(vi) Distributive law [বণ্টন বিধি] :

left : $\forall\, a, b, c \in \mathbb{R} \Rightarrow a(b+c) = ab + ac$

right : $\forall\, a, b, c \in \mathbb{R} \Rightarrow (b+c)a = ba + ca$

Example : $\mathbb{Z}$ is a ring with usual addition and multiplication.

1. $\forall\, a, b \in \mathbb{Z} \Rightarrow a + b \in \mathbb{Z}$
2. $\forall\, a, b, c \in \mathbb{Z} \Rightarrow (a+b)+c = a+(b+c)$
3. $\exists\, 0 \in \mathbb{Z} \Rightarrow a+0 = 0+a = a\ \forall\, a \in \mathbb{Z}$
4. $\forall\, a \in \mathbb{Z}, \exists -a \in \mathbb{Z} \Rightarrow a+(-a) = (-a)+a = 0$
5. $\forall\, a, b \in \mathbb{Z} \Rightarrow a+b = b+a$
6. $\forall\, a, b \in \mathbb{Z} \Rightarrow ab \in \mathbb{Z}$
7. $\forall\, a, b, c \in \mathbb{Z} \Rightarrow (ab)\,c = a(bc)$
8. $\forall\, a, b, c \in \mathbb{Z} \Rightarrow a(b+c) = ab + ac$
9. $\forall\, a, b, c \in \mathbb{Z} \Rightarrow (b+c)\,a = ba + ca$

Thus $\mathbb{Z}$ is called the ring of integers with respect to usual addition and multiplication.

Example : $\mathbb{R}, \mathbb{Z}, \mathbb{Q}$ are ring under addition (+) and multiplication (.).

6.4 Field
ফিল্ড

A ring F with at least two elements and satisfying the following conditions is called a field. [একটি রিং F-কে ফিল্ড বলা হবে যদি কমপক্ষে এর দুইটি উপাদান থাকে এবং নিম্নোক্ত শর্তাবলি সিদ্ধ করে :]

(i) $\forall\, a, b \in F \Rightarrow ab = ba$

(ii) $\exists\, 1 \in F, 1 \neq 0$ such that $a.1 = 1.a = a\ \ \forall\, a \in F$

(iii) $a \in F, a \neq 0, \exists\, a^{-1} \in F \Rightarrow aa^{-1} = a^{-1}a = 1\ \forall\, a \in F$

Example : $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are field with respect to usual addition and multiplication. [সচরাচর যোগ ও গুণ প্রক্রিয়ায় $\mathbb{Q}, \mathbb{R}$ এবং $\mathbb{C}$ প্রত্যেকটি ফিল্ড ।]

6.5 Vector Space
ভেক্টর জগত

[DUH 2016, '15, '14, '13; NUH '10, '08, '03, '98; NU (Preal), '09, '05]

Let F be a field of scalars and V be an arbitrary non-empty set of objects on which two operations, vector addition and scalar multiplication are defined, then

V is called vector space over the field F if the following axioms are holds : [মনে করি, F একটি স্কেলার এবং V উপাদানসমূহের একটি অশূন্যক সেট যাতে দুইটি কার্যক্রম, ভেক্টর যোগ এবং স্কেলার গুণন সংজ্ঞায়িত, তবে F-এ বর্ণিত V-কে ভেক্টর জগত বলা হবে যদি নিম্নোক্ত শর্তগুলো মেনে চলে :]

Operations on vector addition [ভেক্টর যোগের কার্যক্রম]

A_1 : Closure law [আবদ্ধতা বিধি] : $\forall\, \mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$

A_2 : Associative law [সংযোগ বিধি] : $\forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V \Rightarrow (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

A_3 : Existence of **0** (zero vector) [**0** (শূন্য ভেক্টরের) অস্তিত্ব বিধি] : $\exists\ \mathbf{0} \in V \Rightarrow \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \quad \forall\, \mathbf{u} \in V$

A_4 : Existence of negative vector [ঋণাত্মক ভেক্টরের অস্তিত্ব বিধি] : $\forall \mathbf{u} \in V, \exists\, (-\mathbf{u}) \in V$ such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

A_5 : Commutative law [বিনিময় বিধি] : $\forall\, \mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Operations on scalar multiplication [স্কেলার গুণনের কার্যক্রম]

M_1 : $\forall\, \alpha \in F$ and $\forall\, \mathbf{u} \in V \Rightarrow \alpha\, \mathbf{u} \in V$

M_2 : $\forall\, \alpha \in F$ and $\forall\, \mathbf{u}, \mathbf{v} \in V \Rightarrow \alpha\, (\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

M_3 : $\forall\, \alpha, \beta \in F$ and $\forall\, \mathbf{u} \in V \Rightarrow (\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$

M_4 : $\forall\, \alpha, \beta \in F$ and $\mathbf{u} \in V \Rightarrow (\alpha\, .\, \beta)\mathbf{u} = \alpha(\beta\mathbf{u})$

M_5 : $\exists\, 1 \in F, 1 \neq 0 \Rightarrow 1 \cdot \mathbf{u} = \mathbf{u} \cdot 1 = \mathbf{u}\ \ \forall\, \mathbf{u} \in V$

Example :

(i) The set of all real numbers x is a vector space with respect to the standard operations of addition and multiplication.

(ii) The set of all pairs of real numbers of the form $(x, 0)$ is a vector space with respect to the standard operations on $\mathbb{R}^2$.

(iii) The set of all n-tuples of real numbers of the form $(x, x, \ldots\ldots, x)$ is a vector space with respect to the standard operations on $\mathbb{R}^n$.

(iv) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is a vector space with respect to the standard operations.

(v) Every plane through the origin is a vector space.

(vi) The set of all $m \times n$ matrices M with real entries over the scalar field F is a vector space.

(vii) The set of all points in space i.e., $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is a vector space over the scalar field with respect to the standard operations.

Example-1 Show that the set M of all 2×2 matrices with real entries is a vector space. [দেখাও যে, বাস্তব উপাদান বিশিষ্ট সকল 2×2 ম্যাট্রিক্সের সেট M একটি ভেক্টর স্পেস।]

Solution The set M of all 2×2 matrices with real entries will be vector space if it satisfies two operations matrix addition and matrix scalar multipliation.

Let $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$

Now $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$

$$= \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \in V$$

Similarly, $\mathbf{v} + \mathbf{u} \in V$

Again $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \left\{ \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \right\}$

$$= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} + w_{11} & v_{12} + w_{12} \\ v_{21} + w_{21} & v_{22} + w_{22} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & u_{22} + v_{22} + w_{22} \end{bmatrix} \in V$$

Similarly, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \begin{bmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & u_{22} + v_{22} + w_{22} \end{bmatrix} \in V$

A_3: $\mathbf{u} + \mathbf{0} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in V$

also $\mathbf{0} + \mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in V$

$\therefore \mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \in V$

A_4: $\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$

$$= \begin{bmatrix} u_{11} - u_{11} & u_{12} - u_{12} \\ u_{21} - u_{21} & u_{22} - u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \in V$$

