Leibnitz theorem ①

(লীবনীজ উপপাদ্য)

Statement: If $u(x)$ and $v(x)$ be a function of x, then the differential co-efficient of their product for n times can be express by

* যদি $u(x)$ এবং $v(x)$, x এর ফাংশন হয়, ~~তবে তাদের অন্তরক সহগ~~ তাহলে n তম বার তাদের গুনফলের অন্তরক সহগকে দ্বারা প্রকাশ করা যেতে পারে।

$$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + {}^nc_r u_{n-r} v_r + \cdots + u v_n \qquad \text{—①}$$

where $u_n = \dfrac{d^n u}{dx^n}$, $v_n = \dfrac{d^n v}{dx^n}$ and $(uv)_n = \dfrac{d^n}{dx^n}(uv)$

প্রথমে $n = 1, 2, 3$ এর জন্য (i) নং টি প্রমাণ করব, তারপর,

ধরে নেব $n = m$ এর জন্য প্রমানটি সত্য এবং যদি $n = m+1$ এর জন্যেও সত্য হয় তাহলে গাণিতিক আরোহবিধি অনুযায়ী সকল স্বাভাবিক সংখ্যার জন্য উপপাদ্যটি সত্য। $\forall n \in N$.

(2)

$\frac{d}{dx}(uv) = \frac{d}{dx}(u)v + u\frac{d}{dx}(v)$

Proof: Let, $y = uv$

$n=1$ হলে. y এর 1 বার differentiate, $y_1 = u_1 v + u v_1$ ← $\frac{d}{dx}(uv)$

$n=2$ " y " 2 " " $y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$

$\frac{d}{dx}(u_1 v + u v_1)$

$\therefore y_2 = u_2 v + 2u_1 v_1 + u v_2$

$n=3$ " y " 3 বার " , $\frac{d}{dx}(u_2 v + 2u_1 v_1 + u v_2)$

$y_3 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + u v_3$

$\therefore y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$

or $y_3 = u_3 v = {}^3c_1 u_{3-1} v_1 + {}^3c_2 u_{3-2} v_2 + u v_3$ —— (ii)

Now, the theorem is true for $n = 1, 2, 3$. Let the theorem is true for $n = m$, then,

$y_m = u_m v + {}^mc_1 u_{m-1} v_1 + {}^mc_2 u_{m-2} v_2 + \cdots +$

$\boxed{{}^mc_{r-1} u_{m-r+1} v_{r-1}} + {}^mc_r u_{m-r} v_r + \cdots + u v_m$ —— (iii)

$\boxed{{}^mc_{r-1} u_{m-(r-1)} v_{r-1} = {}^mc_{r-1} u_{m-r+1} v_{r-1}}$

সুবিধার জন্য

constant

$$y_m = U_m V + {}^mC_1 U_{m-1} V_1 + {}^mC_2 U_{m-2} V_2 + \cdots + {}^mC_{n-1} U_{m-n+1} V_{n-1} + {}^mC_n U_{m-n} V_n + \cdots + U V_m$$

Now, differentiating both sides w.r. to x the we get,

$$\cancel{y_{m+1} = U_{m+1} V + U_m V_1 + {}^mC_1 (U_{m-2+1} V_2 + U_{m-2} V_3) + \cdots}$$

$$y_{m+1} = U_{m+1} V + U_m V_1 + {}^mC_1 (U_{m-1+1} V_1 + U_{m-1} V_2) + {}^mC_2 (U_{m-2+1} V_2 + U_{m-2} V_3) + \cdots + {}^mC_{n-1} (U_{m-n+1+1} V_{n-1} + U_{m-n+1} V_{n-1+1}) + {}^mC_n (U_{m-n+1} V_n + U_{m-n} V_{n+1}) + \cdots + U_1 V_m + U V_{m+1}$$

$$= U_{m+1} V + U_m V_1 + {}^mC_1 (U_m V_1 + U_{m-1} V_2) + {}^mC_2 (U_{m-1} V_2 + U_{m-2} V_3) + \cdots + {}^mC_{n-1} (U_{m-n+2} V_{n-1} + U_{m-n+1} V_n) + {}^mC_n (U_{m-n+1} V_n + U_{m-n} V_{n+1}) + \cdots + U_1 V_m + U V_{m+1}$$

$$= U_{m+1} V + \boxed{U_m V_1} + {}^mC_1 \boxed{U_m V_1} + {}^mC_1 \boxed{U_{m-1} V_2} + {}^mC_2 \boxed{U_{m-1} V_2} + {}^mC_2 U_{m-2} V_3 + {}^mC_{n-1} U_{m-n+2} V_{n-1} + {}^mC_{n-1} \boxed{U_{m-n+1} V_n}$$

