Model 1: $\{(M/M/1) : (\infty/FCFS)\}$

Assumption:

(I) The arrival follow poisson distribution with mean arrival rate λ

(II) There service time " exponential distribution " average service rate μ

(III) arrival are infinit population

(IV) customer are served on a FCFS

(V) A server

(VI) Single waiting line and each arrival waits to be served regardless of the length of queue. (No bulking, No Reneging)

Following events may occur during a small interval of time, Δt, just before time t. It is assumed that, the system is in (n) state at time t.

(I) The system is in 'n' (no of customer) and no arrival & no departure, leaving the total to n customers.

(II) The system is in state n+1 and no arrival and one departure reducing the total to 'n' customers,

(III) The system is in state n-1 and one arrival and no departure bringing the total to n customer.

We will determine the probability P_n of n customers. in the system at time t and value of its various operating system characteristics.

Step 1: Obtain System of differential equation

If $P_n(t)$ is the probability of n customers at time t in the system. Then the probability that the system will contain n customers at time $(t+\Delta t)$ can be expressed as the sum of the joint probabilities of three mutually exclusive and collectively exhaustive cases as mentioned above.

That is,

For $n \geq 1$ and $t \geq 0$

$$P_n(t+\Delta t) = P_n(t) \cdot \text{Prob (no arrival in } \Delta t \text{ \& no departure in } \Delta t)$$
$$+ P_{n+1}(t) \cdot \text{Prob (no arrival in } \Delta t \text{ \& 1 departure in } \Delta t)$$
$$+ P_{n-1}(t) \cdot \text{Prob (one arrival in } \Delta t \text{ \& no departure in } \Delta t)$$

$$P_n(t+\Delta t) = P_n(t) \{ \text{Prob. (no arrival in } \Delta t) \times \text{Prob (no departure in } \Delta t) \}$$
$$+ P_{n+1}(t) \{ \text{Prob. (no arrival in } \Delta t) \times \text{Prob. (1 departure in } \Delta t) \}$$
$$+ P_{n-1}(t) \{ \text{Prob. (1 arrival in } \Delta t) \times \text{Prob. (1 departure in } \Delta t) \}$$

$$= P_n(t)\{1-\lambda\Delta t\}\{1-\mu\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}\{\mu\Delta t\}$$
$$+ P_{n-1}(t)\lambda\Delta t\{1-\mu\Delta t\}$$

$$= P_n(t) - (\lambda+\mu)\Delta t P_n(t) + P_{n+1}(t)\mu\Delta t + P_{n-1}(t)\lambda\Delta t$$
$$+ \text{terms involving } (\Delta t)^2$$

=

[Since Δt is very small $\therefore (\Delta t)^2$ part can be neglated]

Substructing $P_n(t)$ b/s and dividing by Δt, we get

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t)$$

By taking limit on both sides as $\Delta t \to 0$ we get

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t) \quad \text{—— (1)}$$

Similarly if there is no customer in the system at time $(t+\Delta t)$, Then there will be no service completion during Δt.

Thus for $n=0$ and $t \geq 0$ we get,

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)(1-\lambda\Delta t)\mu\Delta t$$

$$= P_0(t) - P_0(t)\lambda\Delta t + P_1(t)\mu\Delta t + O(\Delta t)$$

higher order term

$$\Rightarrow \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)\Delta t + \mu P_1(t)\Delta t + \frac{O(\Delta t)}{\Delta t}$$

$$\Rightarrow P_0'(t) = -\lambda P_0(t)\Delta + \mu P_1(t) \quad ; n=0 \quad \text{—— (2)}$$

Step 2: Obtain the system of steady-state equation:

In steady-state $P_n(t)$ is independent of time and the rate of change $P_n(t)$ can be considered to be zero. That is

$$\lim_{t\to\infty} P_n(t) = P_n$$

and $\lim_{t\to\infty} [P_n'(t)] = 0 \quad ; \quad n = 0, 1, 2 \ldots$

Consequently Eq (1) and (2) can be rewritten as

$$\lambda P_{n-1}(t) + \mu P_{n+1} - (\lambda+\mu) P_n = 0 \quad ; \quad n \geq 1 \quad \text{——— (3)}$$

$$\mu P_1 - \lambda P_0 = 0 \quad ; \quad n = 0 \quad \text{——— (4)}$$

Step 3 : Solve the system of steady-state difference equation
Solve

From, Eq (4) we get,

$$\mu P_1 = \lambda P_0$$

$$\text{or, } P_1 = \frac{\lambda}{\mu} P_0 \quad \text{——— (5)}$$

Putting $n=1$ in eq (3) we get,

$$\cancel{\lambda P_{n-1}(t) + \mu P_{n+1}} - (\lambda+\mu) P_n = 0$$

$$\text{or, } 0 = -(\lambda+\mu)P_1 + \lambda P_0 + \mu P_2$$

$$\text{or, } \mu P_2 = (\lambda+\mu)P_1 - \lambda P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda+\mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 - \left(\frac{\lambda}{\mu}\right) P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \quad \text{——— (6)}$$

Putting $n=2$ in equation ~~ee~~ we get,

$$0 = -(\lambda+\mu)P_2 + \lambda P_1 + \mu P_3$$

$$\text{or, } \mu P_3 = (\lambda+\mu)P_2 - \lambda P_2$$

$$\text{or, } P_3 = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 P_0 - \left(\frac{\lambda}{\mu}\right)^2 P_2$$

$$\text{or, } P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_2$$

$$\text{Similarly} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

To obtain the values of P_0, we make use of the fact that sum of all probabilities

$$\sum_{n=0}^{\infty} P_n = P_0 + P_1 + \cdots + P_n = 1$$

$$\text{or, } \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\text{or, } P_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

$$= \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 \cdots}$$

$$\text{or, } P_0 = 1 - \frac{\lambda}{\mu}$$

$$\rho = \frac{\cancel{\lambda}}{\mu} \frac{1/\mu}{\cancel{1/\lambda}} = \frac{\lambda}{\mu}$$

hence $P_0 = 1 - \rho$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n (1-\rho)$$

$$= \rho^n (1-\rho) \qquad \rho < 1, \; n = 0, 1, 2 \cdots$$

P_ω = Probability that an arrival customer has to wait

$$= 1 - P_0 = \frac{\lambda}{\mu}$$

This experience gives the required P.O of exactly n customer

Model III $\{(M/M/1) : N/FCFS\}$

Assumption:

(I) Arrival – Poison Distribution
(II) Service – exponential Distribution
(III) Server – 1
(IV) No of customer – N
(V) Diciplne – FCFS
(VI) Not more than N customers at time time t

Following events may ~~occu~~ occurs during a small inter arrival of time Δt, just before time t, It is assume that the system is in state 'n' at t

(I) in state 'n' – no arrival & no departure
(II) In state '$n+1$' – no arrival & 1 departure
(III) In state '$n-1$' – 1 arrival & no depature

Step 1: Obtain differential equation

If $P_n(t)$ is the probability of n customers at time t, then the probability in the system will contain n customers at time $(t+\Delta t)$ can be expressed as the sum of the joint probability of 3 mutually exclusive and collectively exhustive case mentioned above,

For, $n = 1, 2 \ldots N-1$ and $t \geq 0$

$$P_n(t+\Delta t) = P_n(t)\{1-\lambda\Delta t\}\{1-\mu\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}\mu\Delta t + P_{n-1}\{\lambda\Delta t\}\{1-\mu\Delta t\}$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + P_{n+1}(t)\mu\Delta t + P_{n-1}(t)\lambda\Delta t$$

$$\Rightarrow P_n'(t) = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t) \quad \text{———} \ (1)$$

$n = 1, 2 \ldots N-1$

Similarly if there are no customer in the system at time $(t+\Delta t)$ Then there will be no service completion during Δt,