Similarly, $(-\mathbf{u}) + \mathbf{u} = \mathbf{0} \in V$

i.e., $(-\mathbf{u}) + \mathbf{u} = \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \in V$

$$A_5 : \mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

$$\therefore \mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u} \in V$$

M_1 : For any scalar k and vector $\mathbf{u}$

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \in V$$

$$M_2 : k(\mathbf{u}+\mathbf{v}) = k\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11}+kv_{11} & ku_{12}+kv_{12} \\ ku_{21}+kv_{21} & ku_{22}+kv_{22} \end{bmatrix}$$

$$= \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} + \begin{bmatrix} kv_{11} & kv_{12} \\ kv_{21} & kv_{22} \end{bmatrix}$$

$$= k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$= k\mathbf{u}+k\mathbf{v} \in V$$

$$M_3 : (k_1+k_2)\mathbf{u} = (k_1+k_2)\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= k_1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + k_2\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= k_1\mathbf{u}+k_2\mathbf{u} \in V$$

$$M_4 : k_1k_2\,\mathbf{u} = k_1k_2\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= k_1\begin{bmatrix} k_2u_{11} & k_2u_{12} \\ k_2u_{21} & k_2u_{22} \end{bmatrix}$$

$$= k_1(k_2\,\mathbf{u}) \in V$$

$$M_5 : 1.\mathbf{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}.1 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$\Rightarrow 1.\mathbf{u} = \mathbf{u}.1 = \mathbf{u} \in V$$

Thus **u**, **v**, **w** satisfies all the axioms of vector space. So M is a vector space.
[সুতরাং **u**, **v**, **w** ভেক্টর জগতের সকল শর্ত মেনে চলে। অতএব M একটি ভেক্টর জগত।]

[Showed]

6.7 Subspace
উপজগত

[DUH 2016, 2015, 2013; JNU 2015; 2014, 2013, 2011 NUH '10, '08, '03, '01; NUH (NM) '02, '04; B. Sc(pass) '07]

If W be a non-empty subset of a vector space $V(F)$, then W is called a subspace of V if W satisfies all the axioms of vector space V with respect to vector addition and scalar multiplication. [যদি W, $V(F)$ এর একটি অশূন্য উপসেট হয় তবে W-কে V এর উপজগত বলা হবে, যদি W ভেক্টর যোগ ও স্কেলার গুণনের সাপেক্ষে ভেক্টর জগত V এর সকল শর্ত সিদ্ধ করে।]

Or, *A* subset W of a vector space V is called a subspace of V if W itself a vector space under the addition and scalar mutiplication defined on V. [ভেক্টর জগত V এর উপসেট W-কে উপজগত বলা হবে যদি W, V-তে বর্ণিত যোগ ও স্কেলার গুণনের সংজ্ঞায় নিজে ভেক্টর জগত হয়।]

Example :

(i) Lines through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

(ii) The set of all continuous functions is a subspace of the vector space of all real valued functions.

(iii) Let $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ be a vector space. Then $w_1 = \{(x, 0) : x \in \mathbb{R}\}$, $w_2 = \{(0, y) : y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$.

(iv) Let $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ then the set of all points on the line $y = x$ i.e., $w = \{(x, y) : x = y \text{ and } x, y \in \mathbb{R}\}$ is a subspace.

(v) Let $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be a vector space. Then $w_1 = \{(0, y, z) : y, z \in \mathbb{R}\}$, $w_2 = \{(x, 0, z) : x, z \in \mathbb{R}\}$ and $w_3 = \{(x, y, 0) : x, y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^3$, where the sets w_1, w_2 and w_3 represents the sets of all points in the Y-Z, Z-X and X-Y plane respectively.

Example-2 Determine whether the following sets are subspace of $\mathbb{R}^3$ or not. [নিম্নের সেটগুলি $\mathbb{R}^3$ এর উপজগত কি না পরীক্ষা কর :]

(i) $S = \{(x, y, 0) : x, y \in \mathbb{R}\}$

(ii) $T = \{(x, y, 1) : x, y \in \mathbb{R}\}$

Solution

(i) For $\mathbf{0} \in \mathbb{R}^3 \Rightarrow \mathbf{0} = (0, 0, 0) \in S$

$\Rightarrow S$ is non-empty.

Again for any vectors $\mathbf{u} = (x_1, y_1, 0)$ and $\mathbf{v} = (x_2, y_2, 0)$ where $\mathbf{u}, \mathbf{v} \in S$ and for any scalars α, β

we have
$$\begin{aligned}\alpha\,\mathbf{u} + \beta\,\mathbf{v} &= \alpha(x_1, y_1, 0) + \beta(x_2, y_2, 0)\\ &= (\alpha x_1, \alpha y_1, 0) + (\beta x_2, \beta y_2, 0)\\ &= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, 0) \in S\end{aligned}$$

$\Rightarrow \alpha \mathbf{u} + \beta \mathbf{v} \in S$

Thus $S = \{(x, y, 0) : x, y \in \mathbb{R}^3\}$ is a subspace of $\mathbb{R}^3$ (Ans.)

For $\mathbf{0} \in \mathbb{R}^3 \Rightarrow \mathbf{0} = (0, 0, 0) \notin T$

Since 3rd component is zero, not 1.

Again for any $\mathbf{u}, \mathbf{v} \in T$

Let $\mathbf{u} = (x_1, y_1, 1), \mathbf{v} = (x_2, y_2, 1)$

$\Rightarrow \mathbf{u} + \mathbf{v} = (x_1, y_1, 1) + (x_2, y_2, 1)$

$= (x_1 + x_2, y_1 + y_2, 2) \notin T$

Since the 3rd component is 2, not 1.

Hence T is not a subspace of $\mathbb{R}^3$ (Ans.)

eorem-5 : ***First Fundamental theorem of subspace*** [উপজগতের প্রথম মৌলিক পাদ্য] ***[DUH 2014; JNUH 2014, 2011; NUH '07, '08, '12; NU(prel) '09]***

tement : A subset W of a vector space $V(F)$ is called a subspace of $V(F)$ if কটি উপসেট W, $V(F)$ এর উপজগত বলা হবে যদি ও কেবল মাত্র যদি]

W is non-empty [W অশূন্যক হয়] i.e., $W \neq \phi$]

) W is closed under addition [W যোগ প্রক্রিয়ায় আবদ্ধশীল হয়]

i.e., $\forall\, \mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$

i) W is closed under scalar multiplication [W স্কেলার গুণন প্রক্রিয়ায় আবদ্ধশীল হয়]

i.e., $\forall\, k \in F$ and $\mathbf{u} \in \mathrm{W} \Rightarrow k\,\mathbf{u} \in W$

irst let the subset W be a non-empty and closed under addition and scala ultiplication. Then W satisfies the following rules.

roof : $A_1 : \forall\, \mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$

$A_2 : \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in W \Rightarrow (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

$A_3 : \exists\, \mathbf{u}, 0 \in W \Rightarrow \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \quad \forall\, \mathbf{u} \in W$

$A_4 : \forall\, \mathbf{u} \in W, \exists\, (-\mathbf{u}) \in W$ such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$

$A_5 : \forall\, \mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$M_1 : \forall\, \alpha \in F$ and $\mathbf{u} \in W \Rightarrow \alpha\, \mathbf{u} \in W$

$M_2 : \forall\, \alpha \in F$ and and $\mathbf{u}, \mathbf{v} \in W \Rightarrow \alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

$M_3 : \forall\, \alpha, \beta \in F$ and $\mathbf{u} \in W$

$\Rightarrow (\alpha + \beta)\, \mathbf{u} = \alpha\, \mathbf{u} + \beta\mathbf{u}$

$M_4 : \forall\, \alpha, \beta \in F$ and $\mathbf{u} \in W$

$\Rightarrow (\alpha \quad \beta)\mathbf{u} = \alpha(\beta\, \mathbf{u})$

$M_5 : \exists\, 1 \in F, 1 \neq 0, 1.\, \mathbf{u} = \mathbf{u}.1 = \mathbf{u} \quad \forall\, \mathbf{u} \in W$

Since W satisfies all the laws of vector space, so W is a subspace of $V(F)$. Conversely let W is a subspace of $V(F)$. So it will satisfy all the axioms of vector space. [যেহেতু W ভেক্টর জগতের সকল শর্ত সিদ্ধ করে, অতএব W, $V(F)$ এর উপজগত। বিপরীতক্রমে ধরি W, $V(F)$ এর উপজগত। তাহলে ইহা ভেক্টর জগতের সকল শর্ত মেনে চলবে।]

i.e., (i) $W \neq \phi$

(ii) W is closed under addition

(iii) W is closed under scalar multiplication

Hence the theorem is proved.

