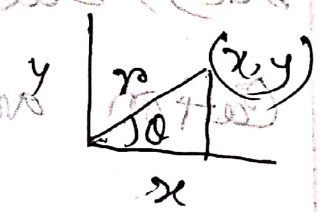
1. a) Define complex numbers, Is complex number algebra? Explain why or why not. For any complex number $z_1, z_2, \ldots, z_n$ prove that $|z_1+z_2| \le |z_1|+|z_2|$.

Answer: A complex number is a number that can be expressed in the form $a+bi$ where "a" and "b" are real numbers, and "i" is the imaginary unit, defined as the square root of -1.

a complex variable $z$ is an ordered pair $(x, y)$ where $x$ & $y$ are real number. It is written as $z = x+iy$ ; $i=\sqrt{-1}$

$x$ is called real part and $y$ is called imaginary part.

in polar coordinates,




$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2 \therefore r = \sqrt{x^2+y^2} \therefore |z| = \sqrt{x^2+y^2}$

Called modulus of $z$.

$\tan\theta = \frac{y}{x}$ ∴ $\theta = \tan^{-1}(y/x)$ ; ∴ $\theta$ is called argument of $z$.

Yes, complex numbers are part of algebra. Algebra is a branch of mathematics that deals with symbols and the rules for manipulating those symbols. It encompasses various topics such as equations, expressions, functions and mathematical structures.

Complex number are an extension of the real numbers and form a mathematical structure called a field. The field of complex numbers, denoted by C, includes both the real numbers (R) and the imaginary unit (i). Complex numbers obey the same algebraic rules as real numbers, such as addition, subtraction, multiplication and divition.

Complex numbers also have algebraic proper-
ties such as commutativity, associativity
and distributivity. These properties
allows for the manipulation and simpli-
fication of complex expressions, equations
and inequalities using algebraic techni-
ques.

Complex numbers are an integral part of
algebra, and their properties and opera-
tions are studied within the framework
of algebraic systems.

The conjugate of complex number a+bi
is obtained by changing the sign
of the imaginary part resulting
in a-bi.

Proof:

$$|z_1+z_2| \le |z_1|+|z_2|$$

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} \quad \left[\because |z|^2 = z\cdot\bar{z}\right]$$

$$= (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{z_1\cdot\bar{z}_2}$$

$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) \quad \left[\because \mathrm{Re}(z) = \frac{z+\bar{z}}{2}\right]$$

$$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \quad \mathrm{Re}\,z \le |z|$$

$$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1|\cdot|z_2| \quad x \le \sqrt{x^2+y^2}$$

$$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2 \quad |\bar{z}| = |z|$$

$$\therefore |z_1+z_2| \le |z_1|+|z_2| \text{ (Proved)}$$

b) Find all values of $z$ for which $z^5 = -32$, and locate these values in the complex plane.

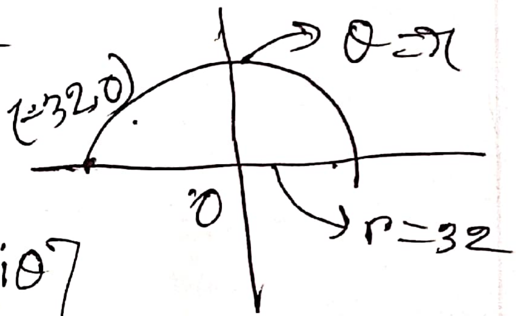
<u>Solution:</u>

$$z^5 = -32$$

Let $z^5 = w$

$w = -32 + 0i \quad \therefore x = r\cos\theta,\ y = r\sin\theta$

$r = \sqrt{(-32)^2 + 0^2} = 32$




$-32 = 32e^{i\pi} \quad [\because re^{i\theta}]$

$= 32e^{i(\pi + 2k\pi)}$

$(-32)^{1/5} = 32^{1/5} e^{i\left(\frac{\pi + 2k\pi}{5}\right)} = 2e^{i\left(\frac{\pi + 2k\pi}{5}\right)}$

