

Ques. 1. a) Define complex numbers. Is complex number algebraic?
Explain why or why not. For any complex numbers z_1, z_2 , prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Answer: A complex number is a number that can be expressed in the form $a+bi$ where "a" and "b" are real numbers, and "i" is the imaginary unit, defined as the square root of -1.

good for blurt out. better enter

that z is a complex variable if z is an ordered pair (x, y) where x and y are real numbers.

It is written as $z = x + iy$; $i = \sqrt{-1}$

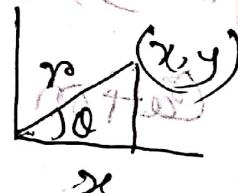
x is called real part and y is called imaginary part.

in polar coordinates, r is the

$$x = r \cos \theta, y = r \sin \theta$$

$$x^2 + y^2 = r^2 \therefore r = \sqrt{x^2 + y^2}$$

Called modulus of z.



$\tan \theta = \frac{y}{x} \therefore \theta = \tan^{-1}(\frac{y}{x})$; θ is called argument of z .

Yes, complex numbers are part of algebra. Algebra is a branch of mathematics that deals with symbols and the rules for manipulating those symbols. It encompasses various topics such as equations, expressions, functions, and mathematical structures.

Complex numbers are an extension of the real numbers and form a mathematical structure called a field. The field of complex numbers, denoted by \mathbb{C} , includes both the real numbers (\mathbb{R}) and (the) imaginary unit (i). Complex numbers obey the same algebraic rules as real numbers, such as addition, subtraction, multiplication, and division.

x

$$x + iy = \sqrt{a^2 + b^2} e^{i\theta} = r e^{i\theta}$$

$$\sqrt{a^2 + b^2} = \sqrt{x^2 + y^2} = r$$

θ to arbitrary levels

Complex numbers also have algebraic properties such as commutativity, associativity and distributivity. These properties allow for the manipulation and simplification of complex expressions, equations, and inequalities using algebraic techniques.

Complex numbers form an integral part of algebra, and their properties and operations are studied within the framework of algebraic systems.

The conjugate of a complex number $a+bi$ is obtained by changing the sign of the imaginary part resulting in $a-bi$.

important Principle states that if $|z_1 + z_2| \leq |z_1| + |z_2|$ then $|z_1 + z_2| \leq |z_1| + |z_2|$

$$\Rightarrow \text{if } |z_1 + z_2|^2 \leq (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \text{ then } |z_1 + z_2| \leq |z_1| + |z_2|$$

- since here we have $(z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + 2\bar{z}_1\bar{z}_2 + 2z_1\bar{z}_2$$

$$= |z_1|^2 + |z_2|^2 + 2\bar{z}_1\bar{z}_2 + 2z_1\bar{z}_2$$

$$\text{Hence largest value} = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \quad (\because \operatorname{Re}(z) = \frac{z + \bar{z}}{2})$$

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad \operatorname{Re}z \leq |z|$$

$$|z_1 + z_2|^n \leq |z_1|^n + |z_2|^n \quad n \in \mathbb{N}, n \geq 1 \quad x \leq \sqrt[n]{x^n}$$

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \quad \sqrt[2]{1} = 1 = |z|$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (\text{proved})$$

Ans

b) Find all values of z for which $z^5 = -32$, and locate these values in the complex plane.

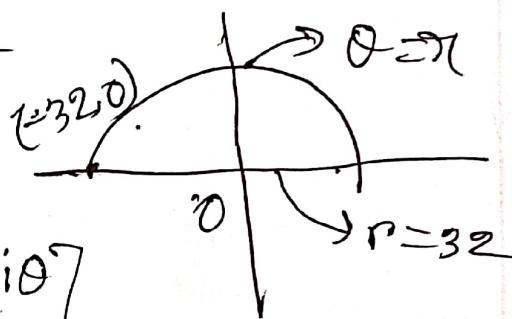
Solution:

$$z^5 = -32$$

$$\text{Let } z^5 = w$$

$$w = -32 + 0i \Rightarrow x = r \cos \theta, y = r \sin \theta$$

$$r = \sqrt{(-32)^2 + 0^2} = 32$$



$$-32 = 32e^{i\pi} [\because re^{i\theta}]$$

$$= 32e^{i(\pi+2k\pi)}$$

$$(-32)^{1/5} = 32^{1/5} e^{i\left(\frac{\pi+2k\pi}{5}\right)} = 2 e^{i\left(\frac{\pi+2k\pi}{5}\right)}$$

$$k=0, z_1 = 2e^{i\frac{\pi}{5}}$$

$$k=1, z_2 = 2e^{i\frac{3\pi}{5}}$$

$$k=2, z_3 = 2e^{i\frac{5\pi}{5}} = 2e^{i\pi} = 2(-1) = -2$$

$$k=3, z_4 = 2e^{i\frac{7\pi}{5}}$$

$$k=4, z_5 = 2e^{i\frac{9\pi}{5}}$$

(c) Represent graphically the set of values of z for which -

$$\left| \frac{z-3}{z+3} \right| = 2 \Leftrightarrow |z-3|^2 = 4|z+3|^2$$

(Q10)(i) $|z-3|^2 = 4|z+3|^2$ after expand
Answer:

$$\left| \frac{z-3}{z+3} \right| = 2 \quad \text{where } z = x+iy$$

$$\Rightarrow \left| \frac{x-3+iy}{x+3+iy} \right| = 2 \quad \text{where } z = x+iy$$

After p. without loss $(0,0)$ satisfies eq.

