+2$ is even, if n is even.

For this reason, we can say that, if $3n+2$ is odd then n is odd.

③ Proof by contradiction: ($\neg q$ prove that p is false) $\neg q$ prove that p is false

Show by contradiction that if $3n+2$ is odd then n is odd.

$\rightarrow p: 3n+2$ is odd
 $q: n$ is odd

$\neg q: n$ is even
 $\therefore n = 2k$

now,

$$3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1) \\ \equiv 2m \quad [m=3k+1]$$

$3n+2$ is even, which contradicts the given statement. So, n should be odd.

Standard Form

Type	Standard Form
Even number	$2k$, where k is an integer
Odd number	$2k+1$ or $2k-1$, where k is an integer.
Multiple of k	kn , where n is an integer
Perfect square	k^2 , where k is an integer
Rational number	$\frac{p}{q}$, where p, q are integers and $q \neq 0$.
Irrational numbers	NO STANDARD FORM

Practice

☐ Use direct proof to show that if n is even, then $7n+4$ is even. (n is an integer)



Given that, n is even.
 $n = 2k$ [k is an integer]

So,

$$7n+4 = 7(2k)+4 = 14k+4 = 2(7k+4) \\ = 2L \quad [L = 7k+4]$$

Let, L is an integer, so $2L$ is even
 $\therefore 7n+4$ is even.

▣ Using direct proof technique, prove that if x is even and y is odd, then xy is even.

→ Here,

x is even

$\therefore x = 2k$ [k is an integer]

y is odd

$\therefore y = 2k+1$ [k is an integer]

$$\text{So, } xy = (2k)(2k+1)$$

$$= 4k^2 + 2k$$

$$= 2(2k^2 + k)$$

$$= 2L \quad [L = \underbrace{2k^2 + k}_{\text{integer}}]$$

~~Though $2L$ is even, so xy is also even.~~

Since, k is an integer, therefore L is an integer.

$2L$ is even

$\therefore xy$ is even.

▣ Using direct proof technique, prove that The sum of two rational numbers is rational.

→ Let,

$$a = \frac{p}{q} ; q \neq 0$$

$$b = \frac{p}{q} ; q \neq 0$$

here, p and q is integer

$$a+b = \frac{p}{q} + \frac{p}{q} = \frac{2p}{q} \quad \text{where } q \neq 0$$

So, there $\frac{2p}{q}$ is rational
for this reason, $a+b$ is rational.

☐ If n is an integer and $7n+2$ is even, then n is even using proof by contraposition.

→ Contrapositive: If n is not even, then $7n+2$ is not even.

Given that,

n is not even, n is odd

$$\therefore n = 2k+1$$

$$\begin{aligned}\text{So, } 7n+2 &= 7(2k+1)+2 = 14k+7+2 = 14k+9 \\ &= 14k+8+1 = 2(7k+4)+1 \\ &= 2L+1 \quad [L = 7k+4, \text{ where } L \text{ is integer}]\end{aligned}$$

So ~~there~~ here $2L+1$ is odd so $7n+2$ is also odd.

Now, we can say that if n is odd then $7n+2$ is also odd.

☐ Using proof by contraposition, prove that: "For all integer n , if n^2+5 is even, then n is odd"

→

It is sufficient to prove that, if n is not odd then n^2+5 is not even.

Given that,

n is even

$$\therefore n = 2k \text{ [} k \text{ is an integer]}$$

Here,

$$\begin{aligned} n^2 + 5 &= (2k)^2 + 5 = 4k^2 + 5 = 4k^2 + 4 + 1 = 2(k^2 + 2) + 1 \\ &= 2L + 1 \text{ [} L = k^2 + 2 \text{]} \end{aligned}$$

Since k is an integer therefore L is an integer

$\therefore 2L + 1$ is odd

For this reason, $n^2 + 5$ is odd.

So, we can say that if $n^2 + 5$ is even, then n is odd.

□ Prove by contraposition, if $n^3 - 1$ is even, then n is odd.

→ It is sufficient to prove that if n is not odd, then $n^3 - 1$ is not even.

~~Contrapos~~

Given that, $n = 2k$ [k is an integer]

$$\begin{aligned} \text{Here, } n^3 - 1 &= (2k)^3 - 1 = 8k^3 - 1 = 2(4k^3) - 1 \\ &= 2L - 1 \text{ [} L = 4k^3 \text{]} \end{aligned}$$

odd

□ Prove that if n is an integer and $3n+2$ is even, then n is even using proof by contradiction.



Assume that,

n is not even. So, n is odd.

$$\therefore n = 2k+1 \text{ [} k \text{ is an integer]}$$

$$\text{Here, } 3n+2 = 3(2k+1)+2 = 6k+3+2 = 6k+5$$

$$= 2(2k+2)+1 = 2L+1 \text{ [} L=2k+2 \text{]}$$

Since k is an integer number therefore L is an integer number.

$2L+1$ is odd. For this reason $3n+2$ is odd.

Here we find $3n+2$ as odd. But $3n+2$ is actually even. So, our assumption lead to a contradiction.

So, we conclude that n is even.

□ Show using proof by contradiction that if n is an integer and n^3+5 is odd, then n is even.

→ Assume that, n is not even means n is odd
i.e. $n = 2k+1$ [k is an integer]

$$\begin{aligned}\text{Here, } n^3+5 &= (2k+1)^3+5 = (2k)^3+(1)^3+3 \cdot 2k \cdot 1(2k+1)+5 \\ &= 8k^3+1+6k(2k+1)+5 \\ &= 8k^3+12k^2+6k+6 \\ &= 2(4k^3+6k^2+3k+3) \\ &= 2L \quad [L=4k^3+6k^2+3k+3]\end{aligned}$$

Since k is an integer number therefore L is an integer number.

$2L$ is even. For this reason, n^3+5 is even. Here we find n^3+5 is even but n^3+5 is actually odd. So our assumption lead to a contradiction.

So, we conclude that n is even.

Not Mentioned

Has standard
Form

Direct Proof

Proof by Contradiction

Proof by Contraposition

No Standard Form

- Prove that if n is a positive integer and n is even, then $7n + 4$ is even. $\rightarrow$ direct proof

indirect
or contraposition proof

- Prove that if x, y are even integers, then $x+z$ is also even.

even.

Proof by contradiction

Assume P is true

Then Q is true

But Q is false

Contradiction

Therefore P is false

Q.E.D.

- Prove that, if x is irrational then $1/x$ is also irrational.

- Prove that if x is an irrational number and $x > 0$, then $\sqrt{x}$ is also irrational.

1. ଆମେ ମାଧ୍ୟମିକ ଓ ଅନ୍ତରାଳ
 ଓ ଏହାଙ୍କର standard form
 କ୍ଷତି ଅନ୍ତରାଳରେ ଉଚିତ
 ଯାହା contradiction ଦର୍ଶାଏ

- Prove that, if n is a perfect square, then $n+2$ is not a perfect square.

☐ Prove that, if x is irrational, then $1/x$ is also irrational.

→

It is sufficient to prove that, if $\frac{1}{x}$ is not irrational then, x is not irrational.

Given that,

$\frac{1}{x}$ is not irrational means $\frac{1}{x}$ is rational

$$\therefore \frac{1}{x} = \frac{p}{q} \quad [q \neq 0]$$

Here,

$$\frac{1}{x} = \frac{p}{q} \Rightarrow x = \frac{q}{p}$$

here, p is an integer, h

p can be 0. That's

why $\frac{q}{p}$ can not be rational.

~~here~~

$\therefore x$ is irrational.

☐ Prove that, if n is a perfect square, then $n+2$ is not a perfect square.

→

Assume that, $n+2$ is perfect square.

$$\therefore n+2 = k^2 \quad [k \text{ is an integer}]$$

$$\Rightarrow n = k^2 - 2$$

$$\Rightarrow k^2 - 2 = m^2 \quad [\text{Assume that } k^2 - 2 \text{ is perfect square}]$$

$$k^2 - 2 = m^2$$

$$\Rightarrow k^2 - m^2 = 2 \Rightarrow (k+m)(k-m) = 2$$

Now,

Case-1:

$$k+m=2$$

$$k-m=1$$

$$\therefore (k+m) + (k-m) = 2+1$$

$$\Rightarrow 2k = 3$$

$$\therefore k = \frac{3}{2}$$

Case-2:

$$k+m=1$$

$$k-m=2$$

$$\therefore (k+m) + (k-m) = 1+2$$

$$\Rightarrow 2k = 3$$

$$\therefore k = \frac{3}{2}$$

we found k as a fraction number but k is an integer. Here, our assumption lead to a contradiction.

So, we conclude our assumption was false.

$\therefore n+2$ is not a perfect square.

Set

Definition: A set is a collection of distinct and well-defined objects or elements.

- $V = \{a, e, i, o, u\}$ is the set of all English vowels.
 - $a \in V$ means a is a member of V .
- $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is the set of all positive integers less than or equal to ten.

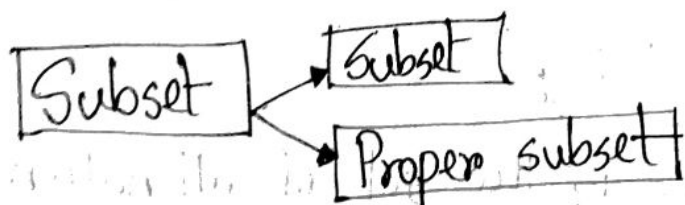
→ Roster method $[S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}]$

→ Set builder method: $S = \{x \in \mathbb{Z}^+ | x \leq 10\}$

• $S_1 = \{x \in \mathbb{Z}^+ | x \text{ is a multiple of } 2 \text{ and } x \leq 4\}$.

$S_2 = \{x \in \mathbb{Z}^+ | x \text{ is a positive integer power of } 2 \text{ and } x \leq 4\}$

$S_1 = S_2$, because both have the members 2 and 4.



Subset ($\subseteq$): A subset is a set where every element is also a member of another set.

A set A is a subset of set B (denoted $A \subseteq B$) if every element of A is also an element of B . This includes the possibility that $A = B$.

$$A = \{1, 2, 3\} ; B = \{1, 2, 3\} \quad \boxed{\rightarrow A \subseteq B}$$

Proper Subset ($\subset$): A proper subset is a subset of a set that contains some, but not all, elements of the original set, meaning the proper subset is always smaller than the original set.

A set A is a proper subset of set B (denoted $A \subset B$) if every element of A is in B and A is not equal to B , meaning A contains fewer elements than B .

$$A = \{1, 3\} ; B = \{1, 2, 3, 4\} \quad \boxed{\rightarrow A \subset B}$$

Some Basic Sets:

- $N = \{0, 1, 2, 3, \dots\}$ is the set of all natural numbers
- $Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all integers
→ $Z^+ = \{1, 2, 3, \dots\}$ is the set of all positive integers

• $\mathbb{Q} = \{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z} \text{ and } q \neq 0 \}$ is the set of all rational numbers

→ $\mathbb{Q}^+$ is the set of all positive rational numbers

• $\mathbb{R}$ is the set of all real numbers.