Theorem-6 : ***Second Fundamental theorem of subspace*** [উপজগতের দ্বিতীয় মৌলিক উপপাদ্য] ***[NUH '10, '09, '04, '06; JNUH 2013]***

Statement : W is a subspace of vector space $V(F)$ if and only if $\mathbf{0} \in W$ or $W \neq \phi$ and $\forall\ \alpha, \beta \in F$ and $\forall\ \mathbf{u}, \mathbf{v} \in W \Rightarrow \alpha\,\mathbf{u} + \beta\,\mathbf{v} \in W$. [$W$, ভেক্টর জগত $V(F)$ এর একটি উপজগত হবে যদি ও কেবল মাত্র যদি $\mathbf{0} \in W$ অথবা $W \neq \phi$ এবং $\forall\ \alpha, \beta \in F$ এবং $\mathbf{u}, \mathbf{v} \in W \Rightarrow \alpha\,\mathbf{u} + \beta\,\mathbf{v} \in W$]

Proof : First let W be a subspace of V, then W must be closed under vector addition and scalar multiplication.

Therefore, $\alpha \in F$ and $\mathbf{u} \in W \Rightarrow \alpha\,\mathbf{u} \in W$

$\beta \in F$ and $\mathbf{v} \in W \Rightarrow \beta\,\mathbf{v} \in W$

which implies $\alpha\,\mathbf{u} + \beta\,\mathbf{v} \in W$

Thus $\alpha, \beta \in F$ and $\mathbf{u}, \mathbf{v} \in W \Rightarrow \alpha\,\mathbf{u} + \beta\,\mathbf{v} \in W$

Conversely let non-emply set W satisfying the condition $\forall\ \alpha, \beta \in F$ and $\mathbf{u}, \mathbf{v} \in W \Rightarrow \alpha\,\mathbf{u} + \beta\,\mathbf{v} \in W$

Let $\alpha = 1$, $\beta = 1$, then $1 \in F$ and $\mathbf{u}, \mathbf{v} \in \mathrm{W}$

$\Rightarrow 1.\mathbf{u} + 1.\mathbf{v} \in \mathrm{W}$

i.e., $\mathbf{u} + \mathbf{v} \in \mathrm{W}$

$\Rightarrow$ W is closed under vector addition [W ভেক্টর যোগ প্রক্রিয়ায় আবদ্ধশীল]

Again let $\alpha = k$, $\beta = 0$, then $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathrm{W}$ becomes

$k.\mathbf{u} + 0.\mathbf{v} \in W$

$\Rightarrow k.\mathbf{u} \in W$

Thus W is closed under scalar multiplication. [W স্কেলার গুণন প্রক্রিয়ায় আবদ্ধশীল]

Hence, W is a subspace of $V(F)$. [সুতরাং W, $V(F)$ এর উপজগত।] [Proved]

Example-9 Show that $W = \{(a, b, c) \in \mathbb{R}^2 : a + b + c = 0\}$ is a subspace of $\mathbb{R}^3$.
[দেখাও যে $W = \{(a, b, c) \in \mathbb{R}^2 : a + b + c = 0\}$, $\mathbb{R}^3$ এর উপজগত ।]

Solution Here $\mathbf{0} = (0, 0, 0) \in W$, since $0 + 0 + 0 = 0$

$\Rightarrow W \neq \phi$ i.e., W is non-empty.

Let $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2) \in W$

Then $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$

Now for any scalar $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha\mathbf{u} + \beta\mathbf{u} = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$$
$$= (\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$$
$$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$$

Now $(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2) = \alpha a_1 + \beta a_2 + \alpha b_1 + \beta b_2 + \alpha c_1 + \beta c_2$

$[\because \{(a, b, c) \in \mathbb{R}^2 : a + b + c = 0\}]$

$$= \alpha(a_1 + b_1 + c_1) + \beta(a_2 + b_2 + c_2)$$
$$= \alpha.0 + \beta.0 = 0$$

Thus $\alpha\mathbf{u} + \beta\mathbf{v} \in W$

Hence W is a subspace of $\mathbb{R}^3$. [Showed]

Example-10 Show that any straight line passing through origin is a subspace of $\mathbb{R}^2$. [দেখাও যে, মূলবিন্দুগামী যে কোন সরলরেখা $\mathbb{R}^2$ এর উপজগত ।]

Solution Let the set of any straight line passing through origin is

$S = \{(x, y) : y = mx\}$

Now we shall have to show that S is a subspace of $\mathbb{R}^2$

$\mathbf{0} = (0, 0) \in \mathbb{R}^+$ and since $(0, 0)$ satisfy $y = mx$

$\therefore (0, 0) \in S$, so that S is non-empty.

Let $\mathbf{u}, \mathbf{v} \in S$ where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$

$\therefore u_2 = mu_1$ and $v_2 = mv_1$

For any scalar α and β, we have

$$\alpha u + \beta v = \alpha(u_1, u_2) + \beta(v_1, v_2)$$
$$= (\alpha u_1, \beta u_2) + (\beta v_1 + \beta v_2)$$
$$= (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2)$$
$$= (w_1, w_2), \text{ where } w_1 = \alpha u_1 + \beta v_1, w_2 = \alpha u_2 + \beta v_2$$

Here $w_2 = \alpha u_2 + \beta v_2$

$= \alpha.mu_1 + \beta.mv_1 \quad [\because v_2 = mv_1, \ u_2 = mu_1]$

$= m(\alpha u_1 + \beta v_1)$

$= mw_1$

$\therefore (w_1, w_2) \in S \Rightarrow \alpha\mathbf{u} + \beta\mathbf{v} \in S$

Thus S is a subspace of $\mathbb{R}^2$.

Hence any straight line passing through origin is a subspace of $\mathbb{R}^2$. [Showed]

Example-11 Show that $W = \{(a, b, c) : a + b = 0\}$ is subspace of $V(\mathbb{R})$. [দেখাও যে $W = \{(a, b, c) : a + b = 0\}$, $V(\mathbb{R})$ এর উপজগত ।]

Solution Here $\mathbf{0} = (0, 0, 0) \in W$, since $0 + 0 = 0$

$\Rightarrow W \neq \phi$, i.e., W is non-empty.