(a)

$$+ {}^{m}C_{n} \boxed{U_{m-n+1} V_n} + {}^{m}C_{n} U_{m-n} V_{n+1} + \cdots + U_1 V_m + U V_{m+1}$$

$$= U_{m+1} V + (1 + {}^{m}C_1) U_m V_1 + ({}^{m}C_1 + {}^{m}C_2) U_{m-1} V_2 + \cdots +$$

$$({}^{m}C_{n-1} + {}^{m}C_n) U_{m-n+1} V_n + \cdots + U V_{m+1} \quad \text{(iv)}$$

We know,

$$1 + {}^{m}C_1 = {}^{m}C_0 + {}^{m}C_1 = {}^{m}C_1$$

$${}^{m}C_1 + {}^{m}C_2 = {}^{m+1}C_2$$

$${}^{m}C_2 + {}^{m}C_3 = {}^{m+1}C_3$$

$${}^{m}C_{n-1} + {}^{m}C_n = {}^{m+1}C_n$$

Now, (iv) $\Rightarrow$

$$y_{m+1} = U_{m+1} V + {}^{m}C_1 U_m V_1 + {}^{m+1}C_2 U_{m-1} V_2 + \cdots$$

$$+ {}^{m+1}C_n U_{m-n+1} V_n + \cdots + U V_{m+1}$$

Since the theorem is true for ~~m=~~ $n=m$, then ~~then~~ the theorem is true for $n=m+1$. ~~i.e. the theorem is true for~~ Thus, by the mathematical induction method, the theorem is true for all ${}^{n}C_N$ or $n \in N$

(5)

Problem Solve:

Example: (1) If $y = \cos\{\ln(1+x)\}$ then show that

$$(1+x)^2 y_{n+2} + (2n+1)(1+x)y_{n+1} + (n^2+1)y_n = 0$$

Solve:

Given that, $y = \cos\{\ln(1+x)\}$ ——(1)

$$\frac{d}{dx}\left(\cos\{\ln(1+x)\}\right) = -\sin\{\ln(1+x)\}$$

$$\frac{d}{dx}\left(\ln(1+x)\right) = \frac{1}{1+x}$$

$$\frac{d}{dx}(1+x) = 0 + 1 = 1.$$

$$y_1 = -\sin\{\ln(1+x)\} \cdot \frac{1}{1+x} \cdot 1$$

$$\therefore y_1 = \frac{-\sin\{\ln(1+x)\}}{1+x}$$

$\Rightarrow (1+x) y_1 = -\sin\{\ln(1+x)\}$ [বজ্রগুণন]

$\Rightarrow (1+x)^2 y_1^2 = \sin^2\{\ln(1+x)\}$ [উভয় পক্ষে বর্গ করে]

$\Rightarrow (1+x)^2 y_1^2 = 1 - \cos^2\{\ln(1+x)\}$ [$\cos\{\ln(1+x)\} = y$]

$\Rightarrow \underbrace{(1+x)^2}_{u}\underbrace{y_1^2}_{v} = 1 - y^2$ [যেহেতু এখানে y_{n+2} আছে তাই y আবার differentiate করতে হবে]

$\Rightarrow (1+x)^2 \cdot 2y_1 \cdot y_2 + 2(1+x) \cdot 1 \cdot y_1^2 = 0 - 2y \cdot y_1$

$$\left[(1+x)^2 \cdot \frac{d}{dx}(y_1^2) \cdot \frac{d}{dx}(y_1) + \frac{d}{dx}(1+x)^2 \cdot \frac{d}{dx}(1+x) \cdot y_1^2 = \frac{d}{dx}(1) - \frac{d}{dx}(y^2) \cdot \frac{d}{dx}(y)\right]$$