Cardinality:

→ Cardinality is the number of elements in a set, representing the "size" of the set.

• $V = \{a, e, i, o, u\}$ → Cardinality of V , $|V| = 5$

• $|\emptyset| = 0$

Power Set:

→ The power set of a set is the set of all possible subsets of that set, including the empty set and the set itself.

$P(\{1, 2\}) = \{ \{1\}, \{2\}, \{1, 2\}, \emptyset \}$ → Cardinality of $P(\{1, 2\})$
↓
 $4 = 2^2$

So, Cardinality of power set = 2^n

n = number of elements.

- $P(\emptyset) = \{\emptyset\} \rightarrow$ cardinality of $P(\emptyset) = 2^0 = 1$
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} \rightarrow$ cardinality of $P(\{\emptyset\}) = 2^1 = 2$

Cartesian Product:

$\rightarrow$ The Cartesian product of two sets A and B is a set formed by pairing every element of A with every element of B , resulting in a collection of ordered pairs (a, b) where a is an element of A and b is an element of B .

- $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

- $A \times B \times C = \{(x, y, z) \mid x \in A \wedge y \in B \wedge z \in C\}$

- $A = \{1, 2, 3\}, B = \{a, b\}$

$$\therefore A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Cardinality of $A \times B$:

$$A = \{1, 2, 3\} \rightarrow 3 \text{ elements}$$

$$B = \{a, b\} \rightarrow 2 \text{ elements}$$

$$A \times B = 3 \times 2 = 6$$

$$\therefore \text{Cardinality of } A \times B : 6$$

Q. $A = \{1, 2\}$, $B = \{a, b\}$, $C = \{\alpha, \beta\}$

Find out $A \times B \times C$.

→ $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$
 $A \times B \times C = \{((1, a), \alpha), ((1, a), \beta), ((1, b), \alpha), ((1, b), \beta), ((2, a), \alpha), ((2, a), \beta), ((2, b), \alpha), ((2, b), \beta)\}$

Wrong way

Right way:

$$A \times B \times C = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$$

Cardinality of $A \times B \times C : 2 \times 2 \times 2 = 8$
 (elements of A) (elements of B) (elements of C)

Union:

→ The union of two sets is the set that includes all elements from both sets, combining them into one set without any duplicates.

• $A = \{1, 2, 3\}$, $B = \{1, 3, 5\} \rightarrow A \cup B = \{1, 2, 3, 5\}$

• $A \cup B = \{x | x \in A \vee x \in B\}$

Intersection:

↳ The intersection of two sets is the set that contains only the elements that are common to both sets.

• $A = \{1, 2, 3\}$, $B = \{1, 3, 5\} \rightarrow A \cap B = \{1, 3\}$

• $A \cap B = \{x \mid x \in A \wedge x \in B\}$

• A member should exist in both of the sets

• Two sets are disjoint if $A \cap B = \emptyset$

↳ Two sets are called disjoint if their intersection is empty.

Cardinality of union set:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

• $A = \{1, 2, 3\}$, $B = \{1, 3, 5, 6\}$

$$A \cup B = \{1, 2, 3, 5, 6\} \quad A \cap B = \{1, 3\}$$

$$|A| = 3, |B| = 4, |A \cap B| = 2, |A \cup B| = 5$$

$$\therefore |A \cup B| = |A| + |B| - |A \cap B| = 3 + 4 - 2 = 5$$

Set operations - Difference:

• $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$

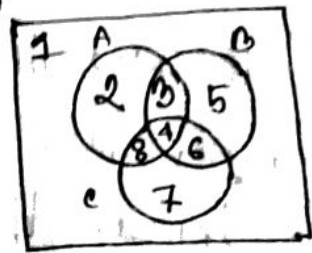
$$A - B = \{2\}$$

• $A - B = \{x \mid x \in A \wedge x \notin B\}$

• A member should exist only in the first set

☐ Draw Venn diagrams for the following sets:
(Assume universal set U if complement is involved)

• $A' \cup (B - C)$



→

$$A = \{2, 3, 4, 8\} \rightarrow A' = U - A = \{1, 2, 3, 4, 5, 6, 7, 8\} - \{2, 3, 4, 8\}$$

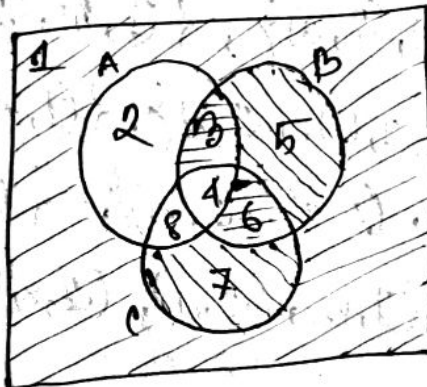
$$B = \{3, 4, 5, 6\} \rightarrow B - C = \{3, 4, 5, 6\} - \{4, 6, 7, 8\}$$

$$C = \{4, 6, 7, 8\}$$

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$B - C = \{3, 4, 5, 6\} - \{4, 6, 7, 8\} = \{3, 5\}$$

$$\therefore A' \cup (B - C) = \{1, 5, 6, 7\} \cup \{3, 5\} = \{1, 2, 5, 6, 7\}$$



• $((A \cap B) \cup C)'$

$$A \cap B = \{2, 3, 4, 8\} \cap \{3, 4, 5, 6\} = \{3, 4\}$$

$$(A \cap B) \cup C = \{3, 4\} \cup \{4, 6, 7, 8\} = \{3, 4, 6, 7, 8\}$$

$$((A \cap B) \cup C)' = U - ((A \cap B) \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8\} - \{3, 4, 6, 7, 8\} = \{1, 2, 5\}$$



Q Suppose you have two sets $A = \{1, 2\}$ and $B = \{a, b\}$

(i) Determine $A \times B$.

(ii) Find the power set $P(A \times B)$

(iii) Show that $|P(A \times B)| = 2^{|A| \cdot |B|}$

(i)

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

(ii)

$$P(A \times B) = \{ \{(1, a)\}, \{(1, b)\}, \{(2, a)\}, \{(2, b)\}, \\ \{(1, a), (1, b)\}, \{(1, a), (2, a)\}, \{(1, a), (2, b)\}, \\ \{(1, b), (2, a)\}, \{(1, b), (2, b)\}, \{(2, a), (2, b)\}, \\ \{(1, a), (1, b), (2, a)\}, \{(1, a), (1, b), (2, b)\}, \\ \{(1, a), (2, a), (2, b)\}, \{(1, b), (2, a), (2, b)\}, \\ \{(1, a), (1, b), (2, a), (2, b)\}, \emptyset \}$$

(iii)

From (ii) we can find, $|P(A \times B)| = 16$

$$|A| = 2, |B| = 2$$

$$\text{here, } 2^{|A| \cdot |B|} = 2^{2 \times 2} = 2^4 = 16$$

$$\therefore |P(A \times B)| = 2^{|A| \cdot |B|}$$

Given that A and B are two sets such that:

$$A \cap B = \{10, 22, 31, 76\}; B = \{50, 64, 97, 84\}$$

(i) Find out $A \cup B$. Order the elements of your set in ascending order.

(ii) Given that $A \cap B = \{50, 64, 97\}$; find out $|A|$.

(iii) Given that Set C is a single-element set containing the letter 'a'. Find out $P((A \cap B) \times C)$.

→ Given: $A \cap B' = \{10, 22, 31, 76\}; B = \{50, 64, 97, 84\}$

(i)

$A \cap B' = \{10, 22, 31, 76\}$ meaning these elements are in A but not in B

$$B = \{50, 64, 97, 84\}$$

The union of these sets is: $A \cup B = \{10, 22, 31, 50, 64, 76, 84, 97\}$

In ascending order: $A \cup B = \{10, 22, 31, 50, 64, 76, 84, 97\}$

(ii)

Here, $A \cap B = \{50, 64, 97\}$ means that these elements are common to both sets A and B.

Since $A \cap B = \{50, 64, 97\}$ and given $A \cap B' = \{10, 22, 31, 76\}$, the full set A is the union of those two:

$$A = (A \cap B) \cup (A \cap B') = \{10, 22, 31, 50, 64, 76, 97\}$$

So, the number of elements in A is: $|A| = 7$

(iii)

Set C contains the element 'a'. The Cartesian product $(A \cap B) \times C$ pairs each element of $A \cap B$ with the single element in C :

Since $A \cap B = \{50, 64, 97\}$, the Cartesian product is:

$$\begin{aligned}(A \cap B) \times C &= \{50, 64, 97\} \times \{a\} \\ &= \{(50, a), (64, a), (97, a)\}\end{aligned}$$

The cardinality $|(A \cap B) \times C|$ is 3 because there are three pairs in the product.

$$\text{Thus, } |P((A \cap B) \times C)| = 3$$

Find the following:

(i) If $A = \{1, 2\}$, then $|P(P(A \times P(A)))| = ?$

(ii) Let $S = \{x \in \mathbb{Z}^+ \mid x \text{ is a divisor of } 40 \text{ and } 10 \text{ and less than } 11\}$, find $P(S)$.

(i)

Here, $A = \{1, 2\}$ where $n = 2$.

$$|P(A)| = 2^n = 2^2 = 4$$

$$|A \times P(A)| = n \times 2^n = 2 \times 4 = 8$$

$$|P(A \times P(A))| = 2^8 = 256$$

$$|P(P(A \times P(A)))| = 2^{256}$$

(ii)

Divisor of 40 = 1, 2, 5, 8, 10, 20, 40

Divisor of 10 = 1, 2, 5, 10

The common divisor of 40 and 10 = {1, 2, 5, 10}
and they all are less than 11.

$$\therefore S = \{1, 2, 5, 10\}$$

$$\therefore P(S) = \{\{1\}, \{2\}, \{5\}, \{10\}, \{1, 2\}, \{1, 5\}, \{1, 10\}, \\ \{2, 5\}, \{2, 10\}, \{5, 10\}, \{1, 2, 5\}, \{1, 2, 10\}, \\ \{1, 5, 10\}, \{2, 5, 10\}, \{1, 2, 5, 10\}, \emptyset\}$$

Session - 21-22

☐ Let $A = \{0, 1, 2, 3\}$ and define relations R, S and T on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\},$$

$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}, \quad T = \{(0, 1), (2, 3)\}$$

(i) Is R reflexive? Symmetric? Transitive?

(ii) Is S reflexive? Symmetric? Transitive?

→ Relation R, S and T on $A = \{0, 1, 2, 3\}$:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$

$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$$

$$T = \{(0, 1), (2, 3)\}$$

(i)

Reflexive: A relation is reflexive if every element is related to itself, i.e., (a,a) is in the relation for every $a \in A$.

R contains $(0,0)$, $(1,1)$, $(2,2)$ and $(3,3)$.

So, R is reflexive.

Symmetric: A relation is symmetric if whenever (a,b) is in the relation, (b,a) is also in the relation.

R contains $(0,1)$ and $(1,0)$, but it also contains $(0,2)$ and $(3,0)$. All pairs have their symmetric counterparts. So, R is symmetric.