Let $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2) \in W$

then $a_1 + b_1 = 0$ and $a_2 + b_2 = 0$

Now for any scalar $\alpha, \beta \in \mathbb{R}$, We have

$\alpha\mathbf{u} + \beta\mathbf{v} = (\alpha a_1, \alpha b_1, \alpha c_1) + (\beta a_2, \beta b_2, \beta c_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2)$

Now $(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2) = (\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2)$

$= \alpha(a_1 + b_1) + \beta(a_2 + b_2)$

$= \alpha.0 + \beta.0 = 0$

$\Rightarrow \alpha\mathbf{u} + \beta\mathbf{v} \in W$

Hence W is a subspace of $V(\mathbb{R})$. [Showed]

Example-12 Show that (i) the xy-plane, (ii) the yz-plane and (iii) the zx-plane are the subspaces of $\mathbb{R}^3$. [দেখাও যে (i) xy সমতল, (ii) yz সমতল এবং (iii) zx সমতল $\mathbb{R}^3$ এর উপজগত ।]

Solution

(i) Let the set of xy plane $P_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$, since third co-ordinate of $\mathbf{0} = (0, 0, 0)$ is zero.

Therefore $\mathbf{0} \in P_1$ implies P_1 is non-empty.

Let $\mathbf{u}, \mathbf{v} \in P_1$ where $\mathbf{u} = (u_1, u_2, 0)$ and $\mathbf{v} = (v_1, v_2, 0)$.

For any scalars α and β

$\alpha\mathbf{u} + \beta\mathbf{v} = \alpha(u_1, u_2, 0) + \beta(v_1, v_2, 0)$

$= (\alpha u_1, \alpha u_2, 0) + (\beta v_1, \beta v_2, 0)$

$= (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, 0)$

Since z co-ordinate of $\alpha\mathbf{u} + \beta\mathbf{v}$ is zero, so $\alpha\mathbf{u} + \beta\mathbf{v} \in P_1$

Thus P_1 is subspace of $\mathbb{R}^3$. Hence xy-plane is a subspace of $\mathbb{R}^3$.

(ii) Let the set of yz plane $P_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$, since first coordinate of $\mathbf{0} = (0, 0, 0)$ is zero. Therefore $\mathbf{0} \in P_2$ implies P_2 is non-empty.

Let, $\mathbf{u}, \mathbf{v} \in P_2$ where $\mathbf{u} = (0, u_2, u_3)$ and $\mathbf{v} = (0, v_2, v_3)$.

For any scalars α and β, we have

$$\begin{aligned}\alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(0, u_1, u_3) + \beta(0, v_2, v_3)\\ &= (0, \alpha u_2, \alpha u_3) + (0, \beta v_2, \beta v_3)\\ &= (0, \alpha u_2 + \beta v_2, \alpha u_3 + \beta v_3)\end{aligned}$$

Since x-co-ordinate of $\alpha\mathbf{u} + \beta\mathbf{v}$ is zero.

$\therefore \alpha\mathbf{u} + \beta\mathbf{v} \in P_2$

Thus P_2 is subspace of $\mathbb{R}^2$. Hence yz-plane is a subspace of $\mathbb{R}^3$.

(iii) Let the set of zx plane $P_3 = \{(x, 0, z) : x, z \in \mathbb{R}\}$, since second co-ordinate of $\mathbf{0} = (0, 0, 0)$ is zero. Therefore $\mathbf{0} \in P_3$ implies P_3 is non-empty.

Let $\mathbf{u}, \mathbf{v} \in P_3$ where $\mathbf{u} = (u_1, 0, u_3)$ and $\mathbf{v} = (v_1, 0, v_3)$. For any scalars α and β

$$\begin{aligned}\alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(u_1, 0, u_3) + \beta(v_1, 0, v_3)\\ &= (\alpha u_1, 0, \alpha u_3) + (\beta v_1, 0, \beta v_3)\\ &= (\alpha u_1 + \beta v_1, 0, \alpha u_3 + \beta v_3)\end{aligned}$$

Since y co-ordinate of $\alpha\mathbf{u} + \beta\mathbf{v}$ is zero, therefor $\alpha\mathbf{u} + \beta\mathbf{v} \in P_3$

Thus P_3 is subspace of $\mathbb{R}^3$. Hence zx-plane is a subspace of $\mathbb{R}^3$. [Showed]

Example-13 Show that the set $T = \{(a, b, c, d) \in \mathbb{R}^4 : 2a - 3b + 5c - d = 0$ is a subspace of $\mathbb{R}^4$. [দেখাও যে, সেট $T = \{(a, b, c, d) \in \mathbb{R}^4 : 2a - 3b + 5c - d) = 0$, $\mathbb{R}^4$ এর একটি উপজগত ।] *[NUH (NM) 2002]*

Solution For $\mathbf{0} \in \mathbb{R}^4$, $\mathbf{0} = (0, 0, 0, 0) \in T$, Since $2.0 - 3.0 + 5.0 - 0 = 0$

Hence T is non-empty.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4) \in T$

Then $2u_1 - 3u_3 + 5u_3 - u_4 = 0$ and $2v_1 - 3v_2 + 5v_3 - v_4 = 0$

Now for any scalars (real numbers) α, β

We have
$$\begin{aligned}\alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(u_1, u_2, u_3, u_4) + \beta(v_1, v_2, v_3, v_4)\\ &= (\alpha u_1, \alpha u_2, \alpha u_3, \alpha u_4) + (\beta v_1, \beta v_2, \beta v_3, \beta v_4)\\ &= (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \alpha u_3 + \beta v_3, \alpha u_4 + \beta v_4)\end{aligned}$$

Now
$$\begin{aligned}&2(\alpha u_1 + \beta v_1) - 3(\alpha u_2 + \beta v_2) + 5(\alpha u_3 + \beta v_3) - (\alpha u_4 + \beta v_4)\\ &= \alpha(2u_1 - 3u_3 + 5u_3 - u_4) + \beta(2v_1 - 3v_2 + 5v_3 - v_4)\\ &= \alpha.0 + \beta.0 = 0\end{aligned}$$

Thus $\alpha\mathbf{u} + \beta\mathbf{v} \in T$. Hence T is a subspace of $\mathbb{R}^4$. [Proved]

Necessary Formulae & Important Informations
প্রয়োজনীয় সূত্র ও গুরুত্বপূর্ণ তথ্যাবলি

- Vector $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V$ (vector space) for any scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ such that $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n$.
- $\mathbf{v} = (1, 1, 1)$ is a linear combination of the vectors $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (0, 1, 2)$ and $\mathbf{v}_3 = (-1, 0, 1)$. That is, $(1, 1, 1) = 2(1, 2, 3) - 3(0, 1, 2) + (-1, 0, 1)$.
- A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n\}$ in a vector space V is called linearly independent if the vector equation
$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$$
has only the trivial solution $\alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$. If there are also nontrivial solution, then S is called linearly dependent.
- The set $S = \{(1, 2), (2, 4)\}$ in $\mathbb{R}^2$ is linearly dependent because $-2(1, 2) + (2, 4) = (0, 0)$.
- The set $S = \{(1, 2, 0), (-2, 2, 1)\}$ is linearly independent because $(1, 2, 0)$ and $(-2, 2, 1)$ are not scalar multiple of each other.
- Vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in $\mathbb{R}^3$ are linearly independent.

7.1 Introduction
ভূমিকা

The concept of linear combination, linear dependence and linear independence plays an important role in linear algebra and in the analysis of pure mathematics. We have discussed in this chapter linear combination, linear dependence and independence of vectors and some theorems on these topics. Some numerical examples are also shown this chapter.

7.2 Linear Combination
যোগাশ্রয়ী সমাবেশ *[DUH 2016, 2014, 2013; JNUH 2016, 2015, 2014, 2013]*

Let $\mathbf{v}$ be a vector space over the field F, then $\mathbf{v} \in V(F)$ is said to be linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V(F)$ if there exists scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n = \sum_{i=1}^{n}\alpha_i\mathbf{v}_i$.