$\Rightarrow (1+x)^2 \, 2y_1 y_2 + 2y_1^2(1+x) + 2yy_1 = 0$

$\Rightarrow 2y_1\{(1+x)^2 y_2 + y_1(1+x) + y\} = 0 \Rightarrow (1+x)^2 y_2 + (1+x)y_1 + y = 0$

(6) Differentiat n times by Lebnitz theorem,

$$\underbrace{(1+x)^2}_{v}\underbrace{y_2}_{u} + \underbrace{(1+x)}_{v}\underbrace{y_1}_{u} + y = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + {}^nc_1 y_{n+2-1} \, 2(1+x) + {}^nc_2 y_{n+2-2} \cdot 2 +$$

$$(1+x) y_{n+1} + {}^nc_1 y_{n+1-1} \cdot 1 + y_n = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + {}^nc_1 (1+x) \cdot 2y_{n+1} + {}^nc_2 \, 2y_n + (1+x)y_{n+1} +$$

$${}^nc_1 y_n + y_n = 0$$

${}^nc_1 = \frac{n}{1!}$

${}^nc_2 = \frac{n(n-1)}{2!}$

$$\Rightarrow (1+x)^2 y_{n+2} + (1+x)2ny_{n+1} + \frac{n(n-1)}{\cancel{2!}} \cdot \cancel{2} y_n + (1+x)y_{n+1}$$

$$+ n \cdot y_n + y_n = 0$$

$2! = 2$

$1! = 1$

$$\Rightarrow (1+x)^2 y_{n+2} + 2n(1+x) y_{n+1} + (n^2 - n) \cdot y_n + (1+x) y_{n+1}$$

$$+ n \cdot y_n + y_n = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + \underline{2n(1+x) y_{n+1}} + n^2 y_n - \cancel{n y_n} +$$

$$\underline{(1+x)y_{n+1}} + \cancel{n y_n} + y_n = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + (1+2n)(1+x) y_{n+1} + y_n(1+n^2) = 0$$

$$\Rightarrow (1+x)^2 y_{n+2} + (2n+1)(1+x) y_{n+1} + (n^2+1) y_n = 0$$

proved

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Example (2) If $y = a\cos(\ln x) + b\sin(\ln x)$ then prove that

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$$

Solve: Given that,

$$y = a\cos(\ln x) + b\sin(\ln x) \longrightarrow (i)$$

$$y_1 = -a\sin(\ln x)\cdot\frac{1}{x} + b\cos(\ln x)\cdot\frac{1}{x}$$

$$y_1 = \frac{b\cos(\ln x)}{x} - \frac{a\sin(\ln x)}{x}$$

$$\underset{v\ u}{x y_1} = b\cos(\ln x) - a\sin(\ln x)$$

$$xy_2 + y_1 = -\frac{b\sin(\ln x)}{x} - \frac{a\cos(\ln x)}{x}$$

$$x(xy_2+y_1) = -[\underbrace{a\cos(\ln x) + b\sin(\ln x)}_{y}]$$

$$x^2 y_2 + xy_1 = -y$$

$$\Rightarrow \underset{v}{x^2}\underset{u}{y_2} + \underset{v}{x}\underset{u}{y_1} + y = 0 \longrightarrow (ii)$$

Diddereneiat n times by Leibnitz theorem,

$$\Rightarrow y_{n+2}\cdot x^2 + {}^nC_1\, y_{n+2-1}\, 2x + {}^nC_2\, y_{n+2-2}\cdot 2 + y_{n+1}\cdot x + {}^nC_1\, y_{n+1-1}\cdot 1 + y_n$$

$$\Rightarrow y_{n+2}\cdot x^2 + y_{n+1}\cdot 2nx + \frac{n(n-1)}{\cancel{2}}\cdot\cancel{2}\cdot y_n + y_{n+1}\cdot x + y_n\cdot n + y_n$$

$$\Rightarrow x^2 y_{n+2} + 2nx\cdot y_{n+1} + n^2 y_n - \cancel{ny_n} + y_{n+1}x + \cancel{ny_n} + y_n$$

$$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0 \text{ (proved)}$$

(8)