$$\Rightarrow \frac{\sqrt{(x-3)^2+y^2}}{\sqrt{(x+3)^2+y^2}} = 2 \quad \text{shorter is st diff}$$

$$\Rightarrow \frac{(x-3)^2+y^2}{(x+3)^2+y^2} = 4 \quad \Rightarrow \frac{x^2-6x+9+y^2}{x^2+6x+9+y^2} = 4$$

$$\Rightarrow (x^2-6x+9+y^2) = 4(x^2+24x+36+y^2)$$

$$\Rightarrow 9-36 = 4x^2+4y^2+24x-x^2-y^2+6x$$

$$\Rightarrow -27 = 3x^2+3y^2+30x \quad \text{(interior)}$$

$$\text{or } 3x^2+3y^2+30x+27=0 \quad \text{new eq}$$

$$\Rightarrow 9x^2+9y^2+10x+9=0 \quad \text{point in (0,0)}$$

$$\Rightarrow x^2 + 2x + 5 + y^2 + 9 + 25 = 0 \rightarrow \text{Discard } 0$$

$$\Rightarrow (x+5)^2 + y^2 = 16 \rightarrow \text{Radius not } 5$$

$$\Rightarrow (x+5)^2 + y^2 = 9^2 \quad \left| \begin{array}{l} r=3 \\ C(-5, 0) \end{array} \right.$$

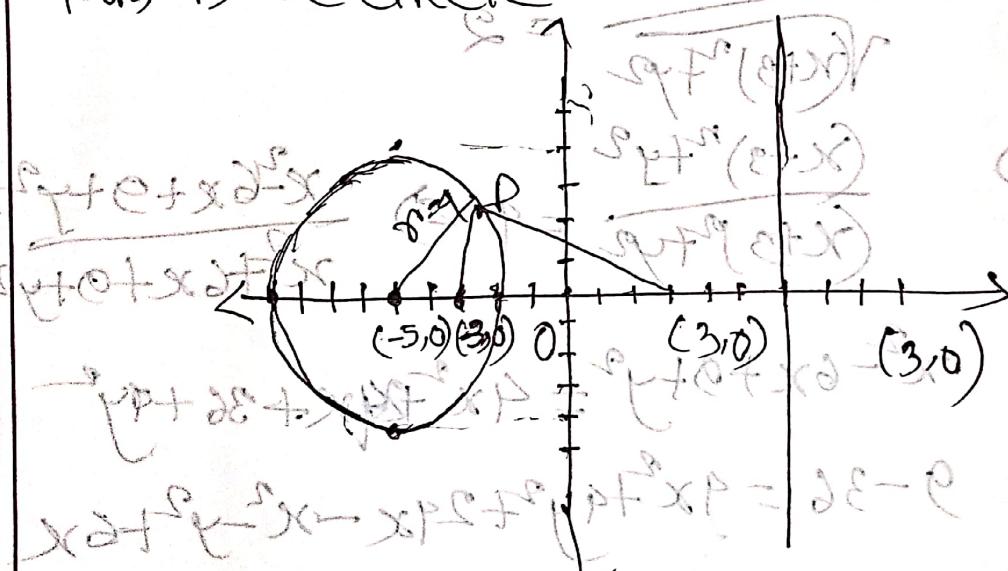
Compare with $(x-a)^2 + (y-b)^2 = r^2$ (circle)

$$(a, b) \rightarrow \text{Centre} \quad r = \left| \frac{r-a}{C-a} \right|$$

$$r = 3 \rightarrow \text{Radius} \quad \left| \begin{array}{l} r = 3 - x \\ r = 3 + x \end{array} \right|$$

so, centre $(-5, 0)$ and radius 3 unit.

this is a circle



Geometrically any point P on this circle is such that the distance from P to $(3, 0)$ is twice the distance P to $(-3, 0)$

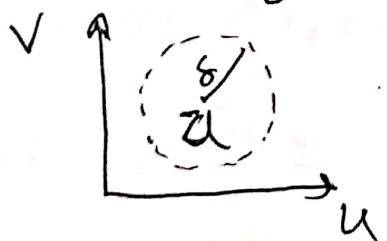
Q. Define limit and continuity of complex functions. Let $w = f(z) = (z^2 + 1)^{1/2}$. Then show that (i) $z = \pm i$ are branch points of $f(z)$ and (ii) a circuit around both branch points produces no change in the branches of $f(z)$.

Ans:

limit: Let $f(z)$ be any function of the complex variable z defined in a bounded and closed domain D . Then ℓ is said to be the limit of $f(z)$ as if for any $\epsilon > 0 \exists \delta > 0$ such that

$$|f(z) - \ell| < \epsilon \text{ for all values of } z \\ \text{i.e. } 0 < |z - a| < \delta$$

in symbols $\lim_{z \rightarrow a} f(z) = \ell$.



Complex continuity: A function $f(z)$ of a complex variable z is defined in the closed bounded domain D .

Said function is continuous at $a \in D$ if and only if for $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(a)| < \epsilon \quad \forall z \in D \text{ with } |z - a| < \delta$$

$z = a$ if $\lim_{z \rightarrow a} f(z) = f(a)$

if $f(z) \neq f(a)$, then $|f(z) - f(a)| > \epsilon$

but $|z - a| < \delta$ so $|f(z) - f(a)| > \epsilon$

then $|f(z) - f(a)| > \epsilon$ and $|z - a| < \delta$

therefore $|z - a| > \delta$ (not true)

so $z \neq a$ (not true)

$$|z - a| > \delta$$

$\Rightarrow z = (x + iy)$ with $x^2 + y^2 > \delta^2$



$$i. w = f(z) = (z^2 + 1)^{\frac{1}{2}} = (re^{i\theta})^{\frac{1}{2}} e^{i\theta/2} = f_1(z)$$