[মনে করি, F ফিল্ডের উপর V একটি ভেক্টর জগত, তাহলে $\mathbf{v} \in V(F)$ কে $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V(F)$ ভেক্টরের যোগাশ্রয়ী সমাবেশ বলা হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in V(F)$ এর জন্য $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n = \sum_{i=1}^{n}\alpha_i\mathbf{v}_i$ হয় ।]

Example-1 Let $v = (5, 9, 5)$ and $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$, $v_3 = (3, 2, 5)$. Show that v can be expressed as a linear combination of the vectors v_1, v_2 and v_3. [মনে কর, $v = (5, 9, 5)$ এবং $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$, $v_3 = (3, 2, 5)$। দেখাও যে v কে v_1, v_2 এবং v_3 এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়।]

Solution v can be expressed as a linear combination of the vectors v_1, v_2 and v_3 if for any scalars α_1, α_2 and $\alpha_3 \in F$ such that [v কে v_1, v_2 এবং v_3 এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যাবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \alpha_3 \in F$ এর জন্য]

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (5, 9, 5) = \alpha_1(2, 1, 4) + \alpha_2(1, -1, 3) + \alpha_3(3, 2, 5)$$

$$\Rightarrow (5, 9, 5) = (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3)$$

$$\Rightarrow (5, 9, 5) = (2\alpha_1 + \alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 + 2\alpha_3, 4\alpha_1 + 3\alpha_2 + 5\alpha_3)$$

Equating the corresponding components, we get the following system of linear equations [অনুরূপ উপাদানসমূহ সমীকৃত করে নিম্নোক্ত সরল সমীকরণ জোট পাই]

$$\begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5 \end{cases} \quad \text{.....} (1)$$

Reducing system (1) to echelon form by using elementary row operations. [প্রাথমিক সারি অপারেশন ব্যবহার করে সমীকরণ জোট (1) কে ইচালন আকারে প্রকাশ করে পাই]

$$\sim \begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ -3\alpha_2 + \alpha_3 = 13 \\ 7\alpha_2 - 3\alpha_3 = -31 \end{cases} \quad \begin{bmatrix} L_2' = 2L_2 - L_1 \\ L_3' = L_3 - 4L_2 \end{bmatrix}$$

$$\sim \begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ -3\alpha_2 + \alpha_3 = 13 \\ -2\alpha_3 = -2 \end{cases} \quad [L_3' = 3L_3 + 7L_2]$$

Using back-substitution method from 3rd equation, we get $\alpha_3 = 1$ [পশ্চাৎ প্রতিস্থাপন পদ্ধতি ব্যবহার করে তৃতীয় সমীকরণ থেকে পাই $\alpha_3 = 1$]

Substituting $\alpha_3 = 1$ in 2nd equation, we get [দ্বিতীয় সমীকরণে $\alpha_3 = 1$ বসিয়ে পাই]

$$-3\alpha_2 + 1 = 13$$

$$\Rightarrow -3\alpha_2 = 12$$

$$\therefore \alpha_2 = -4$$

Again substituting $\alpha_3 = 1$ & $\alpha_2 = -4$ in 1st equation, we get [আবার প্রথম সমীকরণে $\alpha_3 = 1$ এবং $\alpha_2 = -4$ বসিয়ে পাই]

$$2\alpha_1 - 4 + 3 = 5$$

$$\Rightarrow 2\alpha_1 = 6$$

$$\therefore \alpha_1 = 3$$

Thus the solution of the system is [সুতরাং সমীকরণ জোটের সমাধান]

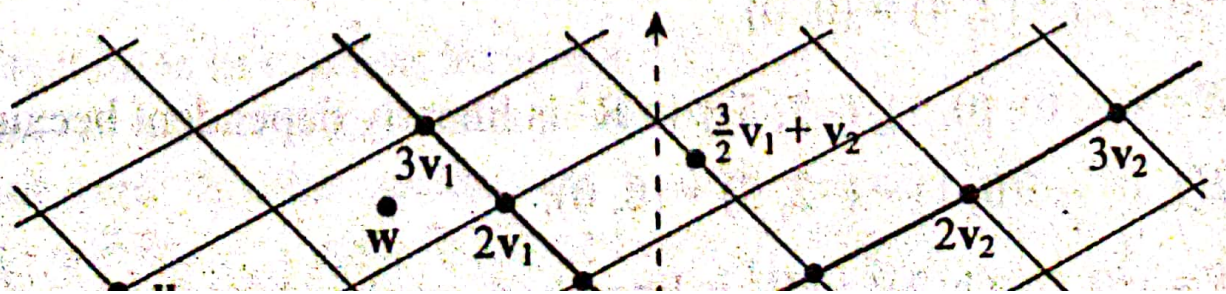
$$\alpha_1 = 3, \alpha_2 = -4, \alpha_3 = 1$$

Therefore, $\mathbf{v} = 3\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3$.

Hence, $\mathbf{v}$ can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. [অতএব $\mathbf{v}$ কে $\mathbf{v}_1$, $\mathbf{v}_2$ এবং $\mathbf{v}_3$ এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায় ।] [Showed]

Example-2 The following figure identifies selected linear combinations of $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (2, 1)$. Estimate the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that generate the vectors $\mathbf{u}$ and $\mathbf{w}$.

[Note that sets of parallel grid lines are drawn through integer multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$.]



7.3 Linear Dependence
যোগাশ্রয়ী নির্ভরশীলতা

[JNU 2016, 2015]

Let $V(F)$ be a vector space over the field F. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V$ is said to be linearly dependent (or simply dependent) if for any scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$, such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots\ldots + \alpha_n\mathbf{v}_n = 0$, where at least one of the scalars of the set $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ is not zero. [মনে করি $V(F)$, F ফিল্ডের উপর একটি ভেক্টর জগত। তাহলে $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V$ ভেক্টরসমূহকে যোগাশ্রয়ীভাবে নির্ভরশীল (সংক্ষেপে নির্ভরশীল) বলা হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ এর জন্য $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots\ldots + \alpha_n\mathbf{v}_n = 0$ হয়, যেখানে স্কেলারসমূহের সেট $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ এর অন্তত একটি উপাদান অশূন্য হবে।]

Examples of linearly dependent sets :

(a) The set $S = \{(1, 2), (2, 4)\}$ in $\mathbb{R}^2$ is linearly dependent because

$$-2(1, 2) + (2, 4) = (0, 0)$$

(b) The set $S = \{(1, 0), (0, 1), (-2, 5)\}$ in $\mathbb{R}^2$ is linearly dependent because

$$2(1, 0) - 5(0, 1) + (-2, 5) = (0, 0).$$

(c) The set $S = \{(0, 0), (1, 2)\}$ in $\mathbb{R}^2$ is linearly dependent because

$$1(0, 0) + 0(1, 2) = (0, 0)$$

Working rule toTesting for Linear Independence and Linear Dependence :

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, perform the following steps.

1. From the vector equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$, write a homogeneous system of linear equations in the variables $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $\alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then is linearly dependent.