Example: 3

If $x = \cosh\left(\frac{1}{a}\ln y\right)$ then show that,

$$(x^2-1)\,y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-a^2)\,y_n = 0$$

Solve:

$$x = \cosh\left(\frac{1}{a}\ln y\right)$$

$$\Rightarrow \cosh^{-1}x = \frac{1}{a}\ln y$$

$$\Rightarrow a\cosh^{-1}x = \ln y$$

$$\Rightarrow \ln y = a\cosh^{-1}x$$

$$\Rightarrow y = e^{a\cosh^{-1}x} \quad \text{--- (1)}$$

[$e^{\ln y} = y$, উভয় পাশে e বসানো হয়েছে]

$$\Rightarrow y_1 = e^{a\cosh^{-1}x} \cdot a \cdot \frac{1}{\sqrt{x^2-1}}$$

$$\left[\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}\right]$$

এই h লিখার ভুলের কারণে $\cos^{-1}x$ এর মান এমন হয়েছে।

$$\Rightarrow y_1 = \frac{a e^{a\cosh^{-1}x}}{\sqrt{x^2-1}}$$

$$(\sqrt{x^2-1})\cdot y_1 = \underbrace{a e^{a\cosh^{-1}x}}_{y}$$

$$\Rightarrow \underbrace{(x^2-1)}_{v}\cdot \underbrace{y_1^2}_{u} = a^2 y^2$$

$$\Rightarrow 2y_1\cdot y_2(x^2-1) + y_1^2\cdot 2x\cdot 1 - 0 = a^2\cdot 2y\cdot y_1$$

$$\Rightarrow 2y_1 y_2(x^2-1) + 2x\cdot y_1^2 - a^2\cdot 2y\cdot y_1 = 0$$

$$= 2y_1\left\{(x^2-1)y_2 + xy_1 - a^2\cdot y\right\} = 0$$

$\Rightarrow (x^2-1)y_2 + xy_1 - a^2 y = 0$ ——— (iii)

Differentiat by Leibnitz theorem,

$(x^2-1)y_{n+2} + {}^nc_1\, y_{n+2-1}\cdot 2x + {}^nc_2\, y_{n+2-2}\cdot 2 + xy_{n+1} + {}^nc_1\, y_{n}\cdot 1 - a^2 y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + ny_{n+1}\cdot 2x + (n^2-n)y_n + xy_{n+1} + ny_n - a^2y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + 2nx\,y_{n+1} + n^2y_n - \cancel{ny_n} + xy_{n+1} + \cancel{ny_n} - a^2y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-a^2)y_n = 0$ (proved)

Example 4: If $y = \tan^{-1}x$ then show that.

(i) $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

(ii) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

Solve:

(1) Given that

$y = \tan^{-1}x$

$y_1 = \frac{1}{1+x^2}$

$(1+x^2)y_1 = 1$

$\Rightarrow (1+x^2)y_1 - 1 = 0$ ——— (1)

${}^nc_1 = \frac{n!}{(n-1)!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

${}^nc_2 = \frac{n(n-1)}{2!} = \frac{n(n-1)}{2}$

Diff. by Leibnitz theorem, differentiate

$$(1+x^2)\, y_{n+1} + {}^nc_1\, y_{n+1-1}\, 2x + {}^nc_2\, y_{n+1-2} \cdot 2.1 + 0 = 0$$

$$\Rightarrow (1+x^2)\, y_{n+1} + n.\, y_n . 2x + \frac{n(n-1)}{2} . 2 . y_{n-1} = 0$$

$$\Rightarrow (1+x^2)\, y_{n+1} + 2nx y_n + n(n-1)\, y_{n-1} = 0$$

(showed)

(2) Solve:

Given that,

$$y = \tan^{-1} x$$

$$y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow \underset{v}{(1+x^2)}\, \underset{u}{y_1} = 1$$

$$\Rightarrow y_2 (1+x^2) + y_1 . 1 . 2x + 2 - 1 = 0$$

$$\Rightarrow (1+x^2)\, y_2 + \underset{v}{2x}\, \underset{u}{y_1} + 1 = 0 \longleftarrow (1)$$