Thus for $n=0$ & $t \geq 0$ we get

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(1-\lambda\Delta t)\mu\Delta t$$
$$= P_0(t) - \lambda P_0(t)\Delta t + P_1(t)\mu\Delta t + O(\Delta t)$$

$$\Rightarrow \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)\Delta t + \mu P_1(t)\Delta t$$

$$\Rightarrow P_0'(t) = -\lambda P_0(t)\cancel{\Delta t} + \mu P_1(t) \quad \text{———} \ (2)$$

$n=0$

$$\lambda_n = \begin{cases} \lambda & ; n = 0, 1 \ldots N-1 \\ 0 & ; n \geq N \end{cases}$$

$$\mu_n = \{\mu ; n = 1, 2 .\}$$

Now, from eqn (1)

$$P_N'(t) = -\mu P_n(t) + \lambda P_{n-1}(t) \quad \text{———} \ (*)$$

Step 2: Obtain steady-state equation

In steady-state, $P_n(t)$ is indepent of time and rate of chang $\to 0$

$$\lim_{t\to\infty} P_n(t) = 0 \quad \text{or,} \ \frac{d}{dt} \cancel{\lim_{t\to\infty}} [P_n(t)] = 0$$

$\therefore$ Steady state equation

$$\lambda P_{n-1}(t) + \mu P_{n+1} - (\lambda+\mu)P_n(t) = 0 \quad \text{———} \ (3)$$
$$\mu P_1(t) - \lambda P_0(t) = 0 \quad \text{———} \ (4)$$
$$\lambda P_{n-1}(t) - \mu P_n(t) = 0 \quad \text{———} \ (5)$$

Step 3: Solve differential equation

From, eq (4)

$$\mu P_1 = \lambda P_0$$

$$\text{or, } P_1 = \frac{\lambda}{\mu} P_0 \quad \text{——— (6)}$$

Putting $n=1$ in eqn 3 we get,

$$0 = -(\lambda+\mu)P_1 + \lambda P_0 + \mu P_2$$

$$\text{or, } \mu P_2 = (\lambda+\mu) P_1 - \lambda P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda+\mu}{\mu}\right) P_1 - \frac{\lambda}{\mu} P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 - \frac{\lambda}{\mu} P_0$$

$$\text{or, } P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Putting $n=2$ in eqn 3 we get,

$$0 = -(\lambda+\mu)P_2 + \lambda P_1 + \mu P_3$$

$$\text{or, } \cancel{\mu} P_3 = \left(\frac{\lambda+\mu}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2 P_0 - \frac{\lambda}{\mu} P_0$$

$$\text{or, } P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

similiarly, $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

Now, in order to obtain the value of P_0

$$\sum_{n=0}^{N} P_n = 1$$

$$\text{or, } \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\text{or, } P_0 \left[1 + \rho + \rho^2 + \cdots \rho^N\right] = P_0 \frac{1-\rho^{N+1}}{1-\rho}$$

$$\text{or, } P_0 = \frac{1-\rho}{1-\rho^{N+1}} \quad \Bigg| \quad P_n = \rho^N \frac{1-\rho}{1-\rho^{N+1}}$$

MODEL: 4 $\{(m/m/s):(\infty/FCFS)\}$

ASSUMPTION: Arrival – poison distribution
Service – exponential Distribution
Server – S
Customer – ∞
Dicipline – FCFS

If there 'n' customer in the system at any point in time, the following 2 cases arrives

(i) If $n<s$ (customers < server) then no queue, ~~(s-n)~~ $(s-n)$ server will not busy, combined service rate $\mu_n = n\mu$

(II) If $n \geq s$, then all server busy maximum no of customer in queue combined service rate $\mu_n = s\mu$

$\lambda_n = \lambda$ for all $n \geq 0$

$\mu_n = \begin{cases} n\mu; & n<s \\ \mu; & n \geq s \end{cases}$

Step 1: Obtain the system of O.D.E

For $1 \leq n < s$

$$P_n(t+\Delta t) = P_n(t)\{1-\lambda\Delta t\}\{1-n\mu\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}\{(n+1)\mu\Delta t\} + P_{n-1}(t)\{\lambda\Delta t\}\{1-(n-1)\mu\Delta t\}$$

$$= P_n(t) - (\lambda + n\mu)\Delta t\, P_n(t) + \lambda P_{n-1}(t)\Delta t + (n+1)\mu P_{n+1}(t)\Delta t + \text{teams involving } \Delta t^2$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu_n)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

$$\Rightarrow P_n'(t) = -(\lambda+n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \quad \text{—} (1)$$

Similarly when $n \geq s$

$$P_n(t+\Delta t) = P_n(t)\{1-\lambda\Delta t\}\{1-sM\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}\{sM\Delta t\}$$

$$+ P_{n-1}(t)\{\lambda\Delta t\}\{1-sM\Delta t\}$$

$$= P_n(t) - (\lambda+sM)P_n(t)\Delta t + sM P_{n+1}(t)\Delta t$$

$$+ \lambda P_{n-1}(t)\Delta t + [\Delta t]^2 \text{ terms}$$

$$\Rightarrow \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+sM)P_n(t) + sM P_{n+1}(t) + \lambda P_{n-1}(t)$$

$$\Rightarrow P_n'(t) = -(\lambda+sM)P_n(t) + sM P_{n+1}(t) + \lambda P_{n-1}(t) \quad \text{--- (2)}$$

For $n=0$ there is no customer

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)M\Delta t$$

$$P_0'(t) = -\lambda P_0(t) + M P_1(t) \quad \text{--- (3)}$$

Step 2: Obtain steady state equation

In steady state equation $P_n(t)$ is indepent of time and rate of change $P_n(t)$ can be considered to be zero.

$$\lim_{t\to\infty} P_n(t) = P_n$$

$$\lim_{t\to\infty} P_n'(t) = 0 \quad ; \quad n = 0, 1, 2, \dots$$

So, from eqn 1, 2, 3, we get

$$-(\lambda+nM)P_n + (n+1)M P_{n+1} + \lambda P_{n-1} = 0 \quad \text{--- (4)}$$

$$-\lambda P_0 + MP_1 = 0 \quad \text{--- (5)}$$

$$-(\lambda+sM)P_n + sM P_{n+1} + \lambda P_{n-1} = 0 \quad \text{--- (6)}$$

Step : 3 Solve Differential equation,

$$\lambda P_0 = \mu P_1$$

Or, $P_1 = \frac{\lambda}{\mu} P_0$; $n = 0$

Putting $n=1$ in eqn 4

$$-(\lambda+\mu)P_1 + 2\mu P_2 + \lambda P_0 = 0$$

$$\Rightarrow 2\mu P_2 = (\lambda+\mu)P_1 - \lambda P_0$$

$$\Rightarrow 2P_2 = \left(\frac{\lambda+\mu}{\mu}\right)P_1 - \frac{\lambda}{\mu}P_0$$

$$\Rightarrow 2P_2 = \left(\frac{\lambda+\mu}{\mu}\right)\cdot\frac{\lambda}{\mu}P_0 - \frac{\lambda}{\mu}P_0$$

$$\Rightarrow P_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

Putting $n=2$ in eqn 4

$$-(\lambda+\mu)P_2 + \lambda P_1 + 3\mu P_3 = 0$$

Or, $3\mu P_3 = (\lambda+\mu)P_2 - \lambda P_1$

Or, $P_3 = \frac{1}{3}\left(\frac{\lambda}{\mu}\right)^3 P_0$

Similarly $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$ $n = 0, 1, 2, \ldots$; $n < s$

$P_n = \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0$; $n \geq s$

In order to find P_0

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{S-1} P_n + \sum_{n=S}^{\infty} P_n = 1 \qquad \rho = \frac{\lambda}{S\mu}$$

$$\Rightarrow \sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 + \sum_{n=S}^{\infty} \frac{1}{S!\,S^{n-S}}\left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow P_0\left[\sum_{n=0}^{S-1} \frac{S^n}{n!}\left(\frac{\lambda}{S\mu}\right)^n P_0 + \sum_{n=S}^{\infty} \frac{S^n}{S!\,(S^{n-S})}\left(\frac{\lambda}{S\mu}\right)^n P_0 = 1\right.$$

$$\Rightarrow P_0\left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{S^S}{S!}\sum_{n=S}^{\infty} \rho^n\right] = 1$$

$$\Rightarrow P_0 = \left[\sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{S^n}{S!}\sum_{n=S}^{\infty} \rho^n\right]^{-1}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 & ; n < S \\ \frac{1}{S!\,(S^{n-S})}\left(\frac{\lambda}{\mu}\right)^n P_0 & ; n \geq S \end{cases}$$

MODEL 5 : $\{(M/E_K/1):\infty/FCFS\}$

Each customer has k phases one by one and a new service does not start until all k-phases have been completed, therefore each arrival increases the number of phases by k in the system.