$k = 0,\ z_1 = 2e^{i\pi/5}$

$k = 1,\ z_2 = 2e^{i\frac{3\pi}{5}}$

$k = 2,\ z_3 = 2e^{i\cdot\frac{5\pi}{5}} = 2e^{i\pi} = 2(-1) = -2$

$k = 3,\ z_4 = 2e^{i\frac{7\pi}{5}}$

$k = 4,\ z_5 = 2e^{i\frac{9\pi}{5}}$

① Represent graphically the set of values of $z$ for which

$$\left|\frac{z-3}{z+3}\right| = 2$$

Answer:

$$\left|\frac{z-3}{z+3}\right| = 2 \qquad |z| = \sqrt{x^2+y^2}$$

$$\Rightarrow \left|\frac{x-3+iy}{x+3+iy}\right| = 2 \qquad z = x+iy$$

$$\Rightarrow \frac{\sqrt{(x-3)^2+y^2}}{\sqrt{(x+3)^2+y^2}} = 2$$

$$\Rightarrow \frac{(x-3)^2+y^2}{(x+3)^2+y^2} = 4 \Rightarrow \frac{x^2-6x+9+y^2}{x^2+6x+9+y^2} = \frac{4}{1}$$

$$\Rightarrow x^2-6x+9+y^2 = 4x^2+24x+36+4y^2$$

$$\Rightarrow 9-36 = 4x^2+4y^2+24x-x^2-y^2+6x$$

$$\Rightarrow -27 = 3x^2+3y^2+30x$$

$$\Rightarrow 3x^2+3y^2+30x+27 = 0$$

$$\Rightarrow x^2+y^2+10x+9 = 0$$

$\Rightarrow x^2+2\cdot x\cdot 5+5^2+y^2+9-25=0$

$\Rightarrow (x+5)^2+y^2=16$

$\Rightarrow (x+5)^2+y^2=4^2$ —— (1)

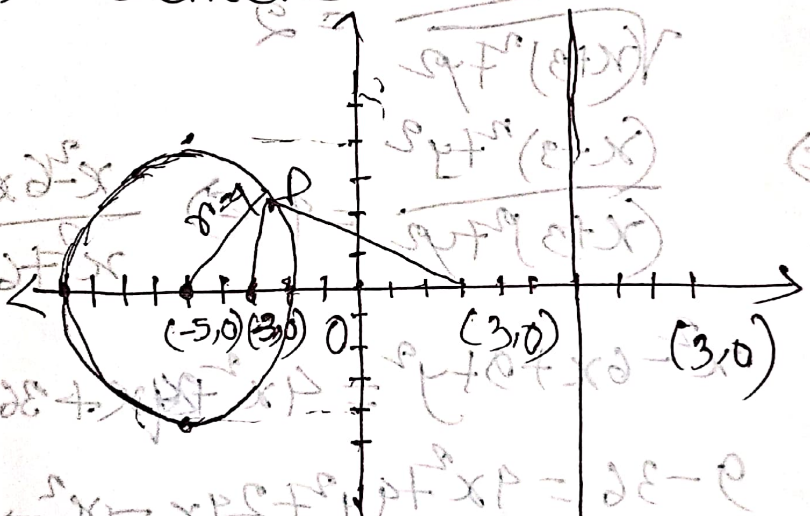
Compare with $(x-a)^2+(y-b)^2=r^2$ (circle)

$(a,b) \Rightarrow$ centre

$r \Rightarrow$ radius

So, centre $(-5,0)$ and radius 4 unit.

this is a circle




Geometrically any point P on this circle is such that the distance from P to $(3,0)$ is twice the distance P to $(-3,0)$

2\. a) Define limit and continuity of complex function. Let $w = f(z) = (z^2+1)^{1/2}$. Then show that (i) $z = \pm i$ are branch points of $f(z)$ and (ii) a circuit around both branch points produces no change in the branches of $f(z)$.

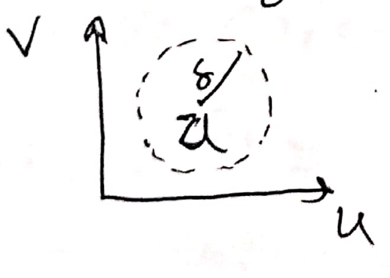
Ans:

limit: Let $f(z)$ be any function of the complex variable $z$ defined in a bounded and closed domain D. Then $l$ is said to be the limit of $f(z)$ as if for any $\varepsilon > 0 \ \exists \ \delta > 0$ such that

$$|f(z) - l| < \varepsilon \text{ for all values of } z$$

i.e $0 < |z - a| < \delta$

in symbols $\lim_{z \to a} f(z) = l$.