Transitive: A relation is transitive if whenever (a,b) and (b,c) are in the relation, (a,c) is also in the relation.

R contains $(0,1)$ and $(1,0)$, but $(0,0)$ is also present, which is not a necessary condition to break transitivity. However, there is no clear case where (a,b) and (b,c) are present without (a,c) . Therefore, R is transitive.

(ii)

Reflexive: S must have (a, a) for all $a \in A$. Since S does not include $(1, 1)$, $(2, 2)$ or $(3, 3)$. S is not reflexive.

Symmetric: S must have (b, a) ~~between~~ whenever (a, b) is present. S contains $(0, 2)$ but does not contain $(2, 0)$. So, S is not symmetric.

Transitive: If (a, b) and (b, c) are in S , then (a, c) must also be in S . In S , $(0, 2)$ and $(2, 3)$ are present, but $(0, 3)$ is already in S . So, S is transitive.

Function

- Natural Numbers

Set of N

$$N = \{0, 1, 2, 3, \dots\}$$

0 $\rightarrow$ Non-negative numbers

- Integers

Set of Z

$$Z = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

- Real Numbers

Set of R

$$R = \{\text{all}\}$$

Function

$\hookrightarrow$ a function is a rule that assigns each input from one set (called the domain) to exactly one output in another set (called the codomain).

If f is a function from A to B , we write

$$f: \underset{\substack{\uparrow \\ \text{Domain}}}{A} \rightarrow \underset{\substack{\uparrow \\ \text{Co-domain}}}{B} \quad (\text{Here } \rightarrow \text{ has nothing to do with if...then})$$

(i) $f: A \rightarrow B$, we can say that A is the domain of f and B is the co-domain of f .

(ii) $f(a) = b$, we can say that b is the image of ' a ' and a is the pre-image of ' b '

Domain

↳ The set of all possible input values for a function. (সমস্ত সম্ভাব্য input value গুলো)

Co-domain

↳ The set of all possible output values that could come from a function, even if some aren't used. (Possibly come out of a function)

Range

↳ The actual set of output values the function produces when you apply the function to every input in the domain. The range is a subset of the co-domain. (Actually comes out of a function)

Example: $x \rightarrow 2x + 1$

Domain: $\{1, 2, 3, 4\}$

Codomain: $\{1, 2, 3, 4, 5, 6, 7, 9, 10\}$

Range: $\{3, 5, 7, 9\}$

How to identify Function??

→ Only two condition.

- One input has one output.
- Every input must have output.

Is it Function or not??

1. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x \rightarrow$ Function (for one input give one output)

2. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \pm \sqrt{x} \rightarrow$ Not a function (for one input get more output)

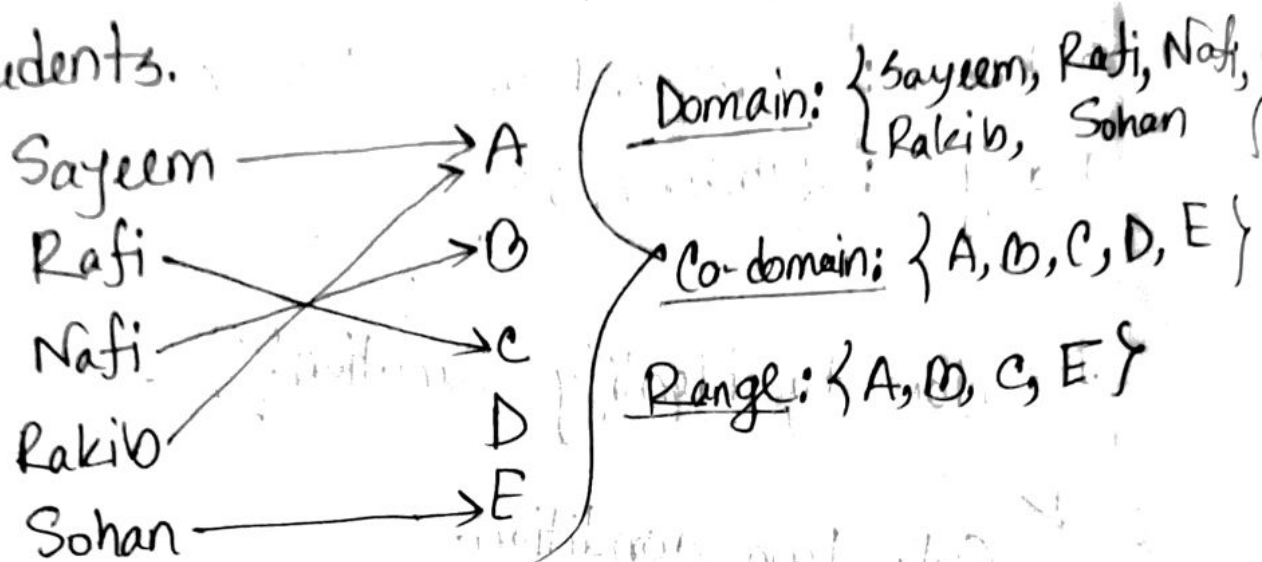
3. $f: \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = \pm \sqrt{x} \rightarrow$ Function (if $x=4$, then $f(x)=+2$, but $\mathbb{R}^+$, so we take only $+2$)

4. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x| \rightarrow$ Function

5. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2 \rightarrow$ Function

6. $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = \pm x \rightarrow$ Not a Function

Q What are the domain, codomain, and range of the function that assigns grades to students.



Q Let us take a look at the function:

$f: P \rightarrow C$ with

$C = \{\text{Boston, New York, Hongkong, Moscow}\}$

$P = \{\text{Alice, Max, Kathy, Peter}\}$

→ $f(\text{Alice}) = \text{Boston}$
 $f(\text{Max}) = \text{New York}$
 $f(\text{Kathy}) = \text{Hongkong}$
 $f(\text{Moscow}) = \text{Moscow}$

Sum and Product of Function:

f_1 and f_2 two function -

① Sum: $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

② Product: $(f_1 f_2)(x) = f_1(x) f_2(x)$

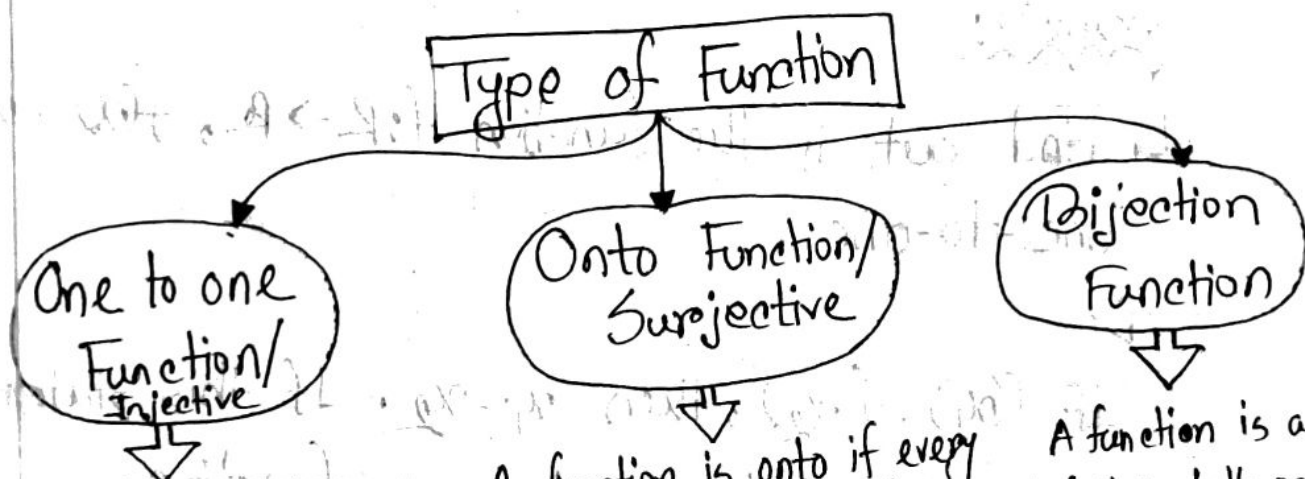
Example:

$$f_1(x) = 3x, f_2(x) = x+5$$

$$\text{Sum: } (f_1 + f_2)(x) = f_1(x) + f_2(x) = (3x)(x+5) \\ = 4x+5$$

$$\text{Product: } (f_1 f_2)(x) = f_1(x) f_2(x) = (3x)(x+5) \\ = 3x^2 + 15x$$

Type of Function



A function is one-to-one if every input gives a different output. No two inputs have the same output.

Different inputs give different outputs

Example: Fingerprint, Email

A function is onto if every possible output has at least one input that gives that output. Nothing in the output is left out.

• Every output has an input.
(Co-domain = Range)

• One output for different input is allowed.

Example: $f(x) = x^2$

A function is a bijection if it is both one-to-one and onto. This means that every input has a unique output and every output comes from a unique input.

One to one + Onto
Bijection

→ One to one
(One for one)

→ Onto
(Co-domain = Range)

Proof of One To One Function:

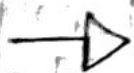
Technique-01: When we are sure that given Function is One to One / we are not sure.

$$f(x) = \dots \dots \dots (x) \dots \dots \dots$$

if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Example:

1. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x+1$ is one-to-one.



If $f(x_1) = f(x_2)$, then $x_1 = x_2$. If this statement is true then this is one-to-one function.

Assume that,

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1 + 1 = x_2 + 1$$

$$\Rightarrow x_1 = x_2 + 1 - 1$$

$$\therefore x_1 = x_2$$

So, we can say that, it is one-to-one function.

2. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is one-to-one.



Assume that,

$$f(x_1) = f(x_2)$$

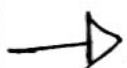
$$\Rightarrow x_1^2 = x_2^2 \Rightarrow \pm \sqrt{x_1^2} = \pm \sqrt{x_2^2} \Rightarrow \pm x_1 = \pm x_2$$

here,

$$+x_1 = +x_2 \text{ or } +x_1 = -x_2 \text{ or } -x_1 = +x_2 \text{ or } -x_1 = -x_2$$

So, it is not one-to-one function.

3. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x^2 - 5x + 1$ is one-to-one.



Assume that,

$$f(x_1) = f(x_2)$$

$$\Rightarrow 2x_1^2 - 5x_1 + 1 = 2x_2^2 - 5x_2 + 1$$

$$\Rightarrow 2(x_1^2 - x_2^2) = 5(x_1 - x_2)$$

$$\Rightarrow 2(x_1 + x_2)(x_1 - x_2) = 5(x_1 - x_2)$$

$$\Rightarrow 2x_1 + 2x_2 = 5$$

$$\Rightarrow 2x_1 = 5 - 2x_2$$

$$\therefore x_1 = \frac{5 - 2x_2}{2}$$

$$\left(\because x_1 \neq x_2 \right.$$

so, it isn't one-to-one function.

4. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \left|\frac{x}{2}\right| + x^4$ is one-to-one.