$\lambda_n = \lambda$ phases arrive per unit.

$\mu_n = K\mu$ phases served per unit (K is positive integer

Step 1 : Obtain system of differential equation

let, $P_n(t)$ be the probability that there are n phases in the system at time t.

$$P_n(t+\Delta t) = P_n(t)\{1-\lambda\Delta t\}\{1-K\mu\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}K\mu\Delta t + P_{n-K}\cdot\lambda\Delta t\{1-K\mu\Delta t\}$$

$$= P_n(t) - (\lambda+K\mu)\Delta t\cdot P_n(t) + P_{n+1}(t)K\mu\Delta t + P_{n-K}\cdot\lambda\Delta t + \text{terms containing } \Delta t^2$$

$$\Rightarrow \frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -(\lambda+K\mu)P_n(t) + K\mu P_{n+1}(t) + \lambda P_{n-1}(t)$$

$$\Rightarrow P_n'(t) = -(\lambda+K\mu)P_n(t) + K\mu P_{n+1}(t) + \lambda P_{n-1}(t) \quad \text{for } n\geq 1 \text{ —— } (1)$$

similary $P_0'(t) = -\lambda P_0(t) + K\mu P_1(t)$ —— 2

for $n=0$

Step 2 Obtain the system of steady state

Since $P_n(t)$ is indepent of time so

$$-(\lambda+K\mu)P_n + K\mu P_{n+1} + \lambda P_{n-K} = 0 \quad ; n \geq 1$$

$$\cancel{\lambda P_0 + K\mu P_{n+1} + \lambda P_{n-K}} =$$

$$\lambda P_0 + K\mu P_1 = 0 \quad ; n=0$$

let, $\rho = \frac{\lambda}{K\mu}$ and divide eqn by $K\mu$, we get

$$(1+\rho)P_n = \rho P_{n-K} + P_{n+1} \quad ; n \geq 1 \text{ --- } (3)$$

$$P_1 = \rho P_0 \quad ; n=0 \text{ --- } (4)$$

Step 3 : Solve equation

We make use of generating function (GF) for solving above eqn $G(x) = \sum_{n=0}^{\infty} P_n x^n \quad ; |x| \leq 1$ --- (5)

multipling equation (3) by x^n and summing over $n=1$ to ∞, we get,

$$(1+\rho)\sum_{n=1}^{\infty} P_n x^n = \rho \sum_{n=1}^{\infty} P_{n-K} x^n + \sum_{n=1}^{\infty} P_{n+1} x^n \text{ --- } (6)$$

Adding and sub. ~~P_0~~ + ρP_0 b/s eqn we get,

$$(1+\rho)\sum_{n=1}^{\infty} P_n x^n + \rho P_0 = P_1 + \rho \sum_{n=1}^{\infty} P_{n-K} \cdot x^n + \sum_{n=1}^{\infty} P_{n+1} x^n$$

$$(1+\rho)\left[\sum_{n=1}^{\infty} P_n x^n + P_0\right] - P_0 = \rho \sum_{n=1}^{\infty} \cdot P_{n-K} + \left[P_1 + \sum_{n=1}^{\infty} P_{n+1} x^n\right]$$

$$(1+\rho)\sum_{n=0}^{\infty} P_n x^n - P_0 = \rho \sum_{n=K}^{\infty} P_{n-K} x^n + \frac{1}{x}\sum_{n=0}^{\infty} P_{n+1} x^{n+1}$$

since, $P_{n-k} = 0$ for $n-k = 0$

or, for $n-k = j$; $n+1 = i$

$$(1+\rho)\sum_{n=0}^{\infty} P_n x^n - P_0 = \rho \sum_{j=0}^{\infty} P_j x^{j+k} + \frac{1}{x}\sum_{i+1}^{\infty} P_i \cdot x^i$$

$$= \rho x^k \sum_{j=0}^{\infty} P_j x^j + \frac{1}{x}\left[\sum_{i=0}^{\infty} x^i - P_0\right.$$

$$(1+\rho) G(x) - P_0 = \rho x^k \cdot G(x) + \frac{1}{x}\left[G(x) - P_0\right]$$

$$G(x) = \cancel{\rho}\ \cancel{\rho x}^k \frac{P_0(1-x)}{(1-x) - \rho x(1-x^k)}$$

$$G(x) = P_0\left[1 - Px\left(\frac{1-x^k}{1-x}\right)\right]^{-1}$$

By Binomial Theorm

$$G(x) = P_0 \sum_{n=0}^{\infty} (x\rho)^n \left(\frac{1-x^k}{1-x}\right)^n \longleftarrow (7)$$

since $\left(\frac{1-K^k}{1-x}\right) = 1 + x + x^2 + \cdots\cdots x^{k-1}$

Therefore, $G(x) = P_0 \cdot \sum_{n=0}^{\infty} (x\rho)^n (1 + x + x^2 + \cdots\cdots x^k)$

$$G(x) = P_0 \sum_{n=0}^{\infty} \rho^n (x + x^2 + \cdots x^k)^n$$

To find Value of P_0 and P_n

let us Put $x = 1$, we get

$$G(1) = P_0 \sum_{n=0}^{\infty} \rho^n (1 + 1^2 + \cdots\cdots + 1^k)^n$$

$$= P \sum_{n=0}^{\infty} \rho^n K^n$$

$$G(1) = P_0 \left[\frac{1}{1-K\rho}\right]$$ sum of infinit GP

for $x=1$ in eq (5) we get

$$G(1) = \sum_{n=0}^{\infty} P_n = 1$$

So, $1 = P_0 \left[\frac{1}{1-K\rho}\right]$

Or, $P_0 = 1-K\rho$

Put, $P_0 = 1-K\rho$ in eq (7) we get,

$$G(x) = (1-K\rho) \sum_{n=0}^{\infty} (x\rho)^n (1-x^k)^n (1-x)^{-n} \quad \text{--- (8)}$$

But, $(1-x^k)^n = 1 - {}^nC_1 x^k + {}^nC_2 (x^k)^2 + \cdots + (-1)^n \, {}^nC_n (x^k)^n$

$$= \sum_{t=0}^{n} (-1)^t \, {}^nC_t \, x^{tk}$$

and $(1-x)^{-n} = \sum_{i=0}^{\infty} (-1)^i \, {}^{-n}C_i \, x^i$

$$= \sum_{i=0}^{\infty} (-1)^{2i} \, {}^{(n+i-1)}C_i \cdot x^i$$

$${}^{-n}C_i = (-1)^i \cdot {}^{(n+i-1)}C_i \, x^{i+tk+n} \quad \text{--- (9)}$$

Combining the co-efficient of x^n b/s of eq 9 we get

$$P_n = (1-K\rho) \sum_{n,t,i} \rho^n (-1)^n \, {}^nC_t \cdot {}^{n+i-1}C_i$$

MODEL : $\{(M/M/S):(N/FCFS)\}$
or, $M/M/S/K$

If there are 'n' customers in the queing system at any point in time then the following two cases arise:

(I) If $n<S$ (customers < server), no queue, $(S-n)$ server are not busy, combined service rate $\mu_n = n\mu$

(II) $N > n \geq S$, all server busy, maximum no of customer in queue, will be $(n-S)$

$$\lambda_n = \begin{cases} \lambda & ; n<N \\ 0 & ; n \geq N \end{cases}$$

and
$$\mu_n = \begin{cases} n\mu & ; n<N \\ S\mu & ; S \leq n \leq N \end{cases}$$
$n \geq N$

Step 1 : Obtain the system of O.D.E

for $1 \leq n < N$

$$P_n(t+\Delta t) = P_n(t)\{1-\lambda\Delta t\}\{1-n\mu\Delta t\} + P_{n+1}(t)\{1-\lambda\Delta t\}\{(n+1)\mu\Delta t\} + P_{n-1}(t)\{\lambda\Delta t\}\{1-(n-1)\mu\Delta t\}$$

$$\cancel{P_n(t+\Delta t)} = P_n(t) - (\lambda + n\mu)P_n(t)\Delta t + (n+1)\mu P_{n+1}(t)\Delta t + \lambda P_{n-1}(t)\Delta t + \text{terms involving } (\Delta t)^2$$

$$\Rightarrow \frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = -(\lambda+n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) + \lambda P_{n-1}(t)$$

$$\Rightarrow P_n'(t) = -(\lambda+n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) + \lambda P_{n-1}(t) \quad \text{--- (1)}$$
$1 \leq n < S$.

no customer in the system then there will be no service completion.

$$P_0(t+\Delta t) = P_0(t)\{1-\lambda \Delta t\} + P_1(t)\mu \Delta t$$

$$\Rightarrow P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \quad \text{--- (3)}$$

Step 2 : Obtain the system of steady state

In the (s.s.t) $P_n(t)$ is independent of time and rate of change $P_n(t)$ can be considered to be zero.

$$\lim_{t\to\infty} P_n(t) = P_n$$

$$\lim_{t\to\infty} P_n'(t) = 0 \quad ; n=0,1,2\ldots$$

$$\therefore \; -\lambda P_0 + \mu P_1 = 0 \quad ; n=0 \quad \text{--- (4)}$$

$$-(\lambda+n\mu)P_n + (n+1)\mu P_{n+1} + \lambda P_{n-1} = 0 \quad \text{--- (5)}$$

$$1<n<s$$

$$-(\lambda+s\mu)P_n + s\mu P_{n+1} + \lambda P_{n-1} = 0 \quad \text{--- (6)}$$

$$n \geq s$$

Thus these are equations constitute the system of steady state differential equation.

Step 3 : Solve equations

$$\lambda P_0 = \mu P_1$$

$$\therefore P_1 = \frac{\lambda}{\mu} P_0 \quad ; n=0$$

Putting $n=1$ in eqn 5

$$-(\lambda+\mu)P_1 + \lambda P_0 + 2\mu P_2 = 0$$

or, $2\mu P_2 = \frac{(\lambda+\mu)}{\mu}\frac{\lambda}{\mu} P_0 - \frac{\lambda}{\mu} P_0 =$

$$P_2 = \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$\therefore P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

similarly, $P_n = \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^2 P_0 \quad ; \; n \geq s$

In order to find P_0

We have

$$\sum_{n=0}^{N} P_n = 1$$

$\rho = \frac{\lambda}{s\mu}$

$$\text{or, } \sum_{n=0}^{s-1} P_n + \sum_{n=s}^{N} P_n = 1$$

$$\text{or, } \sum_{n=0}^{s-1}\left[\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]P_0 + \sum_{n=s}^{n}\left[\frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n\right]P_0 = 1$$

$$\Rightarrow P_0\left\{\left[\sum_{n=}^{s-1}\left[\frac{s^n}{n!}\left(\frac{\lambda}{s\mu}\right)^n\right] + \sum_{n=s}^{\infty}\left[\frac{s^n}{s!\, s^{n-s}}\left(\frac{\lambda}{s\mu}\right)^n\right]\right\} = 1$$

$$\Rightarrow P_0\left\{\sum_{n=0}^{s-1}\frac{s\rho^n}{n!} + \frac{(s\rho)^s}{s!}\,\frac{(1-\rho^{N-s+1})}{1-\rho}\right\} = 1$$

$$\Rightarrow P_0 = \left[\sum_{n=0}^{s-1}\frac{s\rho^n}{n!} + \frac{(s\rho)^n}{s!}\left(\frac{(1-\rho)^{N-s+1}}{1-\rho}\right)\right]^{-1}$$

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 & ; \; 0 \leq n < s \\ \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 & ; \; s \leq n \leq N. \\ 0 & ; \; n > N \end{cases}$$

~~MODEL-1~~

λ = Average Customer arrival rate/ Average no of arrival per unit of time

μ = Average service rate ~~of~~ / Average no of costome service per unit time

MODEL-1

(I) Expected no of customers in the system

$$L_s = \frac{\lambda}{\mu-\lambda}$$

(II) Expected no of customer in the queue

$$L = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(III) Average waiting time in the system

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$$

(IV) Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

(V) Average length of non-empty-queue

$$L = \frac{\mu}{\mu-\lambda}$$

(VI) Probability that there are 'n' customers in the system

$$P_n = \left[\frac{\lambda}{\mu}\right]^n \left[1-\frac{\lambda}{\mu}\right]$$

(7) Traffic intensity $\rho = \frac{\lambda}{\mu}$

(8) Probability that there are in nobody in the system

$$P_0 = 1-\frac{\lambda}{\mu}$$

(9) Prob. that there ~~are~~ is at least one customer in the system / queue is busy $P_1 = 1 - P_0$

Problem 1 : A television repairman find that the time spent on his job has an exponential distrubution with a mean of 30 min. If he repairs the sets in the order in which the came in and if the arrival of sets follows P.D with an approx. average rate of 10 per 8-hour day. Find

1. Expected no of TV sets in the system ?
2. Average time T.V. sets has to wait before being served ?
3. Expected idle time of repairmen each day
4. Average queue length ?

Sol

$$\mu = \frac{1}{30} \times 60 = 2 \text{ sets per hr}$$

$$\lambda = 10/8 = 5/4 \text{ sets per hr}$$

(1) Expected no of TV sets in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{5/4}{2 - 5/4} = 2 \text{ TV sets}$$

(2) the average time spent by TV sets in the system before being served is

$$W_s = \frac{L_s}{\lambda} = \frac{2}{5/4} = 1.6 h$$

average waiting time for TV sets in queue is:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{5/4}{2(2 - 5/4)}$$

$$= 0.83 h$$

(3) Expected Idle time of repairman each day

No of hour for which repairman remain busy in 8-hour day

$$8 \times \frac{\lambda}{\mu} = 8 \times 5/4 = 5 \text{ hour}$$

Hence Idle time for rep.m 8-hour day will be

$$8-5 = 3 \text{ hr}$$

4) Average queue length : $\frac{\mu}{\mu-\lambda} = \frac{2}{2-5/4} = 2.66$

$= 3$ custom

Problem 2 : Trucks at a platform wight-bridge arrive according to Poison Distribution. The time required to weigh the truck follows an exponential distribution - the mean arrival rate is 12 trucks per day. and the mean service rate is 18 trucks per day.

1) what is the probability that no trucks are in the system?

2) Find Average no trucks waiting for service?

3) Find average time a truck wait for weighing service begin?

(4) Find Prob that an arriving truck will have to wait for service?

$\lambda = 12$ trucks per day

$\mu = 18$ truck per day.

(1) Probability that no trucks are in the system.