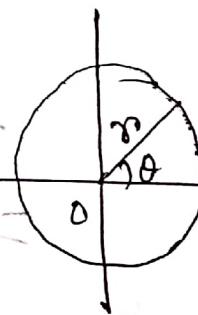
Let, $z = 0$

$$f(z) = z^{\frac{1}{2}} = (re^{i(\theta+2k\pi)})^{\frac{1}{2}} = (re^{\frac{i\theta}{2}})^{\frac{1}{2}}$$

$$= r^{\frac{1}{2}} e^{i\theta/2}$$

$$= -r^{\frac{1}{2}} e^{i\theta/2}$$

$$= f_2(z)$$



$$i. f(z) = (z^2 + 1)^{\frac{1}{2}} = (z^2 - i^2)^{\frac{1}{2}}$$

$$= \sqrt{(z+i)(z-i)}^{\frac{1}{2}} = \sqrt{2i} \cdot \sqrt{z-i}$$

Let, $z+i = r_1 e^{i\theta_1}$ & $z-i = r_2 e^{i\theta_2}$

$$\sqrt{2i} = \sqrt{4} e^{i\theta/2}, \sqrt{z-i} = \sqrt{r_2} e^{i\theta_2/2}$$

Encircle $z=i$, not $z=-i$

$$\sqrt{r_1} e^{i\frac{(\theta_1+2\pi)}{2}}$$

$$= \sqrt{r_1} e^{i\frac{\theta_1}{2}} e^{\pi i}$$

$$z = \sqrt{r_1} e^{i\theta_1/2} (\cos \theta_1 + i \sin \theta_1)$$

$$z = -\sqrt{r_1} e^{i\theta_1/2} (\cos \theta_1 + i \sin \theta_1)$$

Encircle $z=-i$, not $z=i$

$$\sqrt{r_2} e^{i\frac{(\theta_2+2\pi)}{2}}$$

$$= \sqrt{r_2} e^{i\theta_2/2} e^{\pi i}$$

$$z = -\sqrt{r_2} e^{i\theta_2/2} (\cos \theta_2 + i \sin \theta_2)$$

The function is changed.

$$= 2\pi f(a) \quad [0]_0 = 2\pi f(a)$$

so $z^2 = -i$ is a branch point.

$$(S), t =$$

$$0 = S \text{, take}$$

$$(b) f(z) = (z^2 + i)^{\frac{1}{2}} \text{ for } z = (S)$$

$$\begin{aligned} &= \sqrt{(z+i)(z-i)} \\ &= \sqrt{r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}} \quad (S) \text{ st } \\ &= \sqrt{(r_1 r_2)^{\frac{1}{2}} e^{i(\theta_1 + \theta_2)/2}} \end{aligned}$$

$$\sqrt{(r_1 r_2)^{\frac{1}{2}} e^{i(\theta_1 + \theta_2)/2}} = (S) f = i$$

$$\text{Hence, } \sqrt{r_1 r_2} = \sqrt{(r_1 r_2)^{\frac{1}{2}} e^{i(\theta_1 + \theta_2)/2}} =$$

Let, encircle both $z = i$ & $z = -i$.

$$\sqrt{r_1 r_2} = \sqrt{r_1 r_2} e^{i(\theta_1 + 2\pi(i + i\theta_2 + 2\pi i))}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \quad j = S \frac{2\pi i}{2\pi r}$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \quad \text{as } i^2 = -1$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \quad \cos 2\pi i + i \sin 2\pi i$$

$$= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2} \quad = 1 + 0i = 1$$

Value of ϕ major is no 1 minimum is 0
so no V = f not bcoz $\phi = 2\pi - \theta$

The function remain unchanged

B) ~~How to find value of ϕ minor A~~

Solutions: Define harmonic function with
Example: Prove that $U = e^{-x}(x \sin y - y \cos y)$
is harmonic.

Soln: If the 2nd partial derivatives
of U and V with respect to x & y
exists and continuous in R^2 then we
find -

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0; \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

* If G is an open subset of C then
function $U: G \rightarrow R$ is said to be harmonic
if U has continuous second order partial
derivatives and $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$

This equation is called Laplace's equation

$$= 1 \cdot \theta(a) \cdot [0]_0 = 2\pi \cdot \theta(a)$$

i. A function F on a region G is analytic if $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ are harmonic functions which satisfy the Cauchy-Riemann eq.

ii. A region G is simply connected iff each harmonic function u on G such that $\operatorname{grad} u$ is analytic in G .

$$u = e^{-x} (x \sin y - y \cos y)$$

$$\frac{\partial u}{\partial x} = e^{-x} (\sin y) + (-e^{-x}) (x \sin y - y \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y + e^{-x} \sin y + x e^{-x} \sin y - e^{-x} \cos y$$

$$\text{next } \frac{\partial^2 u}{\partial y^2} = -2e^{-x} \sin y + x e^{-x} \sin y - e^{-x} \cos y$$

Again, $\frac{\partial^2 u}{\partial y^2} = e^{-x} (x \cos y + y \sin y - \cos y)$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{-x} (x \cos y + y \sin y - \cos y)$$

$$0 = \frac{\partial^2 u}{\partial y \partial x} = e^{-x} (-\cos y + y e^{-x} \sin y - e^{-x} \cos y)$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} \sin y + e^{-x} \sin y$$

$$+ye^{-x}\cos y + e^{-x}\sin y$$

③ $\partial u / \partial x + \partial v / \partial y = 0$

$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0$

$$(2e^{-x} - e^{-x})\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$v + u = (s)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$$

$\therefore u$ is harmonic

① $\partial u / \partial x + \partial v / \partial y = 0$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Example:

$$f(z) = u + iv$$

$$\text{let } u = x^2 - y^2 \quad \text{then } \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \rightarrow \text{II}$$

eq ① + eq ②

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{harmonic})$$

Laplace's
equ)

$$f(z) = u + iv$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 v}{\partial x^2} \quad \text{---} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y^2} \quad \text{---} \quad ①$$

by ① and ②

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} \quad v + u = 0$$

Example: $v(x, y) = 3x^2y^3$

$$u = \frac{v}{x^2}$$

$$u = \frac{3y^3}{x^2}$$

B. a) Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region.

R. Prove Cauchy's integral formula:

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)} dz$$

Soln:

Eqn of circle C_1

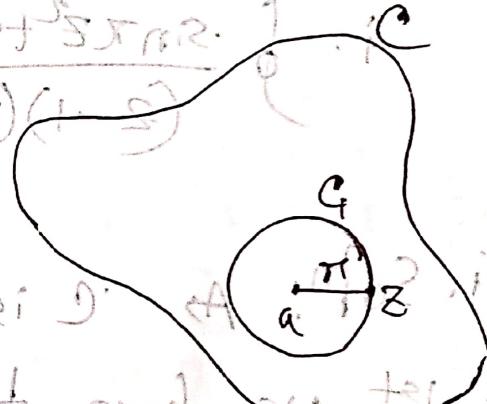
$$|z-a|=r$$

$$\Rightarrow z-a = r e^{i\theta}$$

$$\Rightarrow z = a + r e^{i\theta}$$

$\oint_C f(z) dz = \int_0^{2\pi} f(a + r e^{i\theta}) r i e^{i\theta} d\theta$

$$\frac{\text{start} + \text{end}}{(z-s)(1-s)}$$



when
 $r \rightarrow 0$
then

$$f(z) \rightarrow f(a)$$

$$\oint_C \frac{f(z)}{z-a} dz = \oint_C \frac{f(z)}{z-a} \frac{dz}{z-a} = \int_0^{2\pi} \frac{f(a+r e^{i\theta})}{a+r e^{i\theta}} r i e^{i\theta} d\theta$$

$$\frac{1}{1-\frac{a}{z}} = \int_{C_1} \frac{f(a+r e^{i\theta})}{r e^{i\theta}} \cdot r i e^{i\theta} d\theta$$

$$\frac{1}{1-\frac{a}{z}} = \int_{C_1} \frac{f(a+r e^{i\theta})}{(z-a)(1-\frac{a}{z})} dz = \int_{C_1} \frac{f(a+r e^{i\theta})}{(z-a)(1-\frac{a}{z})} dz = \int_{C_1} f(z) dz$$

$$\int_{C_1} f(z) dz = i f(a) \int_{C_1} dz = i f(a) [z]_0^{2\pi} = 2\pi i f(a)$$

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\Rightarrow f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz \quad (\text{Proved})$$

$$\text{Sb } \frac{(3)t}{(z-s)} \cdot \oint \frac{1}{z-s} dz = (3)t$$

3. Evaluate:

$$\text{i. } \oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

$$\text{ii. } \oint \frac{e^{2z}}{(z+1)^4} dz$$

i. Soln: As C is the circle $|z| = 3$
1st we have to do partial fraction of

$$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)(1-z)} = \frac{1}{(z-1)(z-2)}$$

$$\text{dig it: } \frac{1}{z-2} - \frac{1}{z-1}$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$\text{ob } (1) = \text{ab } \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz$$

By Cauchy's Integral formula with $a=2$ & $\alpha=1$,
we get —

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz = 2\pi i (\sin \pi 2^2 + \cos \pi 2^2)$$

$$= 2\pi i (\sin 4\pi + \cos 4\pi)$$

$$= 2\pi i (0+1) = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz = 2\pi i (\sin \pi i^2 + \cos \pi i^2)$$

$$= 2\pi i (0-1) = -2\pi i$$

$$\text{The required integral} = 2\pi i - (-2\pi i) = 4\pi i$$

ii. Soln: $\oint_C \frac{e^{2z}}{(z+1)^2} dz$ where C is the circle $|z|=3$.

Let $f(z) = e^{2z}$ and $a=-1$ in Cauchy integral formula,

Ques. Comparing both, we get the value of

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (1)$$

$$[2-i]^{n+1} = 9 \Rightarrow n=3$$

$$\begin{aligned} & (z-p_1 + p_2 + p_3) dz = \\ & \therefore f'''(z) = 8e^{2z} dz = \\ & \therefore f'''(2-i) = 8e^{-2} \end{aligned}$$

$$(1) \therefore 8e^{-2} = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4} dz$$

$$\begin{aligned} & \therefore \int_C \frac{(e^{2z}) dz}{(z+1)^4} = \\ & \therefore \int_C \frac{dz}{(z+1)^4} = \frac{8e^{-2} - 2\pi i}{8\pi i e^{-2}} = \frac{8\pi i e^{-2}}{3} \end{aligned}$$

$$\text{imp. } (i\pi) - i\pi = \text{length of arc} = \frac{2\pi r}{3}$$

(Ans.)

$$\text{Ans. } \theta = \arg \int_C \frac{dz}{(z+1)^4} \quad (\text{Ans.})$$

$$\theta = \arg \int_C dz / (z+1)^4$$

$$\theta = \arg \int_C dz / (z+1)^4$$

∴ Answer: $\theta = \arg \int_C dz / (z+1)^4$

9. a) Find Laurent series about the indicated singularity for each of the following functions.

$$\frac{e^{2z}}{(z-1)^3}; z=1 \quad \text{ii. } \frac{z}{(z+1)(z+2)}; z=-2$$

Soln: let $z-1=u \Rightarrow z=1+u$

$$\begin{aligned} \therefore \frac{e^{2z}}{u^3} &= \frac{e^{2+2u}}{u^3} = \frac{e^2 \cdot e^{2u}}{u^3} = \frac{e^2}{u^3} \cdot e^{2u} \\ &= \frac{e^2}{u^3} \left\{ 1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \dots \right\} \\ &= \frac{e^2}{u^3} + \frac{2e^2}{u^2} + \frac{4e^2}{u} + \frac{8e^2}{6} + \dots \end{aligned}$$

$$\therefore \frac{e^2}{(2-u)^3} + \frac{2e^2}{(2-u)^2} + \frac{4e^2}{2(2-u)} + \frac{8e^2}{3} + \dots$$

$\therefore z=1$ is pole of order 3

\therefore The series converges for all value of z without 1.