→

Assume that,

$$f(x_1) = f(x_2)$$

$$\Rightarrow \left|\frac{x_1}{2}\right| + x_1^4 = \left|\frac{x_2}{2}\right| + x_2^4 \Rightarrow \frac{x_1}{2} + x_1^4 = \frac{x_2}{2} + x_2^4$$

$$\Rightarrow \frac{x_1}{2} - \frac{x_2}{2} = x_2^4 - x_1^4 \Rightarrow \frac{x_1 - x_2}{2} = (x_2^2)^2 - (x_1^2)^2$$

$$\Rightarrow \frac{1}{2}(x_1 - x_2) = (x_2^2 + x_1^2)(x_2^2 - x_1^2)$$

$$\Rightarrow \frac{1}{2}(x_1 - x_2) = (x_2^2 + x_1^2)(x_2 + x_1)(x_2 - x_1)$$

$$\Rightarrow -\frac{1}{2}(x_2 - x_1) = (x_2^2 + x_1^2)(x_2 + x_1)(x_2 - x_1)$$

$$\Rightarrow x_2 + x_1 = -\frac{1}{2(x_2^2 + x_1^2)}$$

$$\therefore x_1 = -\frac{1}{2(x_2^2 + x_1^2)} - x_2$$

This is not one-to-one function.

Technique-02: When we are sure that given function is not one-to-one

"Counterexample"

Example:

1. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = |x| + x^2$ is one-to-one using counterexample.

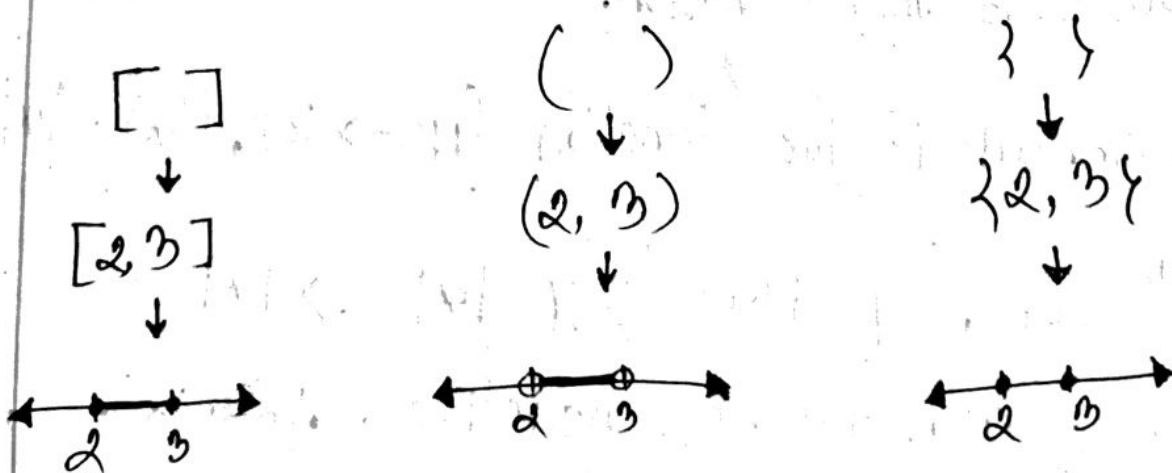


$$f(2) = |2| + 2^2 = 2 + 4 = 6$$

$$f(-2) = |-2| + (-2)^2 = 2 + 4 = 6$$

We are getting same output for different input.
Therefore, we can say $f(x)$ is not a one-to-one function.

☐ How to use close, open, middle Brackets?



Onto Function:

Co domain = Range

$$F: A \rightarrow C, F(x) = x+1$$

$$\begin{array}{ccc} 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \\ & \vdots & \\ 10 & \rightarrow & 11 \end{array}$$

$$\text{Domain, } A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$\text{Codomain, } C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

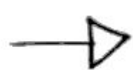
$$\text{Range, } R = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

co domain A
নেই ১
সোমানেই
আছে না

উত্তরে
Co-domain $\neq$ Range
So, it is not
onto function.

Example:

1. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x+1$ is onto.



$$y = f(x) = x+1$$

$$\Rightarrow x = y-1$$

$$\left\{ \begin{array}{l} \text{Range, } y = \text{All Real number} \\ \text{Codomain} = \mathbb{R} \text{ (All Real number)} \end{array} \right.$$

$$\therefore \text{Range} = \text{Codomain}$$

So, it is onto function.

Q. Find out if the function $f: \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = |x|$ is onto



$$\text{here, } y = f(x) \Rightarrow y = |x| \Rightarrow |x| = y$$

So, y can not be negative. $\therefore \text{Range} = [0, +\infty)$

but, Co-domain = $\mathbb{R}^+$ (0 not included)

So not onto

Inverse Function:

↳ An inverse function exists if the original function is bijective, meaning it pairs each input with a unique output^(one-to-one) and each output with a unique input^(onto), allowing the function to reverse the mapping from output back to input.

Example:

1. Find out the inverse of the following functions (if invertible):

i) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$

ii) $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x+1$

iii) $f: \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = e^x$

iv) $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}, f(x) = \frac{2x+1}{x-1}$

iii)

One-to-one:

$$f(x_1) = f(x_2)$$

$$\Rightarrow e^{x_1} = e^{x_2} \Rightarrow \ln e^{x_1} = \ln e^{x_2} \Rightarrow x_1 \ln e = x_2 \ln e$$

$$\Rightarrow x_1 \cdot 1 = x_2 \cdot 1 \quad [\because \ln e = \log_e e = 1]$$

$$\therefore x_1 = x_2$$

Onto :

$$y = f(x) = e^x \Rightarrow \ln(y) = \ln e^x \Rightarrow \ln(y) = x \ln e$$

$$\Rightarrow \ln(y) = x \cdot 1$$

$\therefore x = \ln(y)$ here, y can not be 0 and negative.

$$\therefore \text{Range} = \mathbb{R}^+$$

So, Range = Codomain

Now, e^x is Bijection. So, we can calculate inverse function of e^x .

here,

$$y = f(x) \Rightarrow y = e^x \Rightarrow \ln y = \ln e^x$$

$$\therefore x = \ln(y) \quad \text{--- (1)}$$

Again,

$$y = f(x)$$

$$\Rightarrow f^{-1}(y) = x \Rightarrow f^{-1}(y) = \ln(y) \quad [\text{From 1}]$$

$$\therefore f^{-1}(x) = \ln(x)$$

It is the inverse function.

Floor and Ceiling Functions:

Floor Function: (denoted as $\lfloor x \rfloor$)

- The floor of a real number x is the largest integer less than or equal to x .
- It rounds the number down to the nearest integer.

Example:

$$\rightarrow \lfloor 3.7 \rfloor = 3$$

$$\rightarrow \lfloor -2.3 \rfloor = -3$$

Ceiling Function: (denoted as $\lceil x \rceil$)

- The ceiling of a real number x is the smallest integer greater than or equal to x .
- It rounds the number up to the nearest integer.

Example:

$$\rightarrow \lceil 3.7 \rceil = 4$$

$$\rightarrow \lceil -2.3 \rceil = -2$$



→ The floor function always rounds down.
→ The ceiling function always rounds up.

Mathematical Induction

Mathematical induction is a method of proving that a statement is true for all natural numbers. It works in two steps:

- ① Base Case: First, you prove the statement is true for the smallest number (usually $n=1$ or $n=0$).
- ② Inductive Case: Then, you assume the statement is true for some number $n=k$ and you use this to prove it's true for $n=k+1$.

If both steps are true, the statement is true for all natural numbers.

Arithmetic Progression

3, 5, 7, 9, ...



First term = a

2nd Term = $a + d$

3rd Term = $a + 2d$

n^{th} Term = $a + (n-1)d$

$$\begin{aligned} \text{Sum of} \\ \text{Terms} \\ \text{(Series)} &= n \times \left[\frac{\text{FT} + \text{LT}}{2} \right] \\ &= \frac{n}{2} [2a + (n-1)d] \end{aligned}$$

Geometric Progression

3, 9, 27, 81, ...

$$\frac{9}{3} = 3, \frac{27}{9} = 3 \text{ (r)}$$

First Term = a

2nd Term = ar

3rd Term = ar^2

n^{th} Term = $ar^{(n-1)}$

$$\text{Sum} = \frac{a(1-r^n)}{1-r} \quad [r < 1]$$

Q. Show that if n is a positive integer, then

$$1 + 2 + \dots + n = \frac{n(n+1)}{2} \quad \text{where } n \geq 1$$



$$P(n) = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

Base Case (P(1)):

$$\text{L.H.S.} = P(1) = 1$$

$$\text{R.H.S.} = \frac{1(1+1)}{2} = 1$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Inductive Case: $\forall k, P(k) \rightarrow P(k+1)$

$$P(k) = 1+2+\dots+k = \frac{k(k+1)}{2}$$

$$P(k+1) = 1+2+\dots+k+(k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Assume that, $P(k)$ is true.

$$\begin{aligned} \text{L.H.S.} &= 1+2+\dots+k+(k+1) \\ &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

$= \text{R.H.S.}$

$\therefore \text{L.H.S.} = \text{R.H.S.}$

By the principle of mathematical induction, the given statement is true for all $n \geq 1$.

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

□ Show that if n is odd integer, then

$$1+3+5+\dots+(2n-1)=n^2$$

→

$$P(n) = 1+3+5+\dots+(2n-1) = n^2$$

Base Case (P1): L.H.S. = 1

$$R.H.S. = (1)^2 = 1$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Inductive Case: $\forall_k P(k) \rightarrow P(k+1)$

$$P(k) = 1+3+5+\dots+(2k-1) = k^2$$

$$P(k+1) = 1+3+5+\dots+(2k-1)+(2k+1) = (k+1)^2$$

Assume that, $P(k)$ is true

$$\begin{aligned} \text{L.H.S.} &= 1+3+5+\dots+(2k-1)+(2k+1) \\ &= k^2 + (2k+1) \quad [\text{from } P(k)] \\ &= k^2 + 2 \cdot k \cdot 1 + (1)^2 \\ &= (k+1)^2 \\ &= \text{R.H.S.} \end{aligned}$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

By the principle of mathematical induction, we can say that

$$1+3+5+\dots+(2n-1) = n^2$$

□ Show that if n is a non-negative integer, then $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

→
$$P(n) = 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$$

Base Case ($P(0)$):

$$\text{L.H.S.} = 2^0 = 1$$

$$\text{R.H.S.} = 2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Inductive Case: $\forall k, P(k) \rightarrow P(k+1)$

$$P(k) = 1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

$$P(k+1) = 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

Assume that $P(k)$ is true.

$$\text{L.H.S.} = 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1} \quad [\text{from } P(k)]$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+1+1} - 1 = 2^{k+2} - 1$$

$$= \text{R.H.S.}$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Q Show that if n is an nonnegative integer, then $a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r - 1}$

→

$$P(n) = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Base Case: $P(0)$:

$$\text{L.H.S.} = ar^0 = a \cdot 1 = a$$

$$\text{R.H.S.} = \frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r - 1)}{r - 1} = a$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Inductive Case: $\forall_k P(k) \rightarrow P(k+1)$

$$P(k) = a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

$$P(k+1) = a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}$$

Assume that, $P(k)$ is true

$$\text{L.H.S.} = a + ar + ar^2 + \dots + ar^k + ar^{k+1}$$

$$= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} \quad [\text{from } P(k)]$$

$$= \frac{ar^{k+1} - a + ar^{k+1} \cdot (r - 1)}{r - 1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1} \cdot r - ar^{k+1}}{r - 1}$$

$$= \frac{ar^{k+1+1} - a}{r - 1} = \frac{ar^{k+2} - a}{r - 1} = \text{R.H.S.}$$

Use mathematical induction to prove that $n^3 - n$ is divisible by 3. (n is a positive integer)



$$P(n) = n^3 - n$$

Base Case: $P(1) = 1^3 - 1 = 1 - 1 = 0$

here 0 is divisible by 3, so it's true.