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{12}{18} = 0.333$$

2) Average no of trucks in the queue for service

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{12^2}{18(18-12)} = 1.333 \text{ trucks}$$

3) Average time a truck wait for weighing service begin

$$W_q = \frac{L_q}{\lambda} = \frac{1.3}{12} = 0.111 \text{ hr}$$

4) Prob. that arriving truck will have to wait for service

$$P_b = 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{12}{18} = 0.6667$$

$$= 66.67\%$$

Problem: A road transport company has one ~~repi~~ reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can on an average service 12 customers per hour.

(a) Find average no. of customers waiting for the service of the clerk?

(b) Find average time a customer has to wait before being served.

(c) The mangment is contemplating to inst-

$\lambda = 8$ per hour
$\mu = 12$ per hour

(1) Average no of customers waiting for the service of the desk $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

$$= \frac{8^2}{12(12-8)} = \frac{64}{12 \times 4} = 1.33 \approx 2$$

(2) Average time spent by customer in the system before being served is $W_s = \frac{1}{\mu-\lambda} = \frac{1}{12-8} = \frac{1}{4}$ hour

$= 15$ min

average waiting time for a customer in the queue $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6}h = 10m$

(e) The managment is contemplating to install a computer for handling information and reservation. this is expected to reduce service time from 5 to 3 min. The Aditional ~~service time~~ cost of system works out to Rs 50 per day.. If the cost of goodwill will of having to wait is 12 paise, per minute spent waiting, before being served, should the company install the computer? Assume 8 h working day.

2) Company will install a computer if additional cost pricing + goodwill cost after setting computer system is less than good will cost before installing system

Goodwill cost before installing system $W_s = 15$ m

$= 15 \times \frac{12}{100} = Rs\ 1.8$

Total G.C $= 8 \times 8 \times 1.8 = Rs\ 115.20$

Goodwill cost After installing system

$\mu = 20$

$W_s = \frac{1}{\mu - \lambda} = \frac{1}{20-8} = \frac{1}{12}$ Hr $= 5$ min

Total G.C $= 8 \times 8 \times \frac{12}{100} \times 5 = Rs\ 38.40$

$\therefore$ Total cost = Computer cost + G.C

$= 50 + 38.40 = 88.40$ Rs

This cost is less then good will cost before installing system $115.20 - 88.40 = Rs\ 26.80$

98 Problem. (M/M/1):(N/FCFS)

Consider a single server queuing system with Poisson inputs and exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hr and the maximum permissible calling unit in the system is two. (N)

(a) Derive the steady-state probability distribution of the no of calling units in the system
(b) Cal. expected no of customer in the system.
(d) Expected waiting time of a customer in queue?
(c) Expected queue length

MODEL M/M/1 : N/FCFS

1. Probability that is no customer in the system.

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

2. Probability that there are 'n' customer in the system

$$P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$$

3. Expected no of customer in the system

$$L_s = \sum_{n=1}^{N} nP_n = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

4. Expected queue length or expected no of customer waiting in the queue.

$$L_q = L_s - \frac{\lambda}{\mu} = L_s - \frac{\lambda(1-P_N)}{\mu}$$

(5) Expected waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda(1-P_N)}$$

(6) Expected waiting time of a customer in the queue

$$W_q = W_s - \frac{1}{\mu} \text{ or, } \frac{L_q}{\lambda(1-P_N)}$$

Sol:

$\lambda = 3$ units per hour

$\mu = \frac{1}{0.25}$ hour $= 4$ units per hour

$N = 2$ $\quad \rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

(a) $P_n = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$

$= \frac{(1-0.75)}{1-(0.75)^{2+1}} \cdot (0.75)^n$

$= (0.43)(0.75)^n$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$

$= \frac{1-0.75}{1-(0.75)^3}$

$= 0.431$

(b) $L_s = \sum_{n=1}^{N} n \cdot P_n$

$= \sum_{n=1}^{2} n(0.43)(0.75)^n$

$= (0.43)\left[0.75 + 2 \times (0.75)^2\right]$

$= 0.81$

(c) $L_q = L_s - \frac{\lambda(1-P_N)}{\mu}$

$P_2 = P_0 \rho^n = (0.43)(0.75)^2$
$= 0.2424$

$1 - P_2 = 1 - 0.2424 = 0.7576$

$= 0.81 - \frac{3 \times 0.7576}{4}$

$= 0.2418$

(d) Expected waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda(1-P_N)}$$

$$= \frac{0.81}{3 \times 0.7576}$$

(e) Expected waiting time of a customer in the queue.

$$W_q = \frac{L_q}{\lambda(1-P_N)}$$

$$= \frac{0.2418}{3 \times (0.7576)}$$

$$= 0.1063$$

(f) Probability that there is no queue

$$P_0 + P_1 = (0.431 + 0.3225)$$
$$= 0.7535$$

$P_1 = (0.43)(0.75)^1$
$= 0.3225$

MODEL IV $M/M/S : \infty/FCFS$
$M/M/K : \infty/FCFS$

Important METHODS

Probability that the system have no customer and 'n' customer

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda} \right]^{-1}$$

$$P_n = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0 & : n < s \\ \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0 & ; n \geq s \end{cases}$$

The expected no of customers waiting in the queue (length of line)

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n$$

$$= \left[\frac{1}{(s-1)!} \left(\frac{\lambda}{\mu}\right)^s \frac{\lambda\mu}{(s\mu - \lambda)^2} \right] P_0$$

The expected no of customer in the system

$$L_s = L_q + \frac{\lambda}{\mu}$$

The expected waiting time of a customer in the queue

$$W_q = \frac{L_q}{\lambda}$$

The expected waiting time that a customer spends in the system

$$W_s = W_q + \frac{1}{\mu}$$

$$= \frac{L_q}{\lambda} + \frac{1}{\mu}$$

The probability that all server are simultaneously busy

$$P(n \geq s) = \sum_{n=s}^{\infty} P_n$$

$$= \frac{1}{s!} \left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda} P_0$$

18 **Problem :** A super market has two sales girls at the sales counters. If the service time for each customer is exponential with a mean of 4 minutes and if the people arrive in a poisson fashion at the rate of 10 an hour. then calculate the

(a) Expected no of customer waiting in the queue?

(b) Expected customer waiting time in the queue?

(c) If a customer has to wait is

Solu

$$s = 2$$

$$\mu = \frac{1}{4} \text{ per min}$$

$$\lambda = 10 \text{ per hr.}$$

$$= \frac{10}{60} \text{ per min}$$

$$= \frac{1}{6} \quad " \quad "$$

$$\rho = \frac{\lambda}{s\mu} = \frac{1/6}{2 \cdot 1/4} = \frac{1}{3}$$

(a) Expected no of customer waiting in the queue

$$Lq = \left[\frac{1}{(S-1)!}\left(\frac{\lambda}{M}\right)^S \frac{\lambda M}{(SM-\lambda)^2}\right] P_0$$

$$= \left[\frac{1}{(2-1)!}\left(\frac{1/6}{1/4}\right)^2 \frac{1/6 \times 1/4}{2\times\frac{1}{4}-\frac{1}{6}}\right] P_0 \quad = 0.1674\, P_0$$

(b) $$P_0 = \left[\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{M}\right)^n + \frac{1}{S!}\left(\frac{\lambda}{M}\right)^S \frac{SM}{SM-\lambda}\right]^{-1}$$

$$= \sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{\frac{1}{6}}{\frac{1}{4}}\right)^n + \frac{1}{2!}\left(\frac{1/6}{1/4}\right)^2 \frac{2\times\frac{1}{4}}{\{2\cdot\frac{1}{4}-\frac{1}{6}\}} = \frac{1}{2}$$

$\therefore Lq = 0.1674 \times \frac{1}{2} = 0.0837$

(b) Expected waiting time of a customer in the queue.

$$Wq = \frac{Lq}{\lambda} = \frac{0.0837}{1/6} = 0.5042$$

(c) Expected waiting time for a customer in the system

$$Ws = Wq + \frac{1}{M} = 0.5042 + \frac{1}{1/4} = 4.5 \text{ min}$$

(d) Expected no of customers in the system $Ls = Lq + \frac{\lambda}{M}$

$= 0.0837 + \frac{4}{6}$

$= 0.7497$

(e) Probability of having to wait for being served

$P(\cancel{n \geq S}) = \frac{1}{S!}$

$$P(n \geq S) = \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \frac{S\mu}{S\mu - \lambda} P_0$$

$$= \frac{1}{2!}\left(\frac{1}{6}\right)^2 \frac{8 \times 1/4}{\left[2 \times \frac{1}{4} - \frac{1}{6}\right]} \times \frac{1}{2} = \frac{1}{6}$$

(e) The function of time the server are busy

$$\rho = \frac{\lambda}{S\mu} = 1/3$$

Therefore expected idle time for each sales girls.
is $1 - \frac{1}{3} = 2/3 = 67\%$.