Let $\frac{z}{(z+1)(z+2)}$ for $z = -2$ be a branch point (o.p)

different branches exist for $z = -2$ not distinguishable

$$\text{Let, } z+2 = u \therefore z = u-2$$

$$S = \frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-2+1)u} = \frac{u-2}{(u-1)u} = \frac{2-u}{(1-u)u}$$

$$\Rightarrow \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u} (1+u+u^2+u^3+\dots)$$

$$\begin{aligned} S &= \frac{2}{u} + \frac{u}{u} + \frac{2-u}{u^2} + \frac{2u^2-u^3}{u^3} + \dots \\ &= \frac{2}{u} + \frac{1}{1-u} + \frac{2-u}{u^2} + \frac{2u^2-u^3}{u^3} + \dots \end{aligned}$$

$$\dots + \frac{2}{u} + \frac{1}{1-u} + \frac{2-u}{u^2} + \frac{2u^2-u^3}{u^3} + \dots$$

$$S = \frac{2}{(z+2)} + 1 + (z+2) + (z+2)^2 + \dots$$

for which $(z+2)^0$ is a pole

at $z = -2$ of $(z+2+1)$

D) Evaluate $\oint \frac{e^{zt}}{z^4(z^2+2z+2)} dz$ around the circle C
with equation $|z|=3$.

Soln! $|z|=3 \therefore$ Centre $\equiv (0,0)$ radius $= 3$.

- there is a simple pole at $z=2$ inside C

5. a) i. Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4 \cos \theta} d\theta = \frac{\pi}{12}$

ii. Show that $\int_0^{\pi} \frac{\sin x}{x} dx = \frac{\pi}{2}$

i. $e^{i\theta} = \cos \theta + i \sin \theta$ $\int_0^{2\pi} \frac{e^{i\theta}}{5 - 4 \cos \theta} d\theta = I$

$\cos \theta = \text{Real part}(e^{i\theta})$ $\theta = s - \pi$

$\sin \theta = \text{Im part}(e^{i\theta})$ $\theta = s - \pi$

$I = \text{Real part} \cdot \int_0^{2\pi} \frac{e^{3i\theta}}{5 - 4 \cos \theta} d\theta$ $\text{Let } z = e^{i\theta}$

Putting $\cos \theta = \frac{z + z^{-1}}{2}$ $I = I$

$$d\theta = \frac{dz}{2i} \quad i\pi =$$

$$I = \int_C \frac{z^3}{5 - 4\left(\frac{z + z^{-1}}{2}\right)} \frac{dz}{2i} \quad C \text{ is a circle around } z=0 \text{ with radius } R \rightarrow 0 \quad I = I$$

$$= \int_C \frac{z^2}{5 - \frac{2z^2 + 2}{z}} \frac{dz}{i} \quad I = \int_C \frac{z^3}{5z - 2z^2 - 2} dz$$

$$= -\frac{1}{i} \int_C \frac{z^3}{2z^2 - 5z + 2} dz = \frac{1}{i} \int_C \frac{z^3}{2z^2 - 9z - 2} dz$$

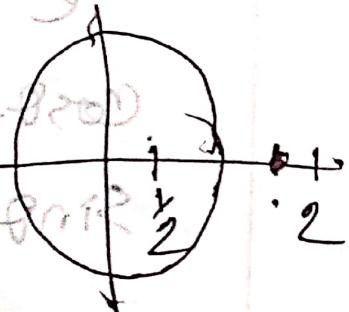
$$\frac{1}{2\pi i} \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz \text{ about point } z = i$$

$$I = \frac{1}{2\pi i} \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz \quad |z=2| = 2\pi i \quad \textcircled{1}$$

$$I_1 = \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz \quad m(2i) + o(2i) = 0$$

Pole $z - \frac{1}{2} = 0 \Rightarrow z = \frac{1}{2}$ lies inside $|z| = 2$

$$z - 2 = 0 \quad (\because z = \frac{1}{2}) \quad m = 0$$



$z = \frac{1}{2}$ lies inside the circle. $\text{Res}_{z=\frac{1}{2}} = \frac{1}{2}$. Total loop = 1

the circle

$$I_1 = \int_C \frac{z^3}{(z-\frac{1}{2})(z-2)} dz = \text{Res}_{z=\frac{1}{2}} + \text{Res}_{z=2}$$

$$= 2\pi i \cdot \left. \frac{z^3}{z-2} \right|_{z=\frac{1}{2}} = 0$$

$$I_1 = 2\pi i \left(\frac{1}{2} \right)^3 \cdot \frac{\frac{1}{2}}{\left(\frac{1}{2} - 2 \right) \pi i} = 0$$

$$\text{Res}_{z=2} = \frac{\pi i}{2} \times \frac{1}{-1} = \frac{\pi i}{2}$$

$$\text{From } \textcircled{1} \quad I = -\frac{1}{2} \times \frac{\pi i}{2} = \frac{\pi i}{4}$$

$$\int_C \frac{e^{3i\theta}}{5-9\cos\theta} d\theta = \frac{\text{Res}(z=2)}{1+9} = \frac{\pi}{12}$$

Real part

$$\text{Re}(\text{Res}(z=2)) = \frac{x \cos^2 \theta - 9}{1+9} = \frac{\cos^2 \theta - 9}{10}$$

$$= \int_0^{2\pi} \frac{\cos^2 \theta - 9}{5-9\cos\theta} d\theta = \int_0^{2\pi} \frac{\cos^2 \theta - 9}{1+9} d\theta = \int_0^{2\pi} \frac{\cos^2 \theta - 9}{10} d\theta$$

i. Soln: $f(a) = \int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ (Where a is parameter)

$$f'(a) = \int_0^\infty \frac{\partial}{\partial a} \left(e^{-ax} \frac{\sin x}{x} \right) dx, a \geq 0$$

$$= \int_0^\infty -x e^{-ax} \frac{\sin x}{x} dx = - \int_0^\infty e^{-ax} x \sin x dx$$

$$I = \int_0^\infty e^{-ax} x \sin x dx = (-a)^{-1} \left[e^{-ax} (\sin x - a \cos x) \right]$$

$$f(a) = \int_0^\infty e^{-ax} \sin x dx = (-a)^{-1} \left[e^{-ax} (\cos x + a \sin x) \right]$$

$$= -e^{-ax} \cos x - a e^{-ax} \sin x - a e^{-ax} \sin x + a^2 e^{-ax} \cos x$$

$$I = -2e^{-ax} (\cos x + a \sin x) - a^2 I$$

$$I = - \frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1}$$

$$-\int e^{-ax} \sin x dx = \frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1} + C$$

$$\therefore - \int_0^B e^{-ax} \sin x dx = \left[\frac{e^{-ax} (\cos x + a \sin x)}{a^2 + 1} \right]_0^B$$