Inductive Case: $\forall k, P(k) \rightarrow P(k+1)$

$$P(k) = k^3 - k$$

$$P(k+1) = (k+1)^3 - (k+1)$$

Assume that, $P(k)$ is true.

$$\text{Now, } P(k+1) = (k+1)^3 - (k+1)$$

$$= k^3 + 1^3 + 3 \cdot k \cdot 1 (k+1) - k - 1$$

$$= k^3 - k + 3k^2 + 3k + 1 - 1$$

$$= k^3 - k + 3k^2 + 3k$$

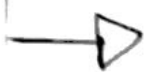
$$= 3m + 3k^2 + 3k \quad [k^3 - k = 3m]$$

$$= 3(m + k^2 + k)$$

Since $m + k^2 + k$ is an integer, we see that $(k+1)^3 - (k+1)$ is divisible by 3.

By mathematical induction, we have proven that $n^3 - n$ is divisible by 3 for all positive integers n .

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n .



$$P(n) = 7^{n+2} + 8^{2n+1}$$

Base Case: $P(0) = 7^{0+2} + 8^{2 \cdot 0 + 1} = 7^2 + 8^1 = 49 + 8 = 57$
here, 57 is divisible by 57. So, it's true.

Inductive Case: $\forall k, P(k) \rightarrow P(k+1)$

$$P(k) = 7^{k+2} + 8^{2k+1}$$

$$P(k+1) = 7^{k+3} + 8^{2k+3}$$

Assume that $P(k)$ is true.

$$\begin{aligned} \therefore 7^{k+3} + 8^{2k+3} &= 7^{k+2} \cdot 7^1 + 8^{2k+1} \cdot 8^2 \\ &= 7^{k+2} \cdot 7 + 8^{2k+1} \cdot 64 = 7^{k+2} \cdot 7 + 8^{2k+1} \cdot (57 + 7) \\ &= 7 \cdot 7^{k+2} + 57 \cdot 8^{2k+1} + 7 \cdot 8^{2k+1} \\ &= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} \\ &= 7 \cdot P(k) + 57 \cdot 8^{2k+1} \\ &= 7 \cdot 57x + 57 \cdot 8^{2k+1} \quad [P(k) = 7^{k+2} + 8^{2k+1} = 57x] \\ &= 57(7x + 8^{2k+1}) \end{aligned}$$

Since $7x + 8^{2k+1}$ is an integer, we see that $7^{k+3} + 8^{2k+3}$ is divisible by 57.

By mathematical induction, we.....

☐ Using Mathematical induction method, show that.

$$\sum_{i=0}^{n-1} 2^i (n-i) = 2^{n+1} - n - 2 \quad \text{where, } n \geq 1$$

→ $P(n) = \sum_{i=0}^{n-1} 2^i (n-i) = 2^{n+1} - n - 2$

Base Case: $P(1)$:

$$\text{L.H.S.} = \sum_{i=0}^{1-1} 2^i (1-i) = 2^0 (1-0) = 1$$

$$\text{R.H.S.} = 2^{1+1} - 1 - 2 = 2^2 - 1 - 2 = 4 - 1 - 2 = 1$$

~~THE~~ L.H.S. = R.H.S., Since both sides are equal, the base case holds.

Inductive Case: $\forall_k P(k) \rightarrow P(k+1)$

$$P(k) = \sum_{i=0}^{k-1} 2^i (k-i) = 2^{k+1} - k - 2$$

$$P(k+1) = \sum_{i=0}^k 2^i ((k+1)-i) = 2^{(k+1)+1} - (k+1) - 2$$

Assume that $P(k)$ is true:

$$\text{L.H.S.} = \sum_{i=0}^k 2^i ((k+1)-i) = \sum_{i=0}^{k-1} 2^i ((k+1)-i) + 2^k ((k+1)-k)$$

$$= \sum_{i=0}^{k-1} 2^i ((k-i)+1) + 2^k (k+1-k)$$

$$= \sum_{i=0}^{k-1} 2^i ((k-i)+1) + 2^k \cdot 1$$

$$= \sum_{i=0}^{k-1} 2^i (k-i) + \sum_{i=0}^{k-1} 2^i + 2^k$$

Now,

$$\textcircled{1} P(k) = \sum_{i=0}^{k-1} 2^i (k-i) = 2^{k+1} - k - 2$$

$\textcircled{2} \sum_{i=0}^{k-1} 2^i$ is the sum of geometric series.

- The first term, $a = 2^0 = 1$
- The common ratio, $r = 2$
- There are k terms.

Now the formula for the sum of a geometric series is: $S_n = \frac{a(r^n - 1)}{r - 1}$ when $r > 1$

Applying this:

$$\sum_{i=0}^{k-1} 2^i = \frac{1 \cdot (2^k - 1)}{2 - 1} = 2^k - 1$$

So, the sum is $2^k - 1$

$$= (2^{k+1} - k - 2) + (2^k - 1) + 2^k$$

$$= 2^{k+1} - k - 2 + 2^k - 1 + 2^k$$

$$= 2^{k+1} + 2^k - k - 3$$

$$= 2^{k+1} + 2^{k+1} - k - 1 - 2$$

$$= 2^1 \cdot 2^{k+1} - (k+1) - 2$$

$$= 2^{(k+1)+1} - (k+1) - 2$$

$$= R.H.S.$$

$$\therefore L.H.S. = R.H.S.$$

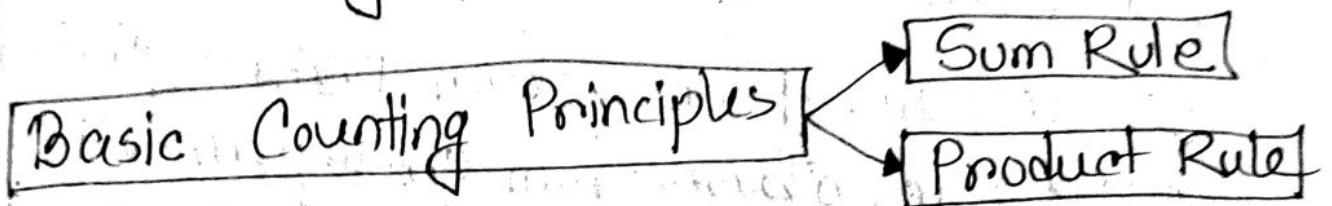
Thus, both sides are equal and the inductive step holds.

By the principle of mathematical induction, the given statement is true for all $n \geq 1$:

$$\sum_{i=0}^{n-1} 2^i (n-i) = 2^{n+1} - n - 2$$

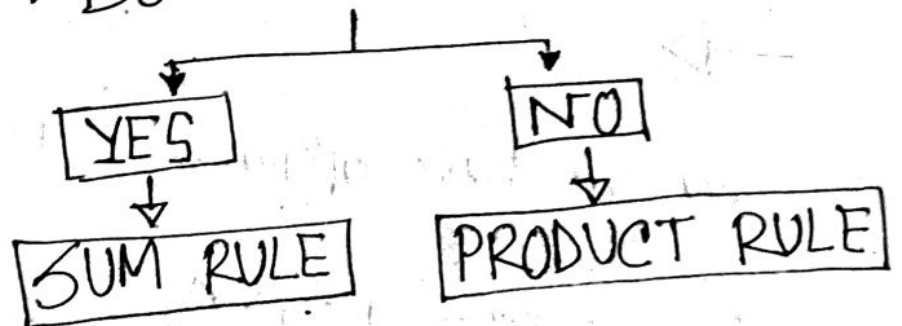
Counting

Counting in discrete mathematics is the process of finding how many ways an event can happen or how many elements are in a set.



How can check to use product or sum rule?

→ Just one condition have to check
→ Do we have Option?



Example:

Tour to Saint Martine:

The tour is divided into three parts:

Dhaka to Chattogram by train, Chattogram to Teknaf by bus and Teknaf to Saint Martins by ship.

① There are 4 trains from Dhaka to Chattogram

② There are 5 buses from Chattogram to Teknaf

③ There are 3 ships from Teknaf to Saint Martins.

How many possible ways to reach Saint Martins from Dhaka?



Do we have options? → No } Product Rule.

$$\begin{aligned}\text{Total way} &= 4 \times 5 \times 3 \\ &= 60\end{aligned}$$

Travelling from Dhaka to Rajshahi:

Available Bus services:

Hanif, Shyamoli, Desh Travels, Ekota → 4 ways

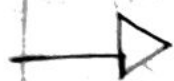
Available Trains:

Silkecity, Padma, Dhumketu, Dhanalata $\rightarrow$ 4 ways

Available Air services:

Biman, US-Bangla, Novoair $\rightarrow$ 3 ways

How many total ways to travel from Dhaka to Rajshahi?



Do you have option? $\rightarrow$ Yes

(SUM RULE)

$$\therefore \text{Total ways} = 4 + 4 + 3 = 11 \text{ ways}$$

☒ I have 2 Book PHYSICS and CHEMISTRY. I have four boxes to put them in. How many way I can put them in these boxes?



If we put the physics book 1st then

Ways for PHYSICS = 4

Ways for CHEMISTRY = 3

$$\therefore \text{Total ways} = 4 \times 3$$

$$= 12$$



Every box can contain 1 book



Do we have option?

$\downarrow$
NO

$\downarrow$
Product Rule

There are twelve empty rooms in the office for MRHS. Sadman and Ancho joins the company. How many way can they be assigned a room each from twelve room?



Ways of Ancho = 12

Ways of Sadman = 11

Total ways = 12×11
 $= 132$

□□□□□□□□□□□□□□

1 room for 1 person



Do we have option ?

↓
 NO
 ↓

PRODUCT RULE

Travelling from Dhaka to Barishal:

You can take a bus (8 services available) directly to Barishal via Padma bridge.

You can also take a bus (5 services available) to Patuakhali cross the ferry and take another bus from Daulatdia to Barishal (7 services available).

You can also take a launch (8 services) directly to Barishal.

How many ways can you go from Dhaka to Barishal?

→ Dhaka → Barishal (8 bus)

Dhaka → Paturia (5 bus) → (1 ferry) → Daulatdia (7 bus) → Barishal

Dhaka → Launch (8 launch) → Barishal

→ No option → Product Rule → $5 \times 1 \times 7 = 35$ ways

Another 2 have option → Sum Rule.

∴ Total ways = $8 + 35 + 8 = 51$ ways.