Diffi. by Leibnitz theorem,

$$(1+x^2)\, y_{n+2} + {}^nc_1\, y_{n+2-1} \cdot 2x + {}^nc_2\, y_{n+2-2} \cdot 2.1 + y_{n+1} . 2x$$

$$+ {}^nc_1\, y_{n+1-1}\, 2.1 + 0 = 0$$

$$\Rightarrow (1+x^2)\, y_{n+2} + n . y_{n+1} . 2x + n(n-1) . y_n + y_{n+1} . 2x$$

$$+ n . y_n . 2 = 0$$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)x y_{n+1} + y_n n^2 - n y_n + 2n y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)x y_{n+1} + n^2 + n y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)x y_{n+1} + n(n+1)y_n = 0$ (showed)

Same as $y = \cot^{-1}x$.

Example: 5 If $y = e^{a\sin^{-1}x}$ then show that,

(i) $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2)y_n = 0$

Solve: Given that,

$$y = e^{a\sin^{-1}x}$$

$$y_1 = e^{a\sin^{-1}x} \cdot a \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{a e^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = a \underbrace{e^{a\sin^{-1}x}}_{y}$$

$$(1-x^2)y_1^2 = a^2 y^2 \quad [\text{বর্গ করে}]$$

$$2y_1 y_2(1-x^2) + y_1^2 (-2x) = a^2 \cdot 2y y_1$$

$$\Rightarrow (1-x^2)2y_1y_2 - 2x y_1^2 - a^2 2y y_1 = 0$$

(13)

$\Rightarrow 2y_1\{(1-x^2)y_2 - xy_1 - a^2y\} = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$ —— (1)

Diffi. by Leibnitz theorem.

$(1-x^2)y_{n+2} + {}^nc_1\, y_{n+2-1}(-2x) + {}^nc_2 \cdot -2\cdot1\cdot y_{n+2-2} - y_{n+1} \cdot x$

$+ {}^nc_1\, y_{n+1-1} \cdot 1 - a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2xn\,y_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n$

$- a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2xn\,y_{n+1} - n^2 y_n + \cancel{ny_n} - xy_{n+1} - \cancel{ny_n}$

$- a^2y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} + x(-2n-1)y_{n+1} + y_n(-n^2-a^2) = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

(showed)

__Example: 6,__ If $y = (\sin^{-1}x)^2$ then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Solve: (19)

Given that,

$$y=(\sin^{-1}x)^2 \quad \text{(i)}$$

$$y_1 = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$(\sqrt{1-x^2})\, y_1 = 2\sin^{-1}x$$

$$(\sqrt{1-x^2})^2 y_1^2 = 2^2 \cdot (\sin^{-1}x)^2$$

$$\Rightarrow (1-x^2)\, y_1^2 = 4\cdot y$$

$$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x)\cdot 1 - 4y_1 = 0$$

$$\Rightarrow 2y_1y_2(1-x^2) - 2y_1^2x - 4y_1 = 0$$

$$\Rightarrow 2y_1\{(1-x^2)y_2 - xy_1 - 2\} = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - 2 = 0 \quad \text{(ii)}$$

Diffi by leibnitz theorem

$$y_{n+2}(1-x^2) + {}^nc_1\, y_{n+2-1}(-2x) + {}^nc_2\, y_{n+2-2}(-2) - y_{n+1}\cdot x - {}^nc_1\, y_{n+1-1}\cdot 1 = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2nx\,y_{n+1} + \{-n(n-1)y_n\} - xy_{n+1} - n\cdot y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2nx\,y_{n+1} - n^2y_n + \cancel{ny_n} - xy_{n+1} - \cancel{ny_n} = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - x(2n+1)y_{n+1} - n^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0 \text{ (showed)}$$

Example 7:

(15)

If $x = \sin\left(\frac{1}{a}\ln y\right)$ and or $y = e^{a\sin^{-1}x}$ then show that, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Solve:

Given that,

$$x = \sin\left(\frac{1}{a}\ln y\right)$$

$$\Rightarrow \sin^{-1}x = \frac{\ln y}{a}$$

$$\Rightarrow \ln y = a\sin^{-1}x$$

$$\Rightarrow y = e^{a\sin^{-1}x}$$

$$y_1 = e^{a\sin^{-1}x} \cdot a \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\left(\sqrt{1-x^2}\right)y_1 = ae^{a\sin^{-1}x}$$

$$\Rightarrow \left(\sqrt{1-x^2}\right)y_1 = a \cdot y$$

$$\Rightarrow (1-x^2)y_1^2 = a^2y^2$$

$$\Rightarrow 2y_1 \cdot y_2(1-x^2) + y_1^2 \cdot (-2x) \cdot 1 - a^2 \cdot 2y \cdot y_1 = 0$$

$$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 - 2yy_1a^2 = 0$$

$$\Rightarrow y_2(1-x^2) - xy_1 - a^2y = 0$$

Diffi. ~~w~~ n times by Leibnitz theorem..

$$y_{n+2}(1-x^2) + {}^nC_1\, y_{n+2-1}(-2x) + {}^nC_2\, y_{n+2-2} \cdot -2 \cdot - y_{n+1}\, x - {}^nC_1\, y_{n+1-1} \cdot 1 - a^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx\, y_{n+1} - n(n-1)y_n - y_{n+1}x - ny_n - a^2 y_n = 0$$

$$= (1-x^2)\, y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n + \cancel{ny_n} - \cancel{ny_n} - a^2 y_n = 0$$

$$= (1-x^2)\, y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2) y_n = 0$$

showed

Example: 8 If $\sin^{-1}y = m\sin^{-1}x$ বা $y = \sin(m\sin^{-1}x)$ then, ~~the~~ show that,

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2-n^2) y_n = 0$$

Solve: Given that

$$\sin^{-1}y = m\sin^{-1}x$$

$$\Rightarrow y = \sin(m\sin^{-1}x)$$

$$y_1 = \cos(m\sin^{-1}x) \cdot \frac{m}{\sqrt{1-x^2}}$$

$$(\sqrt{1-x^2})\, y_1 = m\cos(m\sin^{-1}x)$$

$$(1-x^2)\, y_1^2 = m^2\cos^2(m\sin^{-1}x)$$

$$= m^2\{1-\sin^2(m\sin^{-1}x)\}$$

$$(1-x^2)\, y_1^2 = m^2(1-y^2)$$

$$\Rightarrow 2y_1y_2(1-x^2) + y_1^2 \cdot -2x\cdot 1 = m^2 \cdot -2yy_1$$

$$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 + m^2 2yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2 y = 0 \quad [2y_1 \text{ common নিয়ে}] \quad \text{---} (1)$$

Differenteiat n times by Leibntiz theorem -

$$\Rightarrow y_{n+2}(1-x^2) + {}^nC_1\, y_{n+2-1}\cdot(-2x) + {}^nC_2\, y_{n+2-2}\cdot(-2)\cdot 1 - y_{n+1}\cdot x$$
$$- {}^nC_1\, y_{n+1-1}\cdot 1 + m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2xn\,y_{n+1} - x\,y_{n+1} - n^2y_n + \cancel{ny_n} - \cancel{ny_n}$$
$$+ m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0$$

Showed

Example : 9 :

If $y = \sin\{a\ln(x+b)\}$ then showed that

(i) $(x+b)^2y_2 + (x+b)y_1 + a^2y = 0$

(ii) $(x+b)^2y_{n+2} + (2n+1)(x+b)y_{n+1} + (n^2+a^2)y_n = 0$

(12)

Solve: Given that,

(i)

$$y = \sin\{a \ln(x+b)\}$$

$$y_1 = \cos\{a\ln(x+b)\} \cdot \frac{a}{x+b} \cdot 1$$

$$(x+b) y_1 = a\cos\{a\ln(x+b)\}$$

$$\Rightarrow y_2(x+b) + y_1 \cdot 1 = -a\underbrace{\sin\{a\ln(x+b)\}}_{y} \cdot \frac{a}{x+b} \cdot 1$$

$$\Rightarrow (x+b)y_2 + y_1 = \frac{-a^2 y}{x+b}$$

$$\Rightarrow (x+b)\{(x+b)y_2 + y_1\} = -a^2 y$$

$$\Rightarrow (x+b)^2 y_2 + (x+b)y_1 + a^2 y = 0 \quad \text{(showed)} \quad \text{--- (1)}$$