When $B \rightarrow \infty$ then

$$-\int_0^\infty e^{-ax} \sin x dx = 0 - \frac{1}{a^2 + 1}$$

10. Given $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$, $f(a) = f(a) + C$

$$f(0) = \lim_{x \rightarrow 0} f(x) = f(0) + C \Rightarrow f(0) = \int_0^\infty \frac{\sin x}{x} dx$$

Now when $a \rightarrow \infty$ then

$$f(\infty) = \lim_{a \rightarrow \infty} f(a) = \lim_{a \rightarrow \infty} \int_0^a e^{-ax} \sin x dx$$

Again, $f(a) = f(a) + C$, $f(\infty) = f(\infty) + C$

$$\therefore C = 0 \Rightarrow C = \pi/2$$

$$f(z) = \int_0^{\infty} e^{-xt} \sin x dx \quad \text{where } t > 0$$

Value for $t=0, i.e. z=0$ then $I(i, 0) = \infty$

Other

$\int_0^{\infty} e^{-ax} \frac{\sin bx}{x} dx \quad a > 0$ where a and b are variable. Differentiating a as a constant also diff w.r.t b . $L = e^{-st}$

$$\text{differentiation } I'(b) = \int_0^{\infty} e^{-ax} (\cos bx) \frac{1}{x} dx$$

$$I'(b) = \frac{e^{-ax}}{(a^2 + b^2)} (-a \cos bx + b \sin bx) \Big|_0^{\infty}$$

$$\frac{I'(b)}{(1+i)^2 (a^2 + b^2)} = \frac{(i-a)(i-w)}{(i-w)(i-j)}$$

$$\frac{I'(b)}{(i-w)(i-j)(b/a)} + C = -1$$

$$\text{Substitute } b = jw \quad \Rightarrow C = 0 \quad \text{[equation]}$$

$$I(b) = \frac{(i-j)(j-w)}{(i-w)} \quad \text{[Ans]$$

$$\int_0^{\infty} e^{-ax} \frac{\sin bx}{x} dx = i \operatorname{Im} \left(\frac{b}{a} \right) \frac{i-j}{i-w}$$

$$\text{Substitute } a=0 \text{ and } b=1 \quad \int_0^{\infty} \frac{\sin x}{x} dx = \operatorname{Im} \frac{1}{i} = \frac{1}{2}$$

b) Find a bilinear transformation that maps points $z = 0, -1, -i$ into $w = i, 1, 0$, respectively.

and we know: $z_1 = 0 \rightarrow w_1 = i$
 transformed $z_2 = -1 \rightarrow w_2 = 1$ to $z_3 = -i \rightarrow w_3 = 0$

We know that the bilinear transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_3)(w_2 - w)} \cdot \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\therefore \frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i + 1)}{(-1 - 0)(1 - z)}$$

$$\Rightarrow \frac{(w - i)}{w(1 - i)} = \frac{z(1 - i)}{-i(1 + z)}$$

$$\text{Now, } \Rightarrow \frac{w(1 - i)}{(w - i)} = \frac{-i(1 + z)}{z(1 - i)} = 0$$

$$\frac{1 - i}{w - i} \underset{n \rightarrow \infty}{\lim} i - n = \frac{1 - i}{w - i}$$

$$\text{but } \frac{1 - i}{w - i} \underset{n \rightarrow \infty}{\lim} i - n = \frac{1 - i}{w - i}$$

6. a) Using Gauss-Jordan method solve the system:

$$x+y+2z=9, 2x+4y-3z=1, 3x+6y-5z=0$$

Solve:

Gauss Jordan elimination method

$$x+y+2z=9$$

$$2x+4y-3z=1$$

$$3x+6y-5z=0$$

'in matrix form'

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

Augmented

$$C = [A : B]$$

matrix is given by:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$$A : B \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$[r'_1 = r_2, r'_2 = r_1]$$

$$C = [A : B]$$

$$C = \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Applied operations $R_2 \rightarrow R_2 - 2R_1$; $R_3 \rightarrow R_3 - 3R_1$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -5 & 2 & 1 \\ 0 & 3 & -5 & 0 \end{array} \right] \xrightarrow{\text{Row 2} \leftarrow 2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & 4 & 1 \\ 0 & 3 & -5 & 0 \end{array} \right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & 4 & 1 \\ 0 & 3 & -5 & 0 \end{array} \right] \xrightarrow{\text{Row 3} \leftarrow 3R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

$$C = [A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array} \right]$$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 6 & -5 & -51 \end{array} \right] \xrightarrow{\text{Row 3} \leftarrow R_3 - 3R_2} \textcircled{1}$$

Now, $R_3 \rightarrow R_3 - R_2$, $R_3 \leftrightarrow R_2$

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 1 & -5 & -10 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -5 & -10 \\ 0 & 2 & -7 & -17 \end{array} \right]$$

Now, $R_3 \rightarrow R_3 - 2R_2$; $C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{array} \right]$

$R_2 \rightarrow R_2 + 4R_3$; $C = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{array} \right] = 0$

$R_1 \rightarrow R_1 - 2R_3$; $C = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] = 0$

$R_1 \rightarrow R_1 - R_2$; $C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right] = 0$

eqn: $x = 1$
 $y \leftarrow 2, z \leftarrow 3$, now

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] = 0$

b) Using the result of part (a), determine whether the vector $w = (9, 1, 0)$ can be expressed as a linear combination of the vectors

$$v_1 = (1, 2, 3), v_2 = (1, 4, 6), v_3 = (2, -3, -5)$$

: and, if so, find such a linear combination.