☐ You are at a restaurant and you have to choose a three-course meal (appetizer, main and dessert).

The restaurant offers 15 appetizers

The restaurant offers two cuisines for main-

- 12 dishes from Chinese cuisine

- 16 dishes from Indian cuisine

The restaurant offers 10 desserts

How many ways can you choose a three-course meal?

→ No option → Product rule

Total ways = $15 \times (12 + 16) \times 10 = 4200$ ways

The Pigeonhole Principle:

↳ The pigeonhole principle states that if you have more items (pigeons) than containers (holes) and you try to place each item in a container, at least one container must contain more than one item.

~~If n pigeons~~

If n pigeons occupy k pigeon holes then at least one pigeon hole will have at least z objects.

n = number of pigeon

k = number of hole

$$z = \left\lceil \frac{n}{k} \right\rceil$$

$$z = ??$$

$$\text{If, } n=13, k=12 \quad \left\{ \quad \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{13}{12} \right\rceil = \left\lceil 1.08 \right\rceil = 2 = z \right.$$

▣ Among 100 people, at least z how many are guaranteed to be born in the same month? k

$$n=100, k=12 \quad \left\{ \quad z = \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{100}{12} \right\rceil = \left\lceil 8.33 \right\rceil = 9 \right.$$

What would be the minimum number of students in a class so that at least six receives the same grade?

(Possible grades: A, B, C, D, F)

$$\begin{matrix} k=5 \\ z=6 \\ n=? \end{matrix} \left\{ \begin{array}{l} z = \left\lceil \frac{n}{k} \right\rceil \Rightarrow 6 = \left\lceil \frac{n}{5} \right\rceil \Rightarrow \lceil x \rceil = 6 \left[x = \frac{n}{5} \right] \end{array} \right.$$

$$\text{Now, } 5 < x \leq 6$$

$$\Rightarrow 5 < \frac{n}{5} \leq 6$$

$$\Rightarrow 25 < n \leq 30$$

$\therefore$ Minimum number of student = $25 + 1 = 26$

Card tutorial

There are four suits

- Spades
- Hearts
- Diamonds (Red)
- Clubs (Black)

13 cards of each suit

• Total $4 \times 13 = 52$ cards

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? $Z = ?$ $k = 4$

$n = ?$
 (কোনও একটা প্রকারে 52 cards থেকে আমরা ১২ কার্ড
 Avoid করে,)

$$Z = \lceil n/k \rceil \Rightarrow 3 = \lceil n/4 \rceil \Rightarrow \lceil x \rceil = 3 \quad \{x = n/4\}$$

Now,

$$2 < x \leq 3$$

$$\Rightarrow 2 < \frac{n}{4} \leq 3$$

$$\Rightarrow 8 < n \leq 12$$

Minimum number of
 must be selected
 $= 8 + 1 = 9$

When we can't use Pigeon Hole Rule?

→ Pigeon Hole Rule never specially said something.

→ If something is specially specifically said, then the Pigeon hole principle never works.

How many student must be selected from a school guarantee that we have at least three student from class 4?

class 1	= 10
" 2	= 10
" 3	= 10
" 4	= 10

→

এখন class 4 specifically নয়
হলেও অন্য Pigeon hole principle অনুসারে
করা যাবে না।

একমাত্র আমরা সবচেয়ে worst case চিন্তা-করা,
worst case এছাড়া চিন্তা করতে পারি না,
Class 1-3 পর্যন্ত সবকিছু student search/select করে
সবাই-জামা Class 4 এর 3 জন student select
করবে।

$$\begin{array}{cccc} \text{Class 1} & \text{Class 2} & \text{Class 3} & \text{Class 4} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \text{So, numbers of selected student} & = & 10 + 10 + 10 + 3 \\ & = & 33 \end{array}$$

এই ক্ষেত্রেও সব worst কিছু হতে পারে না।

→

we have to think the worst case.

In the worst case we may pick $10 + 10 + 10 = 30$ student but still not get 3 student from class 4.

But the next 13 students surely the student of class 4. So, we have to select $90+13=33$ students to guarantee that we have at least 13 students from class 4.

☐ How many cards must be selected from a standard deck of 52 cards to guarantee that we have at least three hearts,?
specifically 3♥ 4♥ 5♥



we have to think the worst case.

In the worst case, we may pick 13 spades, 13 Diamonds, 13 clubs but still doesn't get any heart.

Surely the next 3 cards will be the hearts.

∴ We have to select minimum = $13+13+13+3$
= 42 cards.

Principle of Inclusion-Exclusion (PIE):

↳ The principle works by first summing the number of elements in each set, then subtracting the number of elements in the intersections (since they are counted multiple times), and so on, alternating between adding and subtracting terms based on the number of sets involved in the intersection.

For two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Example:

How many bit strings of length 8 either start with 1 or end with 00?

→

Let A be the set of bit strings of length 8 that start with 1.

Let B be the set of bit strings of length 8 that end with 00.

If a bit string ^{starts} with 1, the first bit is fixed as 1. The remaining 7 bits can be any combination of 0 or 1.

Therefore, there are $2^7 = 128$ possible bit strings in set A .

$$\therefore |A| = 128$$

If a bit string ends with 00, the last two bits are fixed as 00. The first 6 bits can be any combination of 0 or 1.

Therefore, there are $2^6 = 64$ possible bit strings in set B .

$$\therefore |B| = 64$$

If a bit string starts with 1 and ends with 00, the first bit is fixed as 1 and the last two bits are fixed as 00. The remaining 5 bits in between can be any condition of 0 or 1.

Therefore, there are $2^5 = 32$ possible bit strings in $A \cap B$.

$$|A \cap B| = 32$$

To find $|A \cup B|$ (the number of bit strings that either starts with 1 or end with 00), we apply the formula:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 128 + 64 - 32$$

$$= 160$$

So, there are 160 bit strings of length 8 that either start with 1 or end with 00.

Find the number of bit strings of length 8 that either have exactly 3 ones, exactly 4 zeros or exactly 2 ones followed by 2 zeros.



Set A: Bit strings of length 8 with exactly 3 ones.

Set B: Bit strings of length 8 with exactly 4 zeros.

Set C: Bit strings of length 8 with exactly 2 ones at the beginning followed by exactly 2 zeros.

Number of bit strings in A (exactly 3 ones):

To find this, we choose 3 positions out of 8 for the ones and the rest will be zeros:

$$|A| = \binom{8}{3} = \frac{8!}{3!(8-3)!} = 56$$

Number of bit strings in B (exactly 4 zeros):

To find this, we choose 4 positions out of 8 for the zeros and the rest will be ones:

$$|B| = \binom{8}{4} = \frac{8!}{4!(8-4)!} = 70$$

Number of bit strings in C (exactly 2 ones at the beginning followed by exactly 2 zeros):

The string looks like: $11xx00yy$, where x and y can be either 0 or 1. There are 4 positions left and these can be filled with any combination of 0s and 1s:

$$|C| = 2^4 = 16$$

Number of bit strings in A ∩ B (exactly 3 ones and exactly 4 zeros):

This is the same as the number of strings in A because having 3 ones means there are exactly 5 zeros:

$$|A \cap B| = |A| = 56$$

Number of bit strings in B ∩ C (exactly 4 zeros and 2 ones at the beginning followed by exactly 2 zeros):

The string looks like: $11xx00zz$, where there are exactly 2 more zeros required in the remaining 4 position. So:

$$|B \cap C| = 2^2 = 4$$

Number of bit strings in CNA (exactly 2 ones at the beginning followed by exactly 2 zeros and exactly 2 ones):

Since 2 ones and 2 zeros are already accounted for, we need 1 more one and 2 more zeros in the remaining 4 positions:

$$|CNA| = \binom{4}{1} = \frac{4!}{1!(4-1)!} = 4$$

Number of bit strings in $AN \cap C$ (exactly 3 ones, exactly 4 zeros and exactly 2 ones at the beginning followed by exactly 2 zeros):

This is essentially CNA because having exactly 2 ones at the beginning followed by exactly 2 zeros is consistent with having 3 ones and 4 zeros:

$$|AN \cap C| = |CNA| = 4$$

Using Inclusion - Exclusion Principle:

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ &= 56 + 70 + 16 - 56 - 4 - 4 + 4 \\ &= 82 \end{aligned}$$

Thus, the number of bit strings of length 8 that either have exactly 3 ones, exactly 4 zeros or exactly 2 ones at the beginning followed by exactly 2 zeros is 82.

Practice Problem

How many cards must be selected from a deck of 52 standard playing cards to guarantee that at least a king, a Queen, and a Red card are picked? Explain with proper reason.

→ There are 4 kings, 4 Queens and 26 Red cards (13 Hearts, 13 Diamonds).

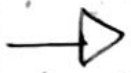
After picking 48 cards, the next card must be a king, Queen or Red card because only 4 kings and 4 Queens (which includes at least some Red cards) are left.

Thus, $(48+1) = 49$ cards must be selected to guarantee at least one king, one Queen and one Red card.

☐ In a game of UNO, there are cards of 4 colors, red, green, blue and yellow. A player is dealt 7 cards in a round.

① Minimum how many cards should be picked to ensure that the player gets at least 3 cards of the same color?

② Explain why there is no guarantee that a player will get at least 2 blue cards.



①

There are 4 colors and the worst-case scenario is that you pick cards of each color equally. You could pick 2 cards from each color (2 red, 2 green, 2 blue and 2 yellow), which would give you 8 cards.

But since only 7 cards are dealt, this guarantees that at least one color will have 3 cards.

So, the minimum number of cards required to ensure

3 cards of the same color is 7.

②

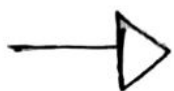
In the worst case, the player could pick 7 cards without picking any blue cards at all. For example, they could pick only red, green and yellow cards.

Therefore, there is no guarantee that the player will get even a single blue card, let alone 2.

☐ Rahat has built a mini library in his house. He has five shelves, one for Bangla novels, one for English novels, one for Maths and one science, one for Current Affairs and one for comics. What is the minimum number of books Rahat must have if he has at least 150 books of the same category?

$\downarrow$
 $z = 150$

$\downarrow$
 $k = 5$



According to Pigeonhole Principle,

$$Z = \lceil \frac{n}{k} \rceil \Rightarrow 150 = \lceil \frac{n}{5} \rceil$$

$$\Rightarrow \lceil x \rceil = 150 \quad \left[x = \frac{n}{5} \right]$$

$$\Rightarrow 149 < x \leq 150$$

$$\Rightarrow 149 < \frac{n}{5} \leq 150$$

$$\Rightarrow 745 < n \leq 750$$

Minimum number
of books Rahat
must have = $745 + 1$
= 746 .

There are 5 shelves (categories).

In the worst case, Rahat could place 149 books on each shelf (satisfying the condition of less than 150 books on each shelf).

To guarantee that at least one shelf has 150 books, he needs to place one more book, making the 150th book fall into one of the categories.

Thus, Rahat must have at least $149 \times 5 + 1 = 746$ books to guarantee that at least 150 books belong to the same category.