Complex tinuity: A function $f(z)$ of a complex variable $z$, defined in the closed and bounded domain $D$ is said to be continuity at $a \in D$, if and only if for $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(z) - f(a)| < \varepsilon \quad \forall \ |z-a| < \delta$$

$z = a$ if $\lim_{z \to a} f(z) = f(a)$

1. $w = f(z) = (z^2+1)^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$
$= f_1(z)$

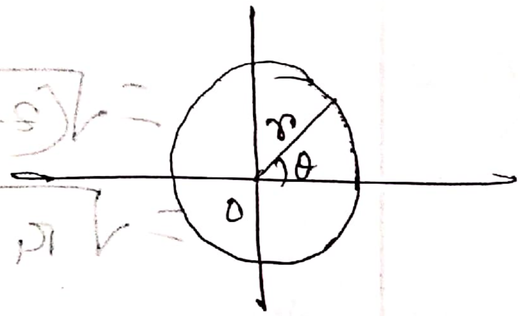
let, $z = 0$

$f(z) = z^{1/2} = (re^{i(\theta+2\pi)})^{1/2}$

$= r^{1/2} e^{i\theta/2} e^{\pi i}$

$= -r^{1/2} e^{i\theta/2}$

$= f_2(z)$



ii. $f(z) = (z^2+1)^{1/2} = (z^2 - i^2)^{1/2}$

$= \{(z+i)(z-i)\}^{1/2} = \sqrt{z+i} \cdot \sqrt{z-i}$

let, $z+i = r_1 e^{i\theta_1}$ & $z-i = r_2 e^{i\theta_2}$

$\sqrt{z+i} = \sqrt{r_1}\, e^{i\theta_1/2}$, $\sqrt{z-i} = \sqrt{r_2}\, e^{i\theta_2/2}$

Encircle $z=i$, not $z=-i$

$\sqrt{r_1}\, e^{i\frac{(\theta_1+2\pi)}{2}}$

$= \sqrt{r_1}\, e^{\frac{i\theta_1}{2}} e^{\pi i}$

$= \sqrt{r_1}\, e^{i\theta_1/2} (\cos\pi + i\sin\pi)$

$= -\sqrt{r_1}\, e^{i\theta_1/2}$

Encircle $z=-i$, not $z=i$

$\sqrt{r_2}\, e^{i\frac{(\theta_2+2\pi)}{2}}$

$= \sqrt{r_2}\, e^{i\theta_2/2} e^{\pi i}$

$= \sqrt{r_2}\, e^{i\theta_2/2}(-1)$

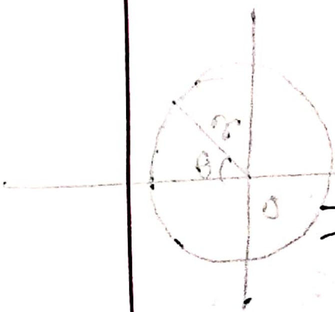
$= -\sqrt{r_2}\, e^{i\theta_2/2}$

$\therefore$ the function is changed.

$= i f(a) [\theta]_0 = 2\pi i f(a)$

so, $z=-i$ is a branch point.

(iii) $f(z) = (z^2+1)^{1/2}$



$$= \sqrt{(z+i)(z-i)}$$

$$= \sqrt{r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}}$$

$$= \sqrt{r_1 r_2 e^{i\theta_1 + i\theta_2}}$$

$$= \sqrt{r_1 r_2}\; e^{\frac{i(\theta_1+\theta_2)}{2}}$$

let, encircle both $z=i$ & $z=-i$.

$$= \sqrt{r_1 r_2 e^{(i\theta_1 + 2\pi i + i\theta_2 + 2\pi i)}}$$

$$= \sqrt{r_1 r_2}\; e^{i(\theta_1+\theta_2)/2}\; e^{2\pi i}$$

$$= \sqrt{r_1 r_2} \cdot e^{i(\theta_1+\theta_2)/2} \cdot 1$$

$$= \sqrt{r_1 r_2}\; e^{i(\theta_1+\theta_2)/2}$$

$$= \sqrt{r_1 r_2}\; e^{i(\theta_1+\theta_2)/2}$$

$e^{2\pi i}$
$= \cos 2\pi + i \sin 2\pi$
$= 1 + 0$
$= 1$

The function remain unchanged

b) Solution: Define harmonic function with example: Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Soln: If the 2nd partial derivatives of u and v with respect to x & y exists and continuous in R, then we find -

$$\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

* If G is an open subset of $\mathbb{C}$ then function $u: G \to R$ is said to be harmonic if u has continuous second order partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