(M/M/s : N/FCFS)
(M/M/C/K model

IMPORTANT METHOD

(1) The Probability of 'n' customers in the system in the steady-state condition is

$$P_n = \begin{cases} \frac{1}{n!}\left[\frac{\lambda}{\mu}\right]^n P_0 & ; 0 < n < s \\ \frac{1}{s!\, s^{n-s}}\left[\frac{\lambda}{\mu}\right]^n P_0 & ; s \le n \le N \end{cases}$$

(2) P_0 (i.e) system shall be idle is

$$P_0 = \left[\sum_{n=0}^{s-N} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=s}^{N} \frac{1}{s!\, s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s (N-s+1)\right]^{-1}$$

(3) The expected number of customers in the queue :

$$L_q = \sum_{n=s}^{N} (n-s) P_n$$

$$= \frac{(s\rho)^s \rho}{s!\,(1-\rho)^2}\left[1-\rho^{N-s+1} - (1-\rho)(N-s+1)\rho^{N-s}\right] P_0$$

(4) The expected number of customers in the system

$$L_s = L_q + \frac{\lambda(1-P_N)}{\mu}$$

$$\lambda_{eff} = \lambda(1-P_N)$$

$$\Rightarrow L_q + s - P_0 \sum_{n=0}^{s-1} \frac{(s-n)}{n!}\left(\frac{\lambda}{\mu}\right)^n$$

(5) Expected waiting time in the system

$$W_s = \frac{L_s}{\lambda(1-P_N)}$$

(6) The Expected waiting time in the queue

$$W_q = W_s - \frac{1}{\mu}$$

$$= \frac{L_q}{\lambda(1-P_N)}$$

Problem

Let there be an automabile inspection situation with three inspection stalls ($S=3$). ~~and~~ Assume that cars waits in such a way that when stall becomes vacant, the car at the head of the line pulls up to it. The station can accomodate almost seven cars at one time ($N=7$). The arrival pattern is Poison with a mean of one car every minute during the peak hour. The service time is exponential with mean of 6 min

Sol: ~~$S=3$~~ (a) Find average no of customer in queue during the peak hr

Sol ~~$S=3$~~

(b) Find the average number of customers in the system during peak hours?

(c) expected waiting time in queue and in the system

(d) Average no of cars per hr that can not enter the station because of full capacity.

Sol: $s=3$, $N=7$

$\lambda = 1$ car per min

$\mu = \cancel{6}\ 1/6$ car per min

$\frac{\lambda}{\mu} = 6$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left[\frac{\lambda}{\mu}\right]^n + \sum_{n=s}^{N} \frac{1}{s!\,s^{n-s}}\left[\frac{\lambda}{\mu}\right]^n\right]^{-1}$$

$$= \left[\sum_{n=0}^{3-1} \frac{1}{n!}[6]^n + \sum_{n=3}^{7} \frac{1}{3!\,3^{n-3}}[6]^n\right]^{-1}$$

$$= \frac{1}{1141}$$

(a) Expected no of customer in the queue

$$L_q = \sum_{n=3}^{7} (n-s) P_n$$

$$= 0P_3 + (4-3)P_4 + (5-3)P_5 + (6-3)P_6 + (7-3)P_7$$

$$= 0P_3 + P_4 + 2P_5 + 3P_6 + 4P_7$$

$$= \left[\frac{1}{3!\,3^{4-3}}(6)^4 + \frac{2}{3!\,3^{5-3}}(6)^5 + \frac{3}{3!\,3^{6-3}}(6)^6 + \frac{4}{3!\,3^{7-3}}(6)^7\right]P_0$$

$$= \left[\frac{1}{3!\,3}(6)^4 + \frac{2\times 6^5}{3!\,3^2} + \frac{3\times 6^6}{3!\,3^3} + \frac{4\times 6^7}{3!\,3^4}\right]\frac{1}{1141}$$

$$= 3.09 \text{ cars}$$

(b) Expected number of customers in the system

$$L_s = L_q + \frac{\lambda(1-P_N)}{\mu} = L_q + 6(1-P_N)$$

$$6(1-P_7) = 6\left[1 - \frac{(6)^7}{3!\,3^4}\times\frac{1}{1141}\right] = \cancel{3}\ 0.5\times 6 = 3$$

$$L_s = 3.09 + 3 = \cancel{3.59}\ 6.09 \text{ cars}$$

(c) Expected waiting time in the system

$$W_s = \frac{L_s}{\lambda(1-P_N)}$$

$$= \frac{6.09}{0.5} = 12.3 \text{ min}$$

(d) Expected waiting time in queue

$$W_q = W_s - \frac{1}{\mu}$$

$$= 12.3 - 6$$
$$= 6.3 \text{ min}$$

(d) The expected no of ~~customers~~ car per hour that can not enter the station

$$60 \times \lambda \times P_N = 60 \times 1 \times P_7$$

$$= 60 \times \frac{6^7}{3!\,3^4} \times \frac{1}{1141} = 30.4 \text{ cars per hour}$$

Full capacity Probability

13. Multiphase service Queing MODEL

$M/E_k/1 : \infty/FCFS$

Important Method

① The expected number of customers in the queue:

$$L_q = \left[\frac{k+1}{2k}\right]\left[\frac{\lambda^2}{\mu(\mu-\lambda)}\right]$$

model - 1st

② The expected waiting time of a customer in the queue

$$W_q = \frac{L_q}{\lambda} = \left[\frac{k+1}{2k}\right]\left[\frac{\lambda}{\mu(\mu-\lambda)}\right]$$

③ Expected waiting time of a customer in the system

$$W_s = W_q + \frac{1}{\mu} = \left[\frac{k+1}{2k}\right]\left[\frac{\lambda}{\mu(\mu-\lambda)}\right] + \frac{1}{\mu}$$

④ Expected number of customer in the system

$$L_s = L_q + \frac{\lambda}{\mu}$$

or, $L_s = \lambda W_s$

Problem: In a factory cafeteria the customers (employees) have to pass through three counters. The customers buy coupons at the first counter, select and colect the snacks at the second counter and collect tea at the third. The server at each counter takes, on an average, 1.5 min athough the distribution of service time is approximatly poison at an average rate of 6 per hour.

(a) The average ~~to~~ number of customer in the queue?

(b) the average time a customer spend waiting in the cafeteria ?

(c) The average time a customer spends in the cafeteria before being served?

(d) The average no of customers in the cafeteria?

(e) The most probable time in getting the service ?

Sol: $K=3$, ~~λ~~.