▣ There are 64 districts in Bangladesh. At least how many students must be enrolled in University of Barishal to ensure that at least 20 students are from the same district?

$$\begin{aligned} \rightarrow n=? \\ k=64 \\ z=20 \end{aligned} \left\{ \begin{aligned} z &= \left\lceil \frac{n}{k} \right\rceil \Rightarrow 20 = \left\lceil \frac{n}{64} \right\rceil \\ &\Rightarrow 19 < \frac{n}{64} \leq 20 \\ &\Rightarrow 1216 < n \leq 1280 \end{aligned} \right.$$

So, at least $(1216+1) = 1217$ students must be enrolled to guarantee that at least 20 students are from the same district.

▣ A company stores product in a warehouse. There are 80 aisles, 120 horizontal locations in each aisle and 6 shelves throughout the warehouse. At least how many products must be in the warehouse so that at least two products must be stored in the same bin?

→

The number of bins is the product of aisles, locations and shelves, $k = 80 \times 120 \times 6$
 $= 57,600$ bins.

To guarantee that at least two products are in the same bin, so $z = 2$

According to Pigeonhole Principle,

$$z = \left\lceil \frac{n}{k} \right\rceil$$

$$\Rightarrow z = \left\lceil \frac{n}{57,600} \right\rceil$$

$$\Rightarrow 1 < \frac{n}{57,600} \leq 2$$

$$\Rightarrow 57,600 < n \leq 115,200$$

Thus, at least $(57,600 + 1) = 57,601$ products must be stored to ensure that at least two are in the same bin.

▣ Among a set of 5 black balls and 3 red balls, how many selections of 5 balls can be made such that at least 3 of them are black?



We break down the problem into cases:

- Case-01: 3 black and 2 red

$$\binom{5}{3} \times \binom{3}{2} = \frac{5!}{3!(5-2)!} \times \frac{3!}{2!(3-2)!} = 10 \times 3 = 30$$

- Case-02: 4 black and 1 red

$$\binom{5}{4} \times \binom{3}{1} = \frac{5!}{4!(5-4)!} \times \frac{3!}{1!(3-1)!} = 5 \times 3 = 15$$

- Case-03: 5 black

$$\binom{5}{5} = 1$$

Adding all the cases:

$$30 + 15 + 1 = 46$$

Thus, there are 46 ways to select 5 balls such that at least 3 are black.

Permutation and Combination

Permutation: The arrangement of a set of objects in a specific order.

Combination: The selection of objects from a set where the order does not matter.

Permutation = Pick and arrange $P(n, r) = \frac{n!}{(n-r)!}$	Combination = Pick $C(n, r) = \frac{n!}{r!(n-r)!}$
---	---

☐ In how many way we can pick 6 person out of 10 and arrange them in order?



Pick and Arrange → Permutation

$$\begin{aligned} n=10 \\ r=6 \quad \left\{ \begin{aligned} P(n, r) &= \frac{n!}{(n-r)!} = \frac{10!}{(10-6)!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4!} \\ &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \\ &= 151,200 \end{aligned} \right. \end{aligned}$$

☐ In how many way we can select 11 students among 20. For creating a football team?



Only pick → Combination

$$\left. \begin{array}{l} n=20 \\ r=11 \end{array} \right\} = \frac{n!}{r!(n-r)!} = \frac{20!}{11!(20-11)!} = \frac{20!}{11!9!}$$

☐ How many ways to award gold, silver and bronze medals from 8 runners in a race?

→ Pick 3 and arrange them → Permutation

$$\left. \begin{array}{l} n=8 \\ r=3 \end{array} \right\} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336 \text{ ways}$$

☐ How many ways to award gold, silver and bronze medals from 8 runners in a race, if one of them is Usain Bolt?



It is sure that Usain Bolt get gold.

$$\text{So, } n=7, r=2 \rightarrow \frac{7!}{(7-2)!} = 7 \times 6 = 42 \text{ ways}$$

How many ways to award gold, silver and bronze medals from 8 runners in a race if one of them is Niloy who is about 95 kg?

→

Niloy never get any medal. So,

$$n=7 \quad r=3 \quad \rightarrow \quad \frac{7!}{(7-3)!} = 7 \times 6 \times 5 = 210 \text{ ways}$$

In how many way we can arrange 6 person in a order?

→ 6 জন ব্যক্তি ৬ জনে - arrange হবে

সুতরাং So, $n=6$ & $r=6 \rightarrow \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! \quad [\because 0! = 1]$

In how many way we can arrange ABCDEF?

→ $n=6, r=6 \therefore P(n,r) = 6!$

How many permutations of the letters ABCDEFGH contain the string ABC?

→ ABCDEFGH → Total 6 বাকী So, $n=6; r=6$

$$\therefore P(n,r) = 6!$$

How many permutations of the letters ABCDEFGH contain the letters A, B and C together?

A, B, C DEFGH
6^{2r}



We can assume that ABC is only a letter then total letters is 6.

We can permute 6 letters in $6!$ ways.

Among ABC we can arrange them in $3!$ ways.

Therefore, total way is $= 6! \times 3!$

How we can re-arrange 4 person in this 10 room?



1st person = 10

2nd person = 9

3rd person = 8

4th person = 7

Total way = $10 \times 9 \times 8 \times 7$

$${}^{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!}$$

$$= 10 \times 9 \times 8 \times 7$$

How many permutations of the letters ABCDEFGH contain all of A, B and C separated?



AB C D E F G H



1 2 3 4 5 6
 [] [D] [] [E] [] [F] [] [G] [] [H] []

We can arrange "DEFGH" in $5!$ ways

We can put "ABC" in 6 spaces in $P(6,3)$ ways.

$$P(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!}$$

$$= 6 \times 5 \times 4 = 120$$

∴ Total ways = $120 \times 5!$

How many ways can the letters of the word "SILHOUETTE" be rearranged, when -

- ① No special condition is given
- ② All the vowels are together
- ③ All the T's are together
- ④ All the E's are together

How many permutations of the letters ABCDEFGH contain all of A, B and C separated?



AB C D E F G H



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 [] D [] E [] F [] G [] H []

We can arrange "DEFGH" in $5!$ ways

We can put "ABC" in 6 spaces in $P(6, 3)$ ways.

$$P(6, 3) = \frac{6!}{(6-3)!} = \frac{6!}{3!}$$

$$= 6 \times 5 \times 4 = 120$$

$$\therefore \text{Total ways} = 120 \times 5!$$

How many ways can the letters of the word "SILHOUETTE" be rearranged, when -

- ① No special condition is given
- ② All the vowels are together
- ③ All the T's are together
- ④ All the E's are together

①

"SILHOUETTE" — { Here 'E' and 'T' are repeated
2 times.
Here 10 letters can be
rearranged in 10 boxes.

SILHOUETTE

We can permute 10 boxes letters in $10!$ ways.

On the other hand,

We can permute 2(E) in $2!$ ways and
2(T) in $2!$ ways.

Therefore, Total way is $= \frac{10!}{2! 2!} - 1$

So, we can rearranged

"SILHOUETTE" - this

word in $\frac{10!}{2! 2!} - 1$ ways.

In this ways repeated
also "SILHOUETTE".

②

All the vowels are together means

IOUEE SLHTT

We can assume that IOUEE is only a letter
then total letter is 6.

We can permute 6 letters in $6!$ ways.

Here, repeated T 2 times means, $2!$

On the other hand,

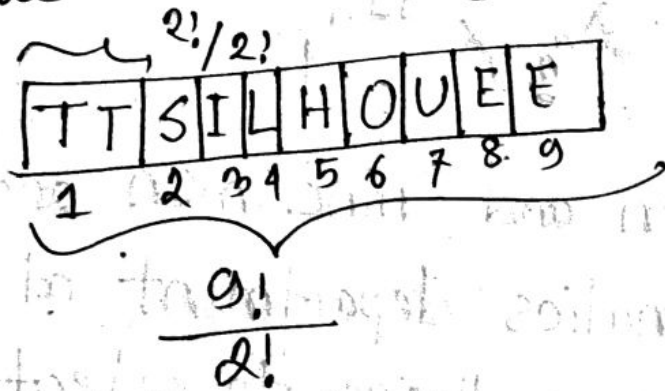
Among IOUVE, we can arrange them in $5!$ ways but here, repeated E 2 times means divided

by $2!$

Therefore, Total way is $= \frac{6!}{2!} \times \frac{5!}{2!}$

(3)

All the T's are together means -



∴ Total way is

$$= \frac{9!}{2!} \times \frac{2!}{2!} - 1$$

$$= \frac{9!}{2!} - 1$$

(4)

All the E's are together means -



$$\Rightarrow \frac{9!}{2!} \times \frac{2!}{2!} = \frac{9!}{2!}$$

There are 9 Math faculties and 11 CS faculties in a university. The authority wants to form a committee to develop a Discrete Math course, such that 3 members of the committee are Math faculties and 4 are CS faculties.

How many ways to form the committee?

$$\rightarrow 9 \rightarrow 3 \Rightarrow {}^9C_3 = \frac{9!}{3!(9-3)!}$$

$$11 \rightarrow 4 \Rightarrow {}^{11}C_4 = \frac{11!}{4!(11-4)!}$$

$$\text{Total way : } {}^9C_3 \times {}^{11}C_4$$

Seven women and nine men are on the faculty in the mathematics department at a school. How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

	<u>Num of women</u>	<u>Num of men</u>	<u>Total way</u>
i)	1	4	${}^7C_1 \times {}^9C_4$
ii)	2	3	${}^7C_2 \times {}^9C_3$
iii)	3	2	${}^7C_3 \times {}^9C_2$
iv)	4	1	${}^7C_4 \times {}^9C_1$

$$\therefore \text{Total way} = ({}^7C_1 \times {}^9C_1) + ({}^7C_2 \times {}^9C_0) + ({}^7C_3 \times {}^9C_2) + ({}^7C_4 \times {}^9C_3)$$

One hundred tickets, numbered 1, 2, 3, ..., 100 are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if -

- ① There are no restrictions?
- ② Ticket 47 wins the grand prize?
- ③ Ticket 47 wins a prize?
- ④ Ticket 47 does not win a prize?
- ⑤ Ticket 10 and 47 both win prizes?
- ⑥ Ticket 10, 47 and 73 all win prizes?
- ⑦ Ticket 10, 47, 73 and 97 all win prizes?

①

Here, 100 people and 4 prizes (1st, 2nd, 3rd, 4th)

$$\therefore \text{Total ways: } {}^{100}P_4 = \frac{100!}{(100-4)!}$$

(2)

Ticket 47 wins the grand prize.

1st



→ 99 people

∴ Total ways: ${}^{99}P_3$

(3)

If Ticket 47 wins any of the 4 prizes, we have already assigned 1 prize to ticket 47. Now, we need to distributed the remaining 3 prizes among the remaining 99 people.