This equation is called laplace's equation

$= i f(a) \cdot [\theta]_0 = 2\pi i f(a)$

i. A function F on a region G is analytic iff ~~P~~ Re f = u and Im f = v are harmonic functions which satisfy the 'Cauchy Riemann eq'

ii. A region G is simply connected iff each harmonic function u on G such that F = u + iv is analytic in G.

* $u = e^{-x}(x\sin y - y\cos y)$

* $\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y)$

$= e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$

$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y - e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$

$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$

Again, $\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$

$= xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y$

$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + e^{-x}\sin y$

$+ye^{-x}\cos y + e^{-x}\sin y$

$= -x^2 e^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$
$\quad - xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$

$= 0$

$\therefore u$ is harmonic

*

Example:

$f(z) = u + iv$

let $u = x^2 - y^2$

$\dfrac{\partial u}{\partial x} = 2x$ $\qquad\qquad$ $\dfrac{\partial u}{\partial y} = -2y$

$\dfrac{\partial^2 u}{\partial x^2} = 2$ — (i) $\qquad$ $\dfrac{\partial^2 u}{\partial y^2} = -2$ — (ii) eq (i) + eq (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$ (harmonic laplace's equ)

$f(z) = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x, y} \text{ --- (i)} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x, y} \text{ --- (ii)}$$

by (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x, y} - \frac{\partial^2 v}{\partial x, y}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example: $v(x, y) = 3x^2 y - y^3$

3. a) Let $f(z)$ be analytic inside and on the boundary $C$ of a simply-connected region $R$. Prove Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}\,dz$$

Soln:

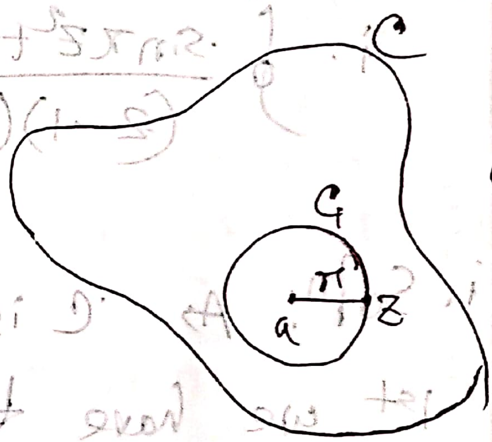
eqn of circle $(C_1)$

$|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$

$\Rightarrow z = a+re^{i\theta}$ ——— (i)

$dz = rie^{i\theta}\,d\theta$ ——— (ii)




when $r \to 0$ then $f(z) \to f(a)$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

$$= \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot rie^{i\theta}\,d\theta$$

$$= i\int_{C_1} f(a+re^{i\theta})\,d\theta = i\int_{C_1} f(z)\,d\theta$$

$$= i\int_{C_1} f(a)\,d\theta = if(a)\int_{C_1} d\theta$$

$$= if(a)\,[\theta]_0^{2\pi} = 2\pi i\, f(a)$$

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$ (Proved)

3. Evalute:

i. $\oint \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$   ii. $\oint \frac{e^{2z}}{(z+1)^4}\,dz$

i. Soln, As C is the circle $|z| = 3$

1st we have to do partial fraction of

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz$$

$$- \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz$$

By cauchy's Integral formula with $a=2$ & $a=1$, we get —

$$\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{z-2} = 2\pi i\left(\sin\pi 2^2+\cos\pi 2^2\right)$$

[a = 2]

$$= 2\pi i(\sin 4\pi+\cos 4\pi)$$
$$= 2\pi i\,(0+1)$$
$$= 2\pi i$$

$$\oint_C \frac{\sin\pi z^2+\cos\pi z^2}{z-1} = 2\pi i\left(\sin\pi 1^2+\cos\pi 1^2\right)$$

[a=1]

$$= 2\pi i\,(0-1)$$
$$= -2\pi i$$

The required integral $= 2\pi i-(-2\pi i) = 4\pi i$

ii. Soln: $\oint \frac{e^{2z}}{(z+1)^4}dz$ where $C$ is the circle $|z|=3$.