Example-3 Show that the vectors $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$ and $\mathbf{v}_3 = (3, 2, 1)$ are linearly dependent. [দেখাও যে $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$ এবং $\mathbf{v}_3 = (3, 2, 1)$ ভেক্টরসমূহ যোগাশ্রয়ীভাবে নির্ভরশীল।]

Solution The given vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are said to be linearly dependent if for any scalars $\alpha_1, \alpha_2, \alpha_3 \in F$ such that [প্রদত্ত ভেক্টরসমূহ $\mathbf{v}_1$, $\mathbf{v}_2$ এবং $\mathbf{v}_3$ যোগাশ্রয়ীভাবে নির্ভরশীল হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \alpha_3 \in F$ এর জন্য]

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0} \text{ (zero vector)}$$

$$\Rightarrow \alpha_1(1, -2, 3) + \alpha_2(5, 6, -1) + \alpha_3(3, 2, 1) = (0, 0, 0)$$

$$\Rightarrow (\alpha_1 + 5\alpha_2 + 3\alpha_3, -2\alpha_1 + 6\alpha_2 + 2\alpha_3, 3\alpha_1 - \alpha_2 + \alpha_3) = (0, 0, 0)$$

Equating the corresponding components, we get the following system of linear equations [অনুরূপ উপাদানসমূহ সমীকৃত করে নিম্নের সরল সমীকরণ জোট পাই]

$$\begin{cases} \alpha_1 + 5\alpha_2 + 3\alpha_3 = 0 \\ -2\alpha_1 + 6\alpha_2 + 2\alpha_3 = 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 = 0 \end{cases}$$

Reducing the system to echelon form by using elementary row operations. [প্রাথমিক সারি অপারেশন ব্যবহার করে সমীকরণ জোটকে ইচালন আকারে প্রকাশ করে পাই]

$$\begin{cases} \alpha_1 + 5\alpha_2 + 3\alpha_3 = 0 \\ 16\alpha_2 + 8\alpha_3 = 0 \\ -16\alpha_2 - 8\alpha_3 = 0 \end{cases} \quad \begin{bmatrix} L_2' = L_2 + 2L_1 \\ L_3' = L_3 - 3L_1 \end{bmatrix}$$

$$\sim \begin{cases} \alpha_1 + 5\alpha_2 + 3\alpha_3 = 0 \\ 16\alpha_2 + 8\alpha_3 = 0 \end{cases}$$

The above system is in echelon form and it has two equations in three unknowns and so it has $(3 - 2) = 1$ free variable. That is there exist at least one variable which is not zero. So the vectors are linearly dependent. [উপরের বর্ণিত জোটটি ইচালন আকারে প্রকাশিত এবং এর 2 টি সমীকরণ এবং 3টি চলক বিদ্যমান। অতএব $(3 - 2) = 1$ টি মুক্ত চলক বিদ্যমান। অর্থাৎ অন্তত একটি চলক আছে যা অশূন্য। অতএব ভেক্টরসমূহ যোগাশ্রয়ীভাবে নির্ভরশীল।] [Showed]

Example-4 Show that the vectors $\mathbf{v}_1 = (3, 0, 1, -1)$, $\mathbf{v}_2 = (2, -1, 0, 1)$, $\mathbf{v}_3 = (1, 1, 1, -2)$ are linearly dependent. [দেখাও যে $\mathbf{v}_1 = (3, 0, 1, -1)$, $\mathbf{v}_2 = (2, -1, 0, 1)$, $\mathbf{v}_3 = (1, 1, 1, -2)$ ভেক্টরসমূহ যোগাশ্রয়ীভাবে নির্ভরশীল।]

Solution The given vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are said to be linearly dependent if for any scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_3 \in F$ such that [প্রদত্ত ভেক্টরসমূহ $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ যোগাশ্রয়ীভাবে নির্ভরশীল হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \ldots\ldots, \alpha_3 \in F$ এর জন্য]

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0} \text{ (zero vector)}$$

$$\Rightarrow \alpha_1 (3, 0, 1, -1) + \alpha_2(2, -1, 0, 1) + \alpha_3 (1, 1, 1, -2) = (0, 0, 0, 0)$$

$$\Rightarrow (3\alpha_1 + 2\alpha_2 + \alpha_3, -\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, -\alpha_1 + \alpha_2 - 2\alpha_3) = (0, 0, 0, 0)$$

Equating corresponding components, we get the following system of linear equations [অনুরূপ পদসমূহ সমীকৃত করে নিম্নোক্ত সরল সমীকরণ জোট পাই]

$$\begin{cases} 3\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_2 + \alpha_3 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 - 2\alpha_3 = 0 \end{cases}$$

Reducing the system to echelon form by using elementary row operations [প্রাথমিক সারি অপারেশন ব্যবহার করে সমীকরণ জোটকে ইচালন আকারে প্রকাশ করে পাই]

$$\sim \begin{cases} 3\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_2 + \alpha_3 = 0 \\ -2\alpha_2 + 2\alpha_3 = 0 \\ \alpha_2 - \alpha_3 = 0 \end{cases} \quad \begin{bmatrix} L_3' = 3L_3 - L_1 \\ L_4' = L_4 + L_3 \end{bmatrix}$$

$$\sim \begin{cases} 3\alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_2 + \alpha_3 = 0 \end{cases} \quad \begin{bmatrix} L_3' = L_3 - 2L_1 \\ L_4' = L_4 + L_2 \end{bmatrix}$$

The above system is in echelon form and it has two equations in three unknowns and so it has $(3 - 2) = 1$ free variable. That is there exist at least one variable which is not zero. So the vectors are linearly dependent. [উপরের বর্ণিত জোটটি ইচালন আকারে প্রকাশিত এবং এর 2 টি সমীকরণ এবং 3টি চলক বিদ্যমান। অতএব $(3 - 2) = 1$ টি মুক্ত চলক বিদ্যমান। অর্থাৎ অন্তত একটি চলক আছে যা অশূন্য। অতএব ভেক্টরসমূহ যোগাশ্রয়ীভাবে নির্ভরশীল।] [Showed]

7.4 Linear Independence
যোগাশ্রয়ী অনির্ভরশীলতা

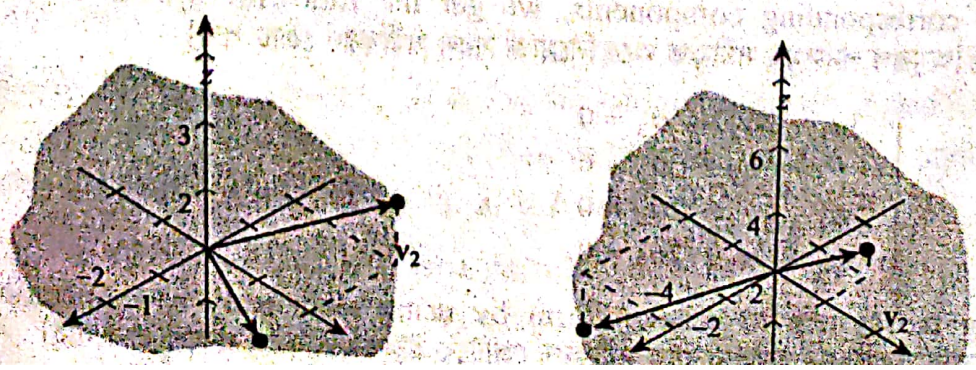
[JNU 2015]

Let V be a vector space over the field F. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in V$ is said to be linearly independent (or simply independent) if for any scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots\ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$ (zero vector), where all the elements of the set $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ are zero. [মনে করি, F ফিল্ডের উপর বর্ণিত V একটি ভেক্টর জগত। তাহলে $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n \in F$ ভেক্টরসমূহ যোগাশ্রয়ীভাবে অনির্ভরশীল হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ এর জন্য $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots\ldots\ldots + \alpha_n\mathbf{v}_n = 0$ (শূন্য ভেক্টর) হয়, যেখানে স্কেলার সেট $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ এর সবকয়টি উপাদান শূন্য।]

Example:

(a) The set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ of vectors $\mathbf{v}_1 = (1, 2, 0)$ and $\mathbf{v}_2 = (-2, 2, 1)$ is linearly independent because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not scalar multiples of each other, as shown in the following Figure 7.2 (a).