(ii) From equation (1) ⇒

$$(x+b)^2 y_2 + (x+b)y_1 + a^2 y = 0$$

Diffi. n times by Leibntiz theorem,

$$y_{n+2}(x+b)^2 + {}^nc_1\, y_{n+2-1}\, 2(x+b) + {}^nc_2\, y_{n+2-2} \cdot 2 \cdot + y_{n+1}(x+b)$$

$$+ {}^nc_1\, y_{n+1-1} \cdot 1 + a^2 y_n = 0$$

$$\Rightarrow y_{n+2}(x+b)^2 + 2n(x+b)y_{n+1} + n^2 y_n - \cancel{n y_n} + (x+b)y_{n+1}$$

$$+ \cancel{n y_n} + a^2 y_n = 0$$

(10)

$$\Rightarrow (x+b)^2 y_{n+2} + (2n+1)(x+b) y_{n+1} + (n^2+a^2) y_n = 0$$

showed

Example 10:

If $y = e^{a\sin^{-1}x} = \sum_{n=0}^{\infty} a_n x^n$ then show that

(i) $(1-x^2)y_2 - xy_1 - a^2 y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2) y_n = 0$

(iii) $(n+1)(n+2)a_{n+2} = (n^2+a^2) a_n$

Solve:

(i) and (ii) Example 5 এ করা হয়েছে।

(iii) Given that,

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{--- (i)}$$

$$y_1 = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{--- (ii)}$$

$$y_2 = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-1-1}$$

$$= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \quad \text{--- (iii)}$$

$\therefore (1-x^2)y_2 - xy_1 - a^2 y = 0$

$= (1-x^2)\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - x\sum_{r=0}^{\infty} r a_r x^{r-1} - a^2\sum_{r=0}^{\infty} a_r x^r = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - x^2\sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - x\sum_{r=0}^{\infty} r a_r x^{r-1} - a^2\sum_{r=0}^{\infty} a_r x^r = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1)a_r \cdot x^{r-2}\cdot x^2 - \sum_{r=0}^{\infty} r a_r x^{r-1}\cdot x^1 - a^2\sum_{r=0}^{\infty} a_r x^r = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2+2} - \sum_{r=0}^{\infty} r a_r x^{r-1+1} - a^2\sum_{r=0}^{\infty} a_r x^r = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1)a_r x^r - \sum_{r=0}^{\infty} r a_r x^r - a^2\sum_{r=0}^{\infty} a_r x^r = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} a_r x^r\{r(r-1) + r + a^2\} = 0$

$= \sum_{r=0}^{\infty} (r-1)r\, a_r x^{r-2} - \sum_{r=0}^{\infty} a_r x^r(r^2 \cancel{-r} \cancel{+r} + a^2) = 0$

$= \sum_{r=0}^{\infty} r(r-1)a_r x^{r-2} - \sum_{r=0}^{\infty} a_r x^r(r^2 + a^2) = 0$

Replacing r by $(n+2)$ in the first term and n in the second term and then equating the coefficients of x^n we get,

$(n+2)\{n+2-1)a_{n+2}x^{n+\cancel{2}-\cancel{2}} - a_n x^n(n^2+a^2) = 0$

$\Rightarrow (n+2)(n+1)a_{n+2}x^n - (n^2+a^2)a_n x^n = 0$

$\therefore (n+2)(n+1)a_{n+2}x^n = (n^2+a^2)a_n x^n$ (showed)

Example 11:

(21)

If $y = \left(x+\sqrt{x^2+1}\right)^m$ then show that

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Solve:

Gives that,

$$y = \left(x+\sqrt{x^2+1}\right)^m$$

$$y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{x^2+1}}\cdot\cancel{2}x\cdot 1\right)$$

$$y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$\left(\sqrt{x^2+1}\right)y_1 = m\left(x+\sqrt{x^2+1}\right)^{m-1}\cdot\left(x+\sqrt{x^2+1}\right)^1$$

$$\left(\sqrt{x^2+1}\right)y_1 = m\underbrace{\left(x+\sqrt{x^2+1}\right)^{m-1+1}}_{y}$$

$$\left(\sqrt{x^2+1}\right)y_1 = my$$

$$(1+x^2)y_1^2 = m^2y^2$$

$$\Rightarrow 2y_1y_2(1+x^2) + y_1^2\cdot 2x\cdot 1 - m^2 2yy_1 = 0$$

$$\Rightarrow 2y_1\left\{(1+x^2)y_2 + xy_1 - m^2y\right\} = 0$$

$$\Rightarrow (1+x^2)y_2 + xy_1 - m^2y = 0 \longrightarrow (1)$$

$$\Rightarrow y_{n+2}(1+x^2) + {}^nc_1 y_{n+2-1}\cdot 2x + {}^nc_2 y_{n+2-2}\cdot 2 + y_{n+1}x + {}^nc_1 y_{n+1-1}\cdot 1 - m^2 y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2nx\,y_{n+1} + n^2 y_n - \cancel{ny_n} + y_{n+1}x + \cancel{ny_n} - m^2 y_n = 0$$

$$\rightarrow (1+x^2)y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0$$

Example 12 If $y = \ln(x+\sqrt{x^2+a^2})$ then show that

$$(x^2+a^2)y_{n+2} + (2n+1)x\,y_{n+1} + n^2 y_n = 0$$

Solve:

Given that,

$$y = \ln(x+\sqrt{x^2+a^2})$$

$$y_1 = \frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{1}{2\sqrt{x^2+a^2}}\cdot(2x+0)\right)$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+a^2}}}\left(\frac{\cancel{\sqrt{x^2+a^2}+x}}{\sqrt{x^2+a^2}}\right)$$

$$y_1 = \frac{1}{\sqrt{x^2+a^2}}$$

$$\Rightarrow (\sqrt{x^2+a^2})y_1 = 1$$

$$\Rightarrow (x^2+a^2)y_1^2 = 1$$

$$\Rightarrow 2y_1y_2(x^2+a^2) + y_1^2\cdot 2x = 0$$

(23)

$\Rightarrow y_2(x^2+a^2) + 2xy_1 = 0$

Diffi. n times by Leibnitz theorem

$\Rightarrow y_{n+2}(x^2+a^2) + {}^nc_1\, y_{n+2-1}\, 2x + {}^nc_2\, y_{n+2-2}\, 2 + y_{n+1}\, x + {}^nc_1\, y_{n+1-1} \cdot 1 = 0$

$\Rightarrow y_{n+2}(x^2+a^2) + n\,2x\,y_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n = 0$

$\Rightarrow (x^2+a^2)y_{n+2} + 2xn\,y_{n+1} + xy_{n+1} + n^2y_n - \cancel{ny_n} + \cancel{ny_n} = 0$

$\Rightarrow (x^2+a^2)y_{n+2} + (2x+1)xy_{n+1} + n^2y_n = 0$

Example 15:

If $y = e^{2\sin^{-1}x}$ then prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+4)y_n = 0$

Solve: Given that

$$y = e^{2\sin^{-1}x}$$

$$\Rightarrow y_1 = e^{2\sin^{-1}x} \cdot 2 \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (\sqrt{1-x^2})y_1 = 2\underline{e^{2\sin^{-1}x}}_{y}$$

$$\Rightarrow (1-x^2)y_1^2 = 4y^2 \quad [\text{বর্গ করে}]$$

$$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = 4 \cdot 2yy_1$$

$$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 - 8yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - 4y = 0 \quad \text{--- (1)}$$

Diffi. n times by leibnitz theorem -

$$y_{n+2}(1-x^2) + {}^nc_1 y_{n+2-1}(-2x) + {}^nc_2 y_{n+2-2}(-2) - y_{n+1} \cdot x - {}^nc_1 y_{n+1-1} - 4y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - 2xny_{n+1} - y_{n+1}x - n^2y_n + ny_n - ny_n - 4y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+4)y_n = 0$$

(showed)