Let, $f(z)=e^{2z}$ and $a=-1$ in cauchy integral formula, Comparing with,

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}\,dz \quad \text{(i)}$$

$n+1=4 \quad \therefore n=3$

$\therefore f'''(z) = 8e^{2z}$

$\therefore f'''(-1) = 8e^{-2}$

$$\therefore 8e^{-2} = \frac{3!}{2\pi i}\oint_c \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\therefore \oint_c \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8e^{-2}\cdot 2\pi i}{3!} = \frac{8\pi i e^{-2}}{3}$$

(Ans.)

4. a) Find laurent series about the indicated singularity for each of the following functions

i. $\dfrac{e^{2z}}{(z-1)^3}$ ; $z=1$  ii. $\dfrac{z}{(z+1)(z+2)}$ ; $z=-2$

Soln: let $z-1=u$ $\therefore z=1+u$

$\therefore \dfrac{e^{2z}}{(z-1)^3} = \dfrac{e^{2+2u}}{u^3} = \dfrac{e^2 \cdot e^{2u}}{u^3} = \dfrac{e^2}{u^3} \cdot e^{2u}$

$= \dfrac{e^2}{u^3}\left\{1+2u+\dfrac{(2u)^2}{2!}+\dfrac{(2u)^3}{3!}+\cdots\right\}$

$= \dfrac{e^2}{u^3}+\dfrac{2e^2}{u^2}+\dfrac{4e^2}{2u}+\dfrac{8e^2}{6}+\cdots$

$= \dfrac{e^2}{(z-1)^3}+\dfrac{2e^2}{(z-1)^2}+\dfrac{4e^2}{2(z-1)}+\dfrac{4e^2}{3}+\cdots$

$z=1$ is a pole of order 3

$\therefore$ The series converges for all value of $z$ without 1.

iii. $\frac{z}{(z+1)(z+2)}$ ; $z=-2$

let, $z+2=u \therefore z=u-2$

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-2+1)u} = \frac{u-2}{(u-1)u} = \frac{2-u}{(1-u)u}$$

$$\Rightarrow \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u}(1+u+u^2+u^3+\cdots)$$

$$= \frac{2-u}{u} + \frac{2u-u^2}{u} + \frac{2u^2-u^3}{u} + \cdots$$

$$= \frac{2}{u} - 1 + 2 - u + 2u - u^2 + 2u^2 - u^3 + \cdots$$

$$= \frac{2}{u} + 1 + u + u^2 + u^3 + \cdots$$

$$= \frac{2}{(z+2)} + 1 + (z+2) + (z+2)^2 + \cdots$$

$\therefore z=-2$ is a pole

$0 < |z+2| < 1$

1) Evalute $\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz$ around the circle C with equation $|z|=3$.

Soln:

$|z|=3 \quad \therefore$ Centre $\equiv (0,0)$ radius $=3$.

There is a simple pole at $z=2$ inside C

5. a) i. Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$

ii. show that $\int_0^{\infty} \frac{\sin x}{x} dx = \pi/2$

i.

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta =$ Real part $(e^{i\theta})$

$\sin\theta =$ Im part $(e^{i\theta})$

$I =$ Real part $\int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta} d\theta$

putting $\cos\theta = \frac{z+z^{-1}}{2}$

$d\theta = \frac{dz}{zi}$

$$I = \int_c \frac{z^3}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{zi}$$

$$= \int_c \frac{z^2}{5 - \frac{2z^2+2}{z}} \frac{dz}{i} = \frac{1}{i}\int_c \frac{z^3}{5z-2z^2+2} dz$$

$$= -\frac{1}{i} \int_c \frac{z^3}{2z^2-5z+2} dz = -\frac{1}{i}\int_c \frac{z^3}{2z^2-4z-z+2} dz$$