(b) The set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ of vectors $\mathbf{v}_1 = (4, -4, -2)$ and $\mathbf{v}_2 = (-2, 2, 1)$ is linearly dependent because $\mathbf{v}_1 = -2\mathbf{v}_2$, as shown in the following Figure 7.2 (b).



Example 5 Show that the vectors $\mathbf{v}_1 = (5, 0, 3)$, $\mathbf{v}_2 = (2, -7, 8)$ and $\mathbf{v}_3 = (-11, 12, 1)$ are linearly independent. [দেখাও যে $\mathbf{v}_1 = (5, 0, 3)$, $\mathbf{v}_2 = (2, -7, 8)$ এবং $\mathbf{v}_3 = (-11, 12, 1)$ ভেক্টরসমূহ যোগাশ্রয়ীভাবে অনির্ভরশীল ।]

Solution The given vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are said to be linearly independent if for any scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_3 \in F$ such that [প্রদত্ত ভেক্টরসমূহ $\mathbf{v}_1$, $\mathbf{v}_2$ এবং $\mathbf{v}_3$ যোগাশ্রয়ীভাবে অনির্ভরশীল হবে যদি যে কোন স্কেলার $\alpha_1, \alpha_2, \ldots\ldots, \alpha_3 \in F$ এর জন্য যেন]

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

$$\Rightarrow \alpha_1(5, 0, 3) + \alpha_2(2, -7, 8) + \alpha_3(-11, 12, 1) = (0, 0, 0)$$

$$\Rightarrow (5\alpha_1 + 2\alpha_2 - 11\alpha_3, -7\alpha_2 + 12\alpha_3, 3\alpha_1 + 8\alpha_2 + \alpha_3) = (0, 0, 0)$$

Equating corresponding components, we get the following system of linear equations [অনুরূপ পদসমূহ সমীকৃত করে নিম্নোক্ত সরল সমীকরণ জোট পাই]

$$\begin{cases} 5\alpha_1 + 2\alpha_2 - 11\alpha_3 = 0 \\ \quad\quad - 7\alpha_2 + 12\alpha_3 = 0 \\ 3\alpha_1 + 8\alpha_2 + \quad \alpha_3 = 0 \end{cases}$$

Reducing the system to echelon form by using elementary row operations [প্রাথমিক সারি অপারেশন ব্যবহার করে সমীকরণ জোটকে ইচালন আকারে প্রকাশ করে পাই]

$$\sim \begin{cases} 5\alpha_1 + 2\alpha_2 - 11\alpha_3 = 0 \\ \quad\quad - 7\alpha_2 + 12\alpha_3 = 0 \quad [L_3' = 5L_3 - 3L_1] \\ \quad\quad 36\alpha_2 + 38\alpha_3 = 0 \end{cases}$$

$$\sim \begin{cases} 5\alpha_1 + 2\alpha_2 - 11\alpha_3 = 0 \\ \quad\quad - 7\alpha_2 + 12\alpha_3 = 0 \quad [L_3' = 7L_3 + 36L_1] \\ \quad\quad 698\alpha_3 = 0 \end{cases}$$

Using back-substitution method we get the solution of the above system [পশ্চাৎ প্রতিস্থাপন পদ্ধতি ব্যবহার করে উপরের সমীকরণ জোটের সমাধান পাই]

$$\alpha_3 = 0, \alpha_2 = 0, \alpha_1 = 0$$

That is all the scalars are zero. [অর্থাৎ সকল স্কেলারের মান শূন্য ।]

So the given vectors are linearly independent. [অতএব প্রদত্ত ভেক্টরসমূহ যোগাশ্রয়ীভাবে অনির্ভরশীল ।] [Showed]

[গুরুত্বপূর্ণ উপপাদ্যসমূহ] :

$$\sim \begin{cases} \alpha_1 + 2\alpha_2 + 2\alpha_3 = 3 \\ 7\alpha_2 + 3\alpha_3 = 15 \\ 17\alpha_3 = -13 \\ -17\alpha_3 = 13 \end{cases} \quad [L_3 = 7L_3 + L_2, L_4' = L_4 - 6L_3]$$

$$\sim \begin{cases} \alpha_1 + 2\alpha_2 + 2\alpha_3 = 3 \\ 7\alpha_2 + 3\alpha_3 = 15 \\ 17\alpha_3 = -13 \end{cases}$$

Now by back substitution method from 3rd equation of the above system, we get

$\alpha_3 = \frac{-13}{17}$

From 2nd equation, we get $7\alpha_2 = 15 + \frac{39}{17}$

$$\Rightarrow \quad 7\alpha_2 = \frac{294}{17}$$

$$\therefore \quad \alpha_2 = \frac{42}{17}$$

and from 1st equation, we get $\alpha_1 + 2\left(\frac{42}{17}\right) + 2\left(-\frac{13}{17}\right) = 3$

$$\therefore \quad \alpha_1 = \frac{-7}{17}$$

Therefore, $\mathbf{v} = \frac{-7}{17} v_1 + \frac{42}{17} v_2 - \frac{13}{17} v_3$

Thus $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. [Showed]

Example-9 Is the vector $\mathbf{v} = (2, 5, 3)$ is a linear combination of the vectors $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$ and $v_3 = (1, -5, 7)$. [ভেক্টর $\mathbf{v} = (2, 5, 3)$ কি $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$ এবং $v_3 = (1, -5, 7)$ ভেক্টরের যোগাশ্রয়ী সমাবেশ?] *[DUH '92]*

Solution In order for $\mathbf{v}$ is a linear combination of v_1, v_2 and v_3 there must be scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (2, 5, 3) = \alpha_1(1, -3, 2) + \alpha_2(2, -4, -1) + \alpha_3(1, -5, 7)$$

$$\Rightarrow (2, 5, 3) = (\alpha_1 + 2\alpha_2 + \alpha_3, -3\alpha_1 - 4\alpha_2 - 5\alpha_3, 2\alpha_1 - \alpha_2 + 7\alpha_3)$$

Equating corresponding components from both sides and forming system of equations, we get

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2 \\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3 = 5 \\ 2\alpha_1 - \alpha_2 + 7\alpha_3 = 3 \end{cases}$$

Reducing the above system to echelon form by the elementary operations, we get

$$\sim \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2 \\ 2\alpha_2 - 2\alpha_3 = 11 \\ -5\alpha_2 + 5\alpha_3 = -1 \end{cases} \quad \begin{bmatrix} L_2' = L_2 + 3L_1 \\ L_3' = L_3 - 2L_1 \end{bmatrix}$$

$$\sim \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2 \\ 2\alpha_2 - 2\alpha_3 = 11 \\ 0 = 53 \end{cases} \quad [L_3' = 2L_3 + 5L_2]$$

From 3rd equation of the above system, we have $0 = 53$, which is not true. Therefore, the above system is inconsistant. Thus **v** is not a linear combination of v_1, v_2 and v_3. (Ans.)