Soln: Suppose $w = (9, 1, 0)$ can be written

as L.C. of $v_1, v_2 \& v_3$

$$(9, 1, 0) = x(1, 2, 3) + y(1, 4, 6) + z(2, -3, -5)$$

$$= (x, 2x, 3x) + (y, 4y, 6y) + (2z, -3z, -5z)$$

$$= (x+2y+2z, 2x+4y-3z, 3x+6y-5z)$$

we get a system of equation

$$x+2y+2z=9 \quad \text{①} \quad \frac{x}{2} \leftarrow \text{①}$$

$$2x+4y-3z=1 \quad \text{④} \\ \text{②} \quad \text{④} \quad \text{②} \quad \text{④}$$

$$3x+6y-5z=0 \quad \text{⑤} \\ \text{③} \quad \text{⑤} \quad \text{③} \quad \text{⑤}$$

Another form of (i) leads to the next part (ii)

$$C = [A : B]$$

Assume $\text{det}(A, B) = w$ where all rows

end in $\begin{bmatrix} 1 & 9 & 3 & 9 \\ 2 & 9 & -3 & 1 \\ 3 & 1 & 6 & -5 \end{bmatrix}$. Then $w \neq 0$

$$(B_1, B_2, B_3) = gV \cdot \begin{bmatrix} 1 & 9 & 3 & 9 \\ 2 & 9 & -3 & 1 \\ 3 & 1 & 6 & -5 \end{bmatrix} \cdot (8, s, D) = Y$$

Performing row operations from left to right: $R_3 \rightarrow R_3 - 3R_1$ or $R_1 \leftrightarrow R_2$: $gV \cdot \begin{bmatrix} 1 & 9 & 2 & 9 \\ 0 & -4 & -7 & -17 \\ 0 & -6 & -11 & -27 \end{bmatrix}$

$$(B_1, B_2, B_3) + (A_1, A_2, A_3)x + (C_1, C_2, C_3)x = (0, 0, 0)$$

$$R_3 \Rightarrow R_3 - R_2 \quad \& \quad (R_3 \leftrightarrow R_2) + (A_1, A_2, A_3)x =$$

$$C = \begin{bmatrix} 1 & 9 & 2 & 9 \\ 0 & -4 & -7 & -17 \\ 0 & -2 & -4 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 2 & 9 \\ 0 & -2 & -9 & -16 \\ 0 & -4 & -7 & -17 \end{bmatrix}$$

~~where for every row left~~

$$R_2 \rightarrow \frac{R_2}{-2} \quad ; \quad C = \begin{bmatrix} 1 & 9 & 2 & 9 \\ 0 & 2 & 4 & 5 \\ 0 & 8 & 4 & -7 \end{bmatrix}$$

~~(P)~~ ~~(M)~~

$$0 = 5Z + 10 + 8E$$

$$R_3 \rightarrow R_3 + R_2;$$

$$C = \begin{bmatrix} 1 & 4 & 2 & : & 9 \\ 0 & 1 & 2 & : & 5 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$$\text{Hence, } z = 3 \quad [\text{3rd equ}]$$

$$y + 2z = 5 \quad [\text{2nd equ}]$$

$$\therefore y = 5 - 6 = -1$$

$$x + 4y + 2z = 9$$

$$\Rightarrow x = 9 + 4(-1) - 6 = 7 \quad (\text{from 1st eqn})$$

$$\text{So, } (9, -1, 0) = 1(1, 2, 3) - 1(1, 4, 6) + 3(2, -3, -5)$$

$$= (7 + 14 + 21) - (1, 4, 6) + (6, -9, -15)$$

$$= (7 - 1 + 6) | 19 - 4 - 9, 21 - 6 - 15 |$$

$$= (12, 0, 0) | 2 2 1$$

$\text{left} \leftrightarrow \text{right}$

$$\begin{bmatrix} e : & 1 & 0 & 1 \\ e : & 0 & 1 & 0 \\ e : & 1 & 0 & 0 \end{bmatrix} = 0$$

[Ans 678] $e = g \cdot \text{left}$

[Ans 678] $e = g + h$

$$h = g - e = g \cdot r$$

$$e = g + h + r$$

- c) Find the inverse of A (if possible) by elementary row operations and also state that rank of

$$A = \begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$$

So $A^{-1} = ?$ (from $AA^{-1} = I$)

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 9 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array} \right) =$$

Step 1: $R_2 \rightarrow R_2 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & -8 & -9 & -2 \\ 0 & 12 & 5 & 0 \end{array} \right] \xrightarrow{\text{Row 2} \leftarrow \text{Row 2} - 2 \cdot \text{Row 1}} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 12 & 5 & 0 \end{array} \right] \xrightarrow{\text{Row 3} \leftarrow \text{Row 3} - 12 \cdot \text{Row 1}} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -11 & 0 \end{array} \right]$$

2: $R_3 \rightarrow R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & -8 & -9 & -2 \\ 0 & 8 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 2} \leftarrow \text{Row 2} - 2 \cdot \text{Row 1}} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 8 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 3} \leftarrow \text{Row 3} - 8 \cdot \text{Row 1}} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

3: ~~$R_1 \rightarrow R_1 - 2R_3$~~ & $R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 8 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 1} \leftarrow \text{Row 1} - 2 \cdot \text{Row 3}} \left[\begin{array}{ccc|c} 1 & 6 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 1} \leftarrow \text{Row 1} - 6 \cdot \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

4: $R_3 \rightarrow R_3 - 8R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 3} \leftarrow \text{Row 3} - 8 \cdot \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \text{different from row 1}$$

[orthogonal] $0 \neq 1$ (AT) : several

5: $R_1 \rightarrow R_1 - 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 1} \leftarrow \text{Row 1} - 4 \cdot \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

\Rightarrow $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$6. R_3 \rightarrow R_3 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & P & 2 & 1 \\ 0 & 1 & 0 & -7 & -9 & 8 & 0 \\ 0 & 1 & 0 & 17 & 9 & -1 & 1 \\ 0 & 0 & 1 & -15 & -8 & 1 & 1 \end{array} \right]$$