The number of ways to do this is the permutation of 99 people taking 3 prizes:

$${}^{99}P_3 = 99 \times 98 \times 97$$

But ticket 47 can win any of the 4 prizes, so we multiply by 4 (since there are 4 possible positions for ticket 47):

$$4 \times {}^{99}P_3 = 3764376$$

(4)

If the ticket 47 does not win a prize, then we are simply distributing all 4 prizes among the remaining 99 people.

$${}^{99}P_4 = 99 \times 98 \times 97 \times 96$$

(5)

Here, both ticket 10 and ticket 47 must win prizes. This means that 2 out of the 4 prizes are already assigned.

$$98P_2 = 98 \times 97$$

However, since ticket 10 and 47 can win any 2 of the 4 prizes, we multiply by the number of ways to choose 2 prizes for them from 4, which is $\binom{4}{2} = 6$:

$$6 \times (98 \times 97)$$

(6)

Here, 10, 47 and 73 all win prizes. So, 3 out of 4 prizes are already assigned.

$$97P_1 = 97$$

There are $\binom{4}{3} = 4$ ways to assign the 3 prizes to tickets to 10, 47 and 73. So that total number of ways is:

$$4 \times 97$$

(7)

Here all 4 prizes are already assigned to specific tickets (ticket 10, 47, 73 and 97), so there is only 1 way to assign these prizes.

Probability Distribution

A probability distribution in discrete mathematics is a function that maps each possible outcome of a discrete random variable to a probability.

$$\text{probability, } P(E) = \frac{|E|}{|S|}$$

complement:

$$P(E) = \frac{|E|}{|S|}$$

$$\text{Now, } |\bar{E}| = |S| - |E| \Rightarrow \frac{|\bar{E}|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|}$$

$$\Rightarrow P(\bar{E}) = 1 - P(E)$$

$$\therefore \boxed{P(E) + P(\bar{E}) = 1}$$

Q What is the probability that a 5 length bit string where last bit is always zero?

→

There are 5-length bit string.

So, sample space, $|S| = 2^5 = 32$

Last bit is zero. So, $|A| = 2^4 = 16$

$$\therefore P(A) = \frac{|A|}{|S|} = \frac{16}{32} = \frac{1}{2}$$

Multiple Events Probability

In the probability theory, the concept of multiple events refers to situations where we are dealing with more than one event and we want to determine the probability of combinations of these events occurring.

When we analyzing multiple events, we often use rules such as the addition rule and the multiplication rule to compute probabilities for different types of events (independent, dependent, mutually exclusive, Non-mutually exclusive events).

① Independent Events:

Two events A and B are independent if the occurrence of one event does not affect the probability of the occurrence of the other. In the other words, knowing that A occurred does not change the probability of B occurring.

The probability of both independent events A and B occurring (called the intersection of A and B, denoted $A \cap B$) is given by:

$$P(A \cap B) = P(A) \cdot P(B)$$

② Dependent Events:

Two events A and B are dependent if the occurrence of one event affects the probability of the other. In this case we need to use "conditional probability" to account for this relationship.

The probability of both dependent events A and B occurring is given by: $P(A \cap B) = P(A) \cdot P(B|A)$ where $P(B|A)$ is the conditional probability of B occurring given that A has already occurred.

conditional probability

The conditional probability of an event A occurring given that event B has occurred is denoted as $P(A|B)$. It is defined as: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

where:

- $P(A|B)$ is the conditional probability of event A given event B
- $P(A \cap B)$ is the probability that both events A and B occur (the intersection of A and B).
- $P(B)$ is the probability that event B occurs.

③ Mutually Exclusive Events

Two events A and B are said to be mutually exclusive (or disjoint) if they cannot occur at the same time. This means that the occurrence of one event excludes the possibility of the other occurring.

The ^{probability that} either event A or event B occurs (called the union of A and B, denoted $A \cup B$) is:

$$P(A \cup B) = P(A) + P(B)$$

since, there ^{is} no overlap (i.e. $P(A \cap B) = 0$)

④ Non-mutually Exclusive Events

If two events A and B are not mutually exclusive, they can both occur at the same time. To find the probability of either A or B occurring we use the general addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The subtraction of $P(A \cap B)$ is necessary because we don't want to double-count the cases where both events occur.

Now, if A and B be events in the sample space S then we should prove that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

So,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$\Rightarrow \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$\Rightarrow P(A \cup B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Q What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

→

Total numbers from 1 to 100 = 100

Numbers divisible by 2: $\left\lfloor \frac{100}{2} \right\rfloor = 50$

Numbers divisible by 5: $\left\lfloor \frac{100}{5} \right\rfloor = 20$

Numbers divisible by both 2 and 5 (i.e. divisible by 10):

$$\left\lfloor \frac{100}{10} \right\rfloor = 10$$

Using inclusion-exclusion:

$$\begin{aligned} P(\text{divisible by 2 or 5}) &= \frac{50 + 20 - 10}{100} \\ &= \frac{70 - 10}{100} = \frac{60}{100} \\ &= 0.6 \end{aligned}$$

So, the probability is 0.6

Bayes' Theorem

Bayes' theorem is a powerful theorem for finding conditional probabilities. It is particularly useful when reversing conditional probabilities, i.e., when we know $P(B|A)$ but need to know find $P(A|B)$. Bayes' theorem states:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

where,

$P(A|B)$ is the probability of event A given that B has occurred (posterior probability)

$P(B|A)$ is the probability of event B given that A has occurred (likelihood)

$P(A)$ is the prior probability of event A.

$P(B)$ is the marginal probability of event B. (the total probability of B)

Now, Prove $\rightarrow$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \cdot P(A)$$

Now, substitute the value of $P(A \cap B)$ into eq. ①

$$\Rightarrow P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Bayes' Theorem

□ "A patient goes to see a doctor. The doctor performs a test with 99 percent reliability - that is, 99 percent of people who are sick test positive and 99 percent of the healthy people test negative. The doctor knows that only 1 percent of the people in the country are sick. Now the question is: if the patient tests positive, what are the chances the patient is sick?"

~~$P(A)$~~

~~probability that a person is sick, $P(A)$~~

Probability of being sick, $P(A)$:

The probability that a person is sick is 1%

$$\therefore P(A) = \frac{1}{100} = 0.01$$

Probability of testing positive given sick, $P(B|A)$:

The probability that a sick person tests positive is 99%, or $P(B|A) = \frac{99}{100} = 0.99$

Probability of testing positive given not sick, $P(B|\bar{A})$:

The probability that a healthy person tests positive is 1% or $P(B|\bar{A}) = \frac{1}{100} = 0.01$

Probability of not being sick, $P(\bar{A})$:

The probability that a person is not sick is 99% or $P(\bar{A}) = \frac{99}{100} = 0.99$

Now, we want to find $P(A|B)$, the probability that the patient is sick given they tested positive

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \quad [\text{Bayes' Theorem}]$$

where,

$$P(B) = P(B \cap A) + P(B \cap \bar{A}) \quad \text{--- (1) [Law of Total Probability]}$$

$$P(B \cap A) = P(B|A) \cdot P(A) \quad \text{--- (2)}$$

$$P(B \cap \bar{A}) = P(B|\bar{A}) \cdot P(\bar{A}) \quad \text{--- (3)}$$

Substitute equations (2) and (3) into equation (1):

$$P(B) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})$$

$$\Rightarrow P(B) = (0.99)(0.01) + (0.01)(0.99)$$

$$\Rightarrow P(B) = 0.0099 + 0.0099$$

$$\therefore P(B) = 0.0198$$

Now substituting back into Baye's theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$\Rightarrow P(A|B) = \frac{(0.99)(0.01)}{0.0198}$$

$$\therefore P(A|B) = \frac{0.0099}{0.0198} \approx 0.5$$

Therefore, the probability that a patient is sick given that they tested positive approximately 50%.

Complexity

□ Proof that a polynomial $P(n) = a_0 + a_1n + a_2n^2 + \dots + a_mn^m$, with degree m , has a time complexity of $O(n^m)$.

→ $P(n)$ is a polynomial of degree m , where the highest power of n is n^m .

Let $b_i = |a_i|$ (the absolute values of the coefficients)

This gives the inequality: $P(n) \leq b_0 + b_1n + b_2n^2 + \dots + b_mn^m$

We can factor out n^m from the terms:

$$P(n) \leq n^m \left(\frac{b_0}{n^m} + \frac{b_1}{n^{m-1}} + \dots + b_m \right)$$

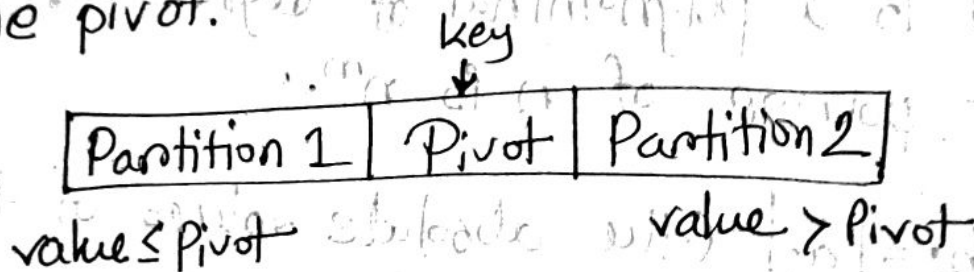
As $n \rightarrow \infty$, the lower-order terms $\left(\frac{b_i}{n^i}\right)$ approach 0, leaving $P(n)$ dominated by n^m .

Thus, $P(n) \leq Mn^m$, where $M = |a_0| + |a_1| + \dots + |a_m|$ is a constant.

This shows that $P(n) = O(n^m)$, meaning the polynomial's time complexity is asymptotically bounded by n^m .

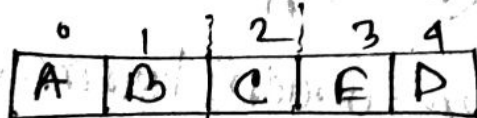
Quick sort:

A highly efficient sorting algorithm that uses a divide-and-conquer approach. It selects a 'pivot' element and partitions the array into elements two halves: one with elements less than the pivot and one with elements greater than the pivot.

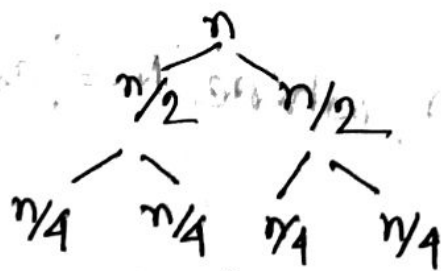


Time Complexity:

Best / Average Case:



↓
partition



$$\frac{n}{2^k} = 1$$

$$\Rightarrow n = 2^k$$

$$\Rightarrow \log_2 n = \log_2 2^k$$

$$\Rightarrow \log_2 n = k \log_2 2$$

$$\therefore k = \log_2 n$$

The partitioning step takes $O(n)$ time.

So, Time complexity: $O(n \log n)$

Worst Case:

0	1	2	3	4
A	B	C	E	D

In this case, the array is split into two parts: one with $n-1$ elements and the other with 0 elements.

$$\begin{array}{c} n \\ \dots \\ n-1 \\ \dots \\ n-2 \\ \dots \\ 1 \end{array} \left\{ \frac{n(n-1)}{2} = \frac{n^2-n}{2} \right.$$

$\therefore$ Time complexity: $O(n^2)$