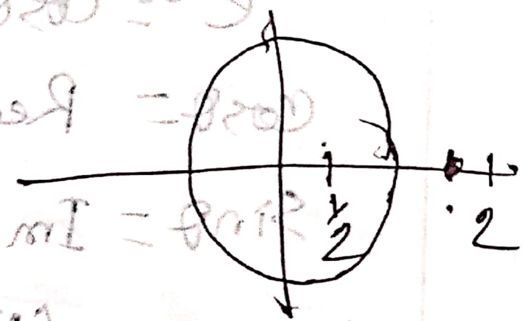
$$= \frac{-1}{i}\int_C \frac{z^3}{(2z-1)(z-2)}\, dz$$

$$I = -\frac{1}{2i}\int_C \frac{z^3}{(z-\frac{1}{2})(z-2)}\, dz, \quad |z|=1 \quad \text{— (1)}$$

$$I_1 = \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)}\, dz$$

Pole $z-\frac{1}{2}=0 \Rightarrow z=\frac{1}{2}$

$z-2=0 \quad \therefore z=2$



$z=\frac{1}{2}$ lies inside the circle

$$I_1 = \int_C \frac{z^3/(z-2)}{(z-\frac{1}{2})}\, dz$$

$$= 2\pi i \left.\frac{z^3}{z-2}\right|_{z=1/2}$$

$$I_1 = 2\pi i \times \left(\frac{1}{2}\right)^3 \frac{1}{\frac{1}{2}-2}$$

$$= \frac{\pi i}{2} \times \frac{1}{-3} = -\frac{\pi i}{6}$$

From (1), $I = -\frac{1}{2i} \times -\frac{\pi i}{6} = \pi/12$

$$\int_c \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = \pi/12$$

Real part

$$\int \frac{\cos 3\theta + i\sin\theta}{5-4\cos\theta}\,d\theta = \pi/12$$

$$= \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \pi/12$$

ii. Sol$^n$: $f(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}dx$ (where a is Parameter, $a \geq 0$)

$$f'(a) = \int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\frac{\sin x}{x}\right)dx$$

$$= \int_0^\infty -x\, e^{-ax}\frac{\sin x}{x} = -\int_0^\infty e^{-ax}\sin x\,dx$$

$$I = \int \underbrace{e^{-ax}\sin x}_{uv}\,dx = e^{-ax}\int \sin x\,dx - \int a e^{-ax}\cos x\,dx$$

$$= -e^{-ax}\cos x - a\left[e^{-ax}\int \cos x\,dx - \int -a e^{-ax}\sin x\,dx\right]$$

$$I = -e^{-ax}(\cos x + a\sin x) - a^2 I$$

$$I = -\frac{e^{-ax}(\cos x + a\sin x)}{a^2+1} + c$$

$$\therefore \int e^{-ax}\sin x\,dx = \frac{e^{-ax}}{a^2+1}(\cos x + a\sin x) + c$$

$$\therefore -\int_0^B e^{-ax}\sin x\,dx = \left[\frac{e^{-ax}(\cos x + a\sin x)}{a^2+1}\right]_0^B$$

When $B \to \infty$ then

$$-\int_0^\infty e^{-ax}\sin x = 0 - \frac{1}{a^2+1}$$

$$f'(a) = -\frac{1}{a^2+1}; \therefore f(a) = -\tan^{-1} + c$$

$$f(a) = -\tan^{-1}(a) + c \Rightarrow f(a) = \int_0^\infty \frac{\sin x}{x}\,dx$$

$$\therefore f(\infty) = 0$$

When $a \to \infty$, then

$$f(\infty) = \lim_{a\to\infty} \int_0^\infty e^{-ax}\,\frac{\sin x}{x}\,dx = 0$$

Again, $f(a) = -\tan^{-1}(a) + c$, $f(\infty) = -\tan^{-1}(\infty) + c$

$$\Rightarrow c - \frac{\pi}{2} = 0 \Rightarrow c = \pi/2$$

$f(0) = c = \frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}\,dx$

Other:-

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx \quad a>0$$ where a and b are variable. after treating a as a constant also diff w.r.t b.