$\lambda = 6$ customer/hour

Dipanwita Roy

service time per phase $= 1.5$

we know, average service time per phase $= \frac{1}{\mu K}$

$\therefore \frac{1}{\mu K} = 1.5$

or, $\mu = \frac{1}{3 \times 1.5} = \frac{1}{4.5}$ per min

$= 13.34$ customer/hour

(a) Average no of customer in the queue.

$$L_q = \left[\frac{K+1}{2K}\right]\left[\frac{\lambda^2}{\mu(\mu-\lambda)}\right] = \frac{3+1}{2\times 3}\left[\frac{6^2}{13.34(13.34-6)}\right]$$

$$= \frac{4}{6} \times \frac{6^2}{13.34 \times 7.34}$$

$$= 0.25$$

(b) We average time a customer spends waiting in the cafeteria

$$W_q = \frac{L_q}{\lambda} = \frac{0.25}{6} = 0.042 \text{ hour}$$
$$= 2.52 \text{ min}$$

(c) The average time a customer spends in the cafeteria before being served.

$$W_s = W_q + \frac{1}{\mu}$$
$$= 0.042 + \frac{1}{13.34}$$
$$= 0.042 + 0.075$$
$$= 0.117 \text{ hour}$$

(d) Average no of customer in the cafeteria :

$$L_s = L_q + \frac{\lambda}{\mu} = \lambda W_s$$
$$= 6 \times 0.117$$
$$= 0.702 \text{ customers}$$

(e) Most probable time in getting service

$$\frac{k-1}{k\mu} = \frac{3-1}{3 \times 13.34} = \frac{1}{20} \text{ hour}$$

last phase [illegible]

Problem: A Barber with a one man shap takes exactly 25 min to complete one hair cut, If customers arrive in a Poisson fashion at an average rate of one every 40 min. How long on an average must a customer wait for service and in the shop?

$M = \frac{1}{25}$ customer per min

$\lambda = \frac{1}{40}$ customer per min

$K \to \infty$ [because service is constant]

$$W_q = \lim_{K\to\infty} \left[\frac{K+1}{2K}\right]\left[\frac{\lambda}{M(M-\lambda)}\right]$$

$$= \lim_{K\to\infty} \left[\frac{1}{2} + \frac{1}{2K}\right]\left[\frac{1/40}{\frac{1}{25}\left(\frac{1}{25} - \frac{1}{40}\right)}\right]$$

$$= 20.8 \text{ min}$$

$$W_s = W_q + \frac{1}{M}$$

$$= 20.8 + 25 = 45.8 \text{ min.}$$

14 $M/E_K/1 : \infty/FCFS$

Problem: A airline maintaince base has facilities for overhauling only one aeroplane engine at a time. Hence, to return the aeroplane into use at the earliest the policy is to stagger the overhauling of the 4 engines of each aeroplane. (i.e) only one engine is overhauled each time.

Under this policy aeroplane have arrivals according to a poison process, at a mean rate of one per day. The time required for an engine overhaul has exponential distribution with mean of half day.

A proposal has been made to change policy so as to overhaul all four engines ~~conseque~~ consecutively each time an aeroplane comes into the shop. This will quadruple the expected service time, plane would need to come into the shop only one-fourth time as often. Compare two alternatives on a meaningful basis

Sol :

The two alternative will be compared on the basis of the cost of waiting time cost of the aeroplane that require overhauling

First Alternative : $M/M/1 : \infty/FCFS$ queueing system

$\lambda = 1$ aeroplane per day

$\mu = 2$ aeroplane per day

Therefore, the average number of aeroplane in the system is $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{1}{2-1} = 1$

Second Alternative :

$K = 4$; service time per aeroplane $= 4 \times \frac{1}{2} = 2$ days.

~~$= 4 \times \frac{1}{2} = 2$ days~~

So, $\mu = \frac{1}{2}$ aeroplane per day

$\lambda = \frac{1}{4}$ aeroplane per day

Thus average no of aeroplane in the system

$$L_s = \frac{K+1}{2K} \times \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$= \frac{4+1}{2\times 4} \times \frac{(\frac{1}{4})^2}{\frac{1}{2}(\frac{1}{2}-\frac{1}{4})}$$

$$= 0.08$$

second Alternative is less than first alternative.

Therefor the waiting cost for requiring overhauling in the 2nd alternative will be less. Hence the proposal be accepted.

Problem: A hospital clinic has a doctor examining every patient brought is for a check-up on an average the doctor spends 4 min on each phase of check-up. the distribution of time spent on each is approx exponential. If each patient goes through four phrases in the check-up and ($k=4$) if the arrival of the patients at the doctor's office approximately poisson at the average rate of 3 per hour, what is the average time spent by a patient waiting in the doctor's office? what is the average time spent in the examination? What is the past probable time spent in the examination?

Sol Service time per phase : 4 min

$K=4$

$\lambda = 3$ patient / hr

$= 3/60 = \frac{1}{20}$ patient / min

Average service time per phase $= \frac{1}{\mu K}$

$= \frac{1}{\mu K} = 4$

$\Rightarrow \mu = \frac{1}{4\times4} = \frac{1}{16}$ Patient / m.

(i) Average time spent by a patient waiting in the doctor's office.

$$W_q = \frac{K+1}{2K} \times \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{4+1}{2\times4} \times \frac{1/20}{1/16\left(\frac{1}{16}-\frac{1}{20}\right)} = 40 \text{ min}$$

(II) Average time spent in the examination

$$\frac{1}{\mu} = \frac{1}{1/16} = 16 \text{ min}$$

(III) most probabable time spent in the examination

last phase $\frac{K-1}{K\mu} = \frac{4-1}{4\times 1/16} = 12 \text{ min}$

15.

FINITE CALLING Population Queuing MODEL or Machine-Repairman MODEL

$\{M/M/1 : M/GD\}$ Single server finit source of arrival

IMPORTANT METHOD

(1) Probability that the system is idle:

$$P_0 = \left[\sum_{n=0}^{M} \frac{M!}{(M-n)!}\left(\frac{\lambda}{M}\right)^n\right]^{-1}$$

(2) The probability that there are 'n' customers in the system:

$$P_n = \left[\frac{M!}{(M-n)!}\left(\frac{\lambda}{M}\right)^n\right]P_0 \qquad n = 1, 2, \ldots M$$

(3) The expected number of customer in the queue,

$$L_q = \sum_{n=1}^{M} (n-1) P_n$$

$$= M - \left(\frac{\lambda + M}{\lambda}\right)(1 - P_0)$$

(4) The expected number of custome is the system

$$L_s = \sum_{n=0}^{M} n P_n$$

$$= L_q + (1 - P_0)$$

(5) The expected waiting time of a customer in the queue.

no of customer finit হবে

$$W_q = \frac{L_q}{\lambda(M - L_s)} \qquad \lambda_{eff} = \lambda(M - L_s)$$

(6) the expected waiting time of a customer in the system

$$W_s = W_q + \frac{1}{M} \quad \text{or,} \quad \frac{L_s}{\lambda(M - L_s)}$$

Problem : A machanic repairs four machines. The mean time between service repuirements is 5 hours for each machine and forms an exponential distribution. The mean repair time is one hour and also follows the same distribution pattern. Machine downtime cost Rs 25 per hour and the mechanic costs Rs 55 per day. Determine the following :

(a) Probability that the service facility will be idle?

(b) Probability of various number of machines [0 to 4] to be out of order and being repaired

(c) Expected no of machines waiting to be repaired and being repaired?