$P = 98 - 2(9) + 1(1) \Rightarrow P = 10$

$$7. R_1 \rightarrow R_1 - 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -41 & P & 2 & 1 \\ 0 & 1 & 0 & 17 & -22 & 0 & 2 \\ 0 & 0 & 1 & -15 & 8 & -1 & 1 \end{array} \right]$$

$P = 98 - 2(9) + 1(1) \Rightarrow P = 10$

$$\therefore A^{-1}$$

$$\left[\begin{array}{ccc|ccc} 0 & -41 & P & -22 & 1 & 2 & 1 \\ 0 & 17 & 1 & 0 & 1 & 0 & 0 \\ 1 & -15 & 0 & 98 & -1 & 0 & 1 \end{array} \right]$$

$P = 98 - 2(9) + 1(1) \Rightarrow P = 10$

Firstly we need to find the possibility of inverse: $|A| \neq 0$ [Condition]

$$\left[\begin{array}{cccc|c} 1 & 6 & 4 & 1 & P \\ 2 & 9 & 5 & 1 & 2 \\ -1 & 2 & 5 & 1 & 2 \end{array} \right]$$

$P = 98 - 2(9) + 1(1) \Rightarrow P = 10$

$$P = 1(20+2) - 6(10-1) + 4(9+32) = 22 - 54 + 32 = 10$$

So we can determine A^{-1} .

~~if $\begin{pmatrix} \epsilon & \rho \\ \rho & 1 \end{pmatrix}$ is not zero then~~ $\det(A) \neq 0$

Rank: As the value of $\det(A)$ is zero (6)
 $\begin{pmatrix} \epsilon & \rho \\ \rho & 1 \end{pmatrix}$

so the rank is not 3 , it may be less than 3 . $\begin{pmatrix} \epsilon & \rho \\ \rho & 1 \end{pmatrix} = A$

Now, we consider $(A)^{-1} = \frac{1}{\det(A)} A^T$

$$\det(A) = \begin{vmatrix} 1 & 6 \\ \rho & 1 \end{vmatrix} = 1 - 6\rho \neq 0$$

So Rank is 2 .

ferred to $\begin{pmatrix} \epsilon & \rho \\ \rho & 1 \end{pmatrix}$ for zero

$$(M^{-1}(I)) = \frac{1}{\det(A)} A^T$$

$$d = (51 - 81) = \begin{vmatrix} \epsilon & \rho \\ \rho & 1 \end{vmatrix}^{-1} (I) = \frac{1}{\det(A)} A^T$$

$$d = (\epsilon - \rho) = \begin{vmatrix} \epsilon & \rho \\ \rho & 1 \end{vmatrix}^{-1} (I) = \frac{1}{\det(A)} A^T$$

$$s = (\rho - 1) = \begin{vmatrix} \epsilon & \rho \\ \rho & 1 \end{vmatrix}^{-1} (I) = \frac{1}{\det(A)} A^T$$

7.

a) find inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 9 & 9 \end{pmatrix}$ by finding
(i) its adjoint To convert $A^{-1} = \frac{1}{|A|} \text{adj } A$

Solution: \exists for if invertible or
Bef, $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 9 & 9 \end{pmatrix}$, \exists mult.

$$A^{-1} = \frac{1}{|A|} \text{adj } A \quad \text{because } |A| \neq 0$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 9 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 12 & 27 \\ 1 & 9 & 9 \end{vmatrix} = 1(18 - 12) - 1(9 - 3) + 1(4 - 1)$$

So, the inverse of matrix is exist.

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 9 & 9 \end{vmatrix} = (18 - 12) = 6$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9 - 3) = -6$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 9 \end{vmatrix} = (9 - 2) = 7$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -\frac{(1-9)}{|A|} = -5$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = \frac{(1-1)}{|A|} = 0$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -\frac{(4-1)}{|A|} = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = (3-2) = 1$$

~~entry gives (1) from there with part (d)~~

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -\frac{(3-1)}{|A|} = -2$$

~~entry gives (1) from there with part (d)~~

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (2-1) = 1$$

~~(1) B2~~

$$\text{So, } \text{Adj } A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & 3 \\ 1 & -2 & 1 \end{pmatrix}$$

~~$A^{-1}A = I$~~

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A \quad |A| = 12$$

$$= \frac{1}{12} \begin{pmatrix} 1 & 1 & 1 \\ -6 & 2 & 1 \\ 8 & -3 & 1 \end{pmatrix} \text{adj}(1-)= \underline{\underline{A^{-1}}}$$

$$= \frac{1}{12} \begin{pmatrix} -5 & 8 & -3 \\ -1 & 12 & 1 \end{pmatrix} \text{adj}(1-)= \underline{\underline{A^{-1}}}$$

$$I = (I - E) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{adj}(1-)= 12A$$

b) Using the result of part (a), solve the system of equations $(I-E) = 12A$

$$I = (I-E) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{adj}(1-)= 12A \quad x+2y+3z=3, \quad x+4y+9z=9, \quad x+9y+27z=6$$

Sol:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 6 \end{bmatrix} \quad \text{adj}(1-)= 12A \quad \text{adj}$$

$\Rightarrow A^{-1}Ax = A^{-1}B$

$$\Rightarrow Ix = A^{-1}B$$

$$\therefore x = A^{-1}B$$

A^{-1} exist if $|A| \neq 0$

$$\begin{aligned}|A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} + (-1)^3 \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} \\ &\quad + (-1)^4 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ &= (18 - 12) - (9 - 3) + (4 - 2) \\ &= 6 - 6 + 2 \\ &= 2\end{aligned}$$

We can take the result of part (a), where

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned}X &= A^{-1}B = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 9 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 9 - 12 + 6 \\ -\frac{15}{2} + 16 - 9 \\ -\frac{3}{2} - 9 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{1}{2} \\ -\frac{2}{2} \end{pmatrix}\end{aligned}$$