$$I'(b) = \int_0^\infty \frac{e^{-ax}(\cos bx)\cancel{x}}{\cancel{x}}\,dx$$

$$I'(b) = \frac{e^{-ax}}{a^2+b^2}\left[-a\cos bx + b\sin bx\right]_0^\infty$$

$$I'(b) = \frac{a}{a^2+b^2}$$

$$I(b) = \tan^{-1}\left(\frac{b}{a}\right) + c \quad \text{(i)}$$

Substitute $b = 0$

$\Rightarrow c = 0$ [equ (i)]

$$I(b) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\left(\frac{b}{a}\right)$$

Substitute $a = 0$ and $b = 1$ $\int_0^\infty \frac{\sin x}{x}\,dx = \tan^{-1}\infty = \frac{\pi}{2}$

b) Find a bilinear transformation that maps points $z = 0, -i, -1$ into $w = i, 1, 0$, respectively.

Soln: $z_1 = 0$ $\quad w_1 = i$

$z_2 = -i$ $\quad w_2 = 1$

$z_3 = -1$ $\quad w_3 = 0$

We know that the bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

$$\Rightarrow \frac{(w-i)}{w(1-i)} = \frac{z(1-i)}{-i(1+z)}$$

Now, $\Rightarrow \dfrac{w(1-i)}{(w-i)} = \dfrac{-i(1+z)}{z(1-i)}$

$$\frac{z-i}{w-i} \quad \lim_{n\to\infty} i-n$$

6. a) Using Gauss-Jordan method solve the system;

$x+y+2z=9$, $2x+4y-3z=1$, $3x+6y-5z=0$

Solve:

Gauss Jordan elimination method

$x+y+2z=9$

$2x+4y-3z=1$

$3x+6y-5z=0$

in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

Augmented matrix is given by:

$C = [A:B]$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$$A:B = \begin{bmatrix} 2 & 4 & -3 & : & 1 \\ 1 & 1 & 2 & : & 9 \\ 3 & 6 & -5 & : & 0 \end{bmatrix} \qquad [r_1' = r_2 \,,\, r_2' = r_1]$$

$C = [A:B]$

$$C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

~~Applying operations $R_2 \to R_2 - 2R_1$ ; $R_3 \to R_3 - 3R_1$~~

$$C \sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & & \\ & & & & \end{bmatrix}$$

$$C \sim \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$C = [A : B] = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 2 & 4 & -3 & : & 1 \\ 3 & 6 & -5 & : & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 3 & -11 & : & -27 \end{bmatrix} \quad (1)$$

$$\cancel{C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 4 & -3 & : & -17 \\ 0 & 6 & -5 & : & 27 \end{bmatrix}}$$

Now, $R_3 \to R_3 - R_2$, $R_3 \leftrightarrow R_2$

$$C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 2 & -7 & : & -17 \\ 0 & 1 & -4 & : & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -4 & : & -10 \\ 0 & 2 & -7 & : & -17 \end{bmatrix}$$

Now,

$R_3 \rightarrow R_3 - 2R_2$ ; $C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & -4 & : & -10 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

$R_2 \rightarrow R_2 + 4R_3$ ; $C = \begin{bmatrix} 1 & 1 & 2 & : & 9 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

$R_1 \rightarrow R_1 - 2R_3$ ; $C = \begin{bmatrix} 1 & 1 & 0 & : & 3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

$R_1 \rightarrow R_1 - R_2$ ; $C = \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

eqn: $x = 1$

$y = 2$

$z = 3$

Ans;

b) Using the result of part (a), determine whether the vector $w=(9,1,0)$ can be expressed as a linear combination of the vectors.

$$v_1=(1,2,3),\ v_2=(1,4,6),\ v_3=(2,-3,-5)$$

and, if so, find such a linear combination.

Soln: Suppose $W(9,1,0)$ can be written as L.C of $v_1$, $v_2$ & $v_3$

$$(9,1,0)=x(1,2,3)+y(1,4,6)+z(2,-3,-5)$$

$$=(x,2x,3x)+(y,4y,6y)+(2z,-3z,-5z)$$

$$=(x+y+2z,\ 2x+4y-3z,\ 3x+6y-5z)$$

we get a system of equation

$$x+y+2z=9 \quad \text{--- (i)}$$

$$2x+4y-3z=1 \quad \text{--- (ii)}$$

$$3x+6y-5z=0 \quad \text{--- (iii)}$$

$$C = [A : B]$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$R_3 \to R_3 - 3R_1$