/(e)

Sol

$$\lambda = \frac{1}{5} = 0.2 \text{ machines/hr}$$

$$\mu = 1 \text{ machine/hr}$$

$$M = 4$$

$$\rho = \frac{\lambda}{\mu} = 0.2$$

(a) The Probability that the system shall be idle (or empty)

$$P_0 = \left[\sum_{n=0}^{m} \frac{M!}{(M-n)!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[\sum_{n=0}^{4} \frac{4!}{(4-n)!} (0.2)^n \right]^{-1}$$

$$= \left[1 + \frac{4!}{3!}(0.2) + \frac{4!}{2!}(0.2)^2 + \frac{4!}{1!}(0.2)^3 + \frac{4!}{0!}(0.2)^4 \right]^{-1}$$

$$= 0.4030$$

(b) The Probability that there shall be various no of machines [0 though 4] in the system.

$$P_n = \frac{M!}{(m-n)!}\left(\frac{\lambda}{M}\right)^n P_0 \; ; \quad n \le M$$

n (a)	$\frac{m!}{(m-n)!}\left(\frac{\lambda}{M}\right)^n$ (b)		Probability $c = (b) \times P_0$
0	1.00	1.00×0.4030	0.4030
1	0.80		0.3224
2	0.48		0.1934
3	0.19		0.0765
4	0.00		0.000

(c) Expected no of machines waiting to be repaired:

$$L_q = \sum_{n=1}^{M} (n-1) P_n$$

$$P_4 = \frac{4!}{(4-4)!}(0.2)^4 P_0 = 0.0155$$

$$= 0 + P_2 + 2P_3 + 3P_4$$

$$= 0 + 0.1934 + 2 \times 0.765 + 3 \times 0.0155$$

$$= 0.3924$$

(d) Expected no of machines to be out of order and being repaired

$$L_s = L_q + (1 - P_0)$$

$$= 0.3924 + 1 - 0.4030$$

$$= 0.99 \text{ machines.}$$

(d) Expected no of waiting time of machines waiting in queue to be repaired and being repaired?

sol

expected waiting machine in the queue

$$W_q = \frac{L_q}{\lambda(M-L_s)}$$

$$= \frac{0.3924}{0.2\times(4-0.99)}$$

$$= 0.65 \text{ hours}$$

Expected waiting time of machines in the system

$$W_s = W_q + \frac{1}{\mu} = 0.65+5 = 5.65 \text{ hr}$$

(e) Expected downtime cost per day would it be economical to engage two machines each repairing only two machines?

sol Expected downtime of machines per day when there is one machine

= Expected no of machines in the system × 8-hour day × no of mechanics

$$= 0.99\times8\times1 = 7.92 \text{ hours/day}$$

Down time cost = 7.92 × 25

= Rs 198 per day

Total cost = 55 + 198 = Rs 253 / day

If there are two mechanics each serving machines M=2

Therefore, $P_0 = \left[\sum_{n=0}^{2} \frac{M!}{(M-n)!} \left[\frac{\lambda}{M}\right]^n \right]^{-1}$

$= \left[\sum_{n=0}^{2} \frac{2!}{(2-n)!} (0.2)^n \right]^{-1} = 0.68$

Cost Analysis

Expected no of machines in the system

$L_s = M - \frac{\mu}{\lambda}(1-P_0)$

$= 2 - \frac{1}{0.2}(1-0.68) = 0.4$ machine

Expected down time of machines per day = 0.4 × 8 × 2

= 6.4 hour/day

Total cost = Machanic cost + Downtime cost

= 2×55 + 6.4 × 25 = Rs 270 par day.

16

FINIT Calling Population

or

Finit source of arrival queuing mode

or

machine repairman model

m/m/s : M/GD   multiple server - Finite source of arrival

IMPORTANT METHODS

(1) The probability that there are 'n' customers in the system

$$P_n = \begin{cases} \frac{M!}{n!(M-n)!}\left[\frac{\lambda}{\mu}\right]^n P_0 & ; 0 \le n < S \\ \frac{M!}{(M-n)!\, S!\, S^{n-S}}\left[\frac{\lambda}{\mu}\right]^n P_0 & ; S < n \le M \end{cases}$$

(2) The probability that the system is idle

$$P_0 = \left\{ \sum_{n=0}^{S-1} \frac{M!}{n!(M-n)!}\left[\frac{\lambda}{\mu}\right]^n + \sum_{n=S}^{M} \frac{M!}{(M-n)!\, S!\, S^{n-S}}\left[\frac{\lambda}{\mu}\right]^n \right\}^{-1}$$

(3) Expected no of customers in the queue:

$$L_q = \sum_{n=S+1}^{M} (n-S) P_n = L_s - S + \sum_{n=0}^{S} (S-n) P_n$$

(4) The expected no of customers in the system:

$$L_s = L_q + \frac{\lambda(M-L_s)}{\mu}$$

$$\lambda_{eff} = \lambda(M-L_s) = \sum_{n=0}^{M} \lambda(M-n)P_n$$

or,

$$L_s = \sum_{n=0}^{S-1} nP_n + \sum_{n=S}^{M} nP_n$$

(5) Expected waiting time of a customer in the queue.

$$W_q = \frac{L_q}{\lambda(m - L_s)} \qquad \lambda_{eff} = \lambda(m - L_s)$$

(6) Expected waiting time of a customer in the system

$$W_s = W_q + \frac{1}{\mu} \text{ or } \frac{L_s}{\lambda(m - L_s)}$$

Problem: There are 5 machines each of which when running suffers breakdown at an average rate of 2 per hour. There are 2 service men and only one man can work on one machine at a time. If 'n' machines are out of order when $n > 2$ then $(n-2)$ of them have to wait until a serviceman is free. Once a service man starts work on a machine the time to complete the repair has an exponential distribution mean of 5 min

(a) Find the distribution of the no of machin out of action at a given time.

(b) Find the average time an out of action machine has to spend waiting for the repairs to start and being repaired.

Sol

(a) $\lambda = 2$ machines per hour

$\mu = \frac{1}{5} \times 60 = 12$ machines per hour

$M = 5$ machines

$S = 2$

$\frac{\lambda}{\mu} = \frac{2}{12} = \frac{1}{6}$

then, $P_0 = \left[\sum_{n=0}^{2-1} \frac{S!}{(S-n)!\,n!} \left[\frac{1}{6}\right]^n + \sum_{n=2}^{S} \frac{S!}{(S-n)!\,2!\,2^{n-2}} \left[\frac{1}{6}\right]^n \right]^{-1}$

$= \frac{648}{1493}$

(a) Average no of machines out of action at a given time

$L_q = \sum_{n=2+1}^{S} (n-2) P_n$

$= P_3 + 2P_4 + 3P_5$

$P_3 = \frac{5!}{2!\,2!\,2} \left(\frac{1}{6}\right)^3 P_0$

$= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 2 \times 216} \times \frac{648}{1493}$

$= 0.029$

$P_4 = \frac{5!}{1!\,2!\,2^2} \left(\frac{1}{6}\right)^4 P_0$

$= 0.01157 \times \frac{648}{1493}$

$= 0.00502$

$P_5 = \frac{5!}{0!\,2!\,2^3} \left(\frac{1}{6}\right)^5 P_0$

$= 0.00096 \times \frac{648}{1493}$

$= 0.00042$

$L_q = 0.0299 + 2 \times 0.00502 + 3 \times 0.0042$

$= 0.0412$ machines

$$L_s = \sum_{n=0}^{S-1} n P_n + \sum_{n=S}^{M} n P_n$$

$$= \sum_{n=0}^{1} n P_n + \sum_{n=2}^{5} n P_n$$

$$= P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5$$

$$= [0.83 + 2\times 0.27 + 3\times 0.0\cancel{6}9 + 4\times 0.1137 + 5\times 0.00096]$$

P_5

$$= 0.887 \text{ machines}$$

(b) Average waiting time an out of action machines has to spend waiting for the repair and being repaired.

$$W_q = \frac{L_q}{\lambda(M - L_s)} = \frac{0.0412}{2(5-0.887)}$$

$$= \frac{0.0412}{0.226}$$

$$= 0.005 \text{ hour}$$

$$W_s = \frac{L_s}{\lambda(M - L_s)} = \frac{0.887}{2(5-0.887)} = 0.108 \text{ hour.}$$