$R_2 \to R_2 - 2R_1$ ; $C = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 9 \\ 0 & -4 & -7 & -17 \\ 0 & -6 & -11 & -27 \end{array}\right]$

$R_3 \Rightarrow R_3 - R_2$ & $R_3 \Leftrightarrow R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 9 \\ 0 & -4 & -7 & -17 \\ 0 & -2 & -4 & -10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 9 \\ 0 & -2 & -4 & -10 \\ 0 & -4 & -7 & -17 \end{array}\right]$$

$R_2 \to \dfrac{R_2}{-2}$ ; $C = \left[\begin{array}{ccc|c} 1 & 4 & 2 & 9 \\ 0 & 1 & 2 & 5 \\ 0 & -4 & -7 & -17 \end{array}\right]$

$R_3 \to R_3 + 4R_2$ ;

$$C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & 1 & 2 & : & 5 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

Here, $z = 3$ [3rd equ]

$y + 2z = 5$ [2nd equ]

$\therefore y = 5 - 6 = -1$

So $x + 4y + 2z = 9$

$\Rightarrow x = 9 + 4 - 6 = 7$

So, $(9, 1, 0) = 7(1, 2, 3) - 1(1, 4, 6) + 3(2, -3, -5)$

$= (7 + 14 + 21) - (1, 4, 6) + (6, -9, -15)$

$= (7 - 1 + 6, 14 - 4 - 9, 21 - 6 - 15)$

$= (12, 1, 0)$

(c) Find the inverse of A (if possible) by elementary row operations and also state that rank of A where

$$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

Soln: Start with $[A \mid I]$ (from $AA^{-1} = I$)

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

Step 1: $R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

2: $R_3 \to R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

3: $\overset{2nd}{\cancel{R_3 \to R_3 + R_2}}$ & $\overset{1st}{R_2 \to \frac{R_2}{-R_3}}$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

4. $R_3 \to R_3 - 8R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array}\right]$$

5. $R_1 \to R_1 - 4R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -7 & -4 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array}\right]$$

6. $R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -7 & -4 & 0 \\ 0 & 1 & 0 & 17 & 9 & -1 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array}\right]$$

7. $R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -41 & -22 & 2 \\ 0 & 1 & 0 & 17 & 9 & -1 \\ 0 & 0 & 1 & -15 & -8 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -41 & -22 & 2 \\ 17 & 9 & -1 \\ -15 & -8 & 1 \end{bmatrix}$$

Firstly we need to find the possibility of inverse: $|A| \neq 0$ [condition]

$$\begin{vmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{vmatrix} = 1(20+2) - 6(10-1) + 4(4+$$

$$= 22 - 54 + 32 = 0$$

So, we can determine $A^{-1}$.

**Rank:** As the value of $|A|_{(3\times3)}$ is zero(0)

So the rank is not 3, it may be less than 3.

Now, we consider $|A|_{2\times2}$ form:

$$|A_1| = \begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4 - 12 = -6 \neq 0$$

So Rank = 2.

7\. a) Find inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ by finding its adjoint

Solve: Let, $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + 1(4-2)$$

$$= 6 - 6 + 2 = 2 \neq 0$$

so, the inverse of matrix is exsist.

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = (18-12) = 6$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = (4-2) = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = (9-1) = 8$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = (3-2) = 1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (2-1) = 1$$

So, $$\text{Adj} A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

$\therefore A^{-1} = \dfrac{1}{|A|} \operatorname{adj} A$

$$= \frac{1}{2}\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

b) Using the result of part (a), solve the system of equations

$x+y+z=3$, $x+2y+3z=4$, $x+4y+9z=6$

Soln:
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$AX = B$

$\Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\therefore X = A^{-1}B$

$A^{-1}$ exist if $|A| \neq 0$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} + (-1)^3 \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= (18-12) - (9-3) + (4-2)$$

$$= 6 - 6 + 2$$

$$= 2$$

we can take the result ~~of part~~ (a), where

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ -1/2 & -1 & 1/2 \end{pmatrix}$$

$$X = A^{-1}B = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ -1/2 & -1 & 1/2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 9-12+6 \\ -\frac{15}{2}+16-9 \\ -3/2-4+3 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{-1}{2} \\ \frac{-2}{\cancel{0}} \end{pmatrix}$$