Example-10 Is the vector $\mathbf{v} = (5, 6, 0)$ linear combination of the vectors $v_1 = (-1, 2, 0)$, $v_2 = (3, 1, 2)$ and $v_3 = (4, -1, 0)$. [ভেক্টর $\mathbf{v} = (5, 6, 0)$ কি $v_1 = (-1, 2, 0)$, $v_2 = (3, 1, 2)$ এবং $v_3 = (4, -1, 0)$ ভেক্টরের যোগাশ্রয়ী সমাবেশ?] *[DUH '83]; JNUH 2016]*

Solution In order for **v** is a linear combination of v_1, v_2 and v_3 there must be scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (5, 6, 0) = \alpha_1(-1, 2, 0) + \alpha_2(3, 1, 2) + \alpha_3(4, -1, 0)$$

$$\Rightarrow (5, 6, 0) = (-\alpha_1 + 3\alpha_2 + 4\alpha_3, 2\alpha_1 + \alpha_2 - \alpha_3, 0.\alpha_1 + 2\alpha_2 + 0.\alpha_3)$$

Equating corresponding components from both sides and forming system of equations

$$\begin{cases} -\alpha_1 + 3\alpha_2 + 4\alpha_3 = 5 \\ 2\alpha_1 + \alpha_2 - \alpha_3 = 6 \\ \alpha_2 = 0 \end{cases}$$

Reducing the above system to echelon form by the elementary operations, we get

$$\sim \begin{cases} -\alpha_1 + 3\alpha_2 + 4\alpha_3 = 5 \\ 7\alpha_2 + 7\alpha_3 = 16 \\ \alpha_2 = 0 \end{cases} \quad [L_2' = L_2 + 2L_1]$$

Now by back substitution method from 3rd & 2nd equations of the above system, we get $\alpha_2 = 0$ and $\alpha_3 = \frac{16}{7}$

Also from 1st equation, we get $-\alpha_1 = 5 - 4\left(\frac{16}{7}\right) = \frac{35-64}{7} = \frac{-29}{7}$

$$\therefore \alpha_1 = \frac{29}{7}$$

Therefore, $\mathbf{v} = \frac{29}{7}v_1 + 0.v_2 + \frac{16}{7}v_3$

Thus $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. (Ans.)

Example-11 Consider the vectors $v_1 = (1, 0, 1)$, $v_2 = (0, -1, 1)$ and $v_3 = (-$ 1, 1) in $\mathbb{R}^3$. Show that $\mathbf{v} = (2, 3, 4)$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. [(1, 0, 1), $v_2 = (0, -1, 1)$ এবং $v_3 = (-1, -1, 1)$ ভেক্টরগুলো বিবেচনা কর। দেখাও যে (2, 3, 4) হলো v_1, v_2 এবং v_3 এর যোগাশ্রয়ী সমাবেশ।] [DU

Solution In order to show that $\mathbf{v}$ is a linear combination of v_1, v_2 and v_3 th must be scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (2, 3, 4) = \alpha_1(1, 0, 1) + \alpha_2(0, -1, 1) + \alpha_3(-1, -1, 1)$$

$$\Rightarrow (2, 3, 4) = (\alpha_1 + 0.\alpha_2 - \alpha_3, 0.\alpha_1 - \alpha_2 - \alpha_3, \alpha_1 + \alpha_2 + \alpha_3)$$

Equating corresponding components from both sides and forming system equations

$$\begin{cases} \alpha_1 \qquad - \alpha_3 = 2 \\ \quad - \alpha_2 - \alpha_3 = 3 \\ \alpha_1 + \alpha_2 + \alpha_3 = 4 \end{cases}$$

$$\sim \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 4 \\ \alpha_1 \qquad - \alpha_3 = 2 \\ \quad - \alpha_2 - \alpha_3 = 3 \end{cases} \quad \text{[Rearranging of the above system]}$$

Reducing the above system to echelon form by the elementary operations, we get

$$\sim \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 4 \\ \quad - \alpha_2 - 2\alpha_3 = -2 \\ \quad - \alpha_2 - \alpha_3 = 3 \end{cases} \quad [L_2' = L_2 - L_1]$$

$$\sim \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 4 \\ \quad - \alpha_2 - 2\alpha_3 = -2 \\ \qquad \alpha_3 = 5 \end{cases} \quad [L_3' = L_3 - L_2]$$

From 3rd equation of the above system, we get $\alpha_3 = 5$. Substituting $\alpha_3 =$ 2nd equation, we get $-\alpha_2 = -2 + 10$

$$\Rightarrow \alpha_2 = -8$$

Also from 1st equation, we get $\alpha_1 = 4 + 8 - 5 = 7$

Therefore, $\mathbf{v} = 7v_1 - 8v_2 + 5v_3$.

Thus $\mathbf{v}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. [Showed]

Example-12 Express the vector $\mathbf{v}$ as a linear combination of the vectors v_1, v_2 and v_3, where [ভেক্টর $\mathbf{v}$ কে v_1, v_2 এবং v_3 ভেক্টরের যোগাশ্রয়ী সমাবেশরূপে প্রকাশ কর, যেখানে]

(i) $\mathbf{v} = (5, 9, 5)$, $v_1 = (2, 1, 4)$, $v_2 = (1, -1, 3)$, $v_3 = (3, 2, 5)$

(ii) $\mathbf{v} = (3, 7, 8)$, $v_1 = (1, 1, 1)$, $v_2 = (0, 1, 1)$, $v_3 = (1, 2, 2)$

(iii) $\mathbf{v} = (19, -4, -2)$, $v_1 = (1, 2, 0)$, $v_2 = (2, -1, 1)$, $v_3 = (-3, 2, 1)$

(iv) $\mathbf{v} = (1, 2, 6)$, $v_1 = (2, 1, 0)$, $v_2 = (1, -1, 2)$, $v_3 = (0, 3, -4)$

(v) $\mathbf{v} = (2, -5, 3)$, $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$, $v_3 = (1, -5, 7)$

Solution

(i) In order to express $\mathbf{v}$ as a linear combination of v_1, v_2 and v_3 there must be scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (5, 9, 5) = \alpha_1(2, 1, 4) + \alpha_2(1, -1, 3) + \alpha_3(3, 2, 5)$$

$$\Rightarrow (5, 9, 5) = (2\alpha_1 + \alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 + 2\alpha_3, 4\alpha_1 + 3\alpha_2 + \alpha_3)$$

Equating corresponding components from both sides and forming system of equations

$$\begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5 \end{cases}$$

Reducing the above system to echelon form by the elementary operations, we get

$$\sim \begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ -3\alpha_2 + \alpha_3 = 13 \\ 7\alpha_2 - 3\alpha_3 = -31 \end{cases} \quad \begin{bmatrix} L_2' = 2L_2 - L_1 \\ L_3' = L_3 - 4L_2 \end{bmatrix}$$

$$\sim \begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ -3\alpha_2 + \alpha_3 = 13 \\ -2\alpha_3 = -2 \end{cases} \quad [L_3' = 3L_3 + 7L_2]$$

Now by back substitution method from 3rd equation of the above system,

we get $\alpha_3 = 1$,

From 2nd equation we get, $-3\alpha_2 + 1 = 13 \therefore \alpha_2 = -4$

Also from 1st equation, we get $2\alpha_1 - 4 + 3 = 5$

$$\Rightarrow 2\alpha_1 = 6$$

$$\therefore \alpha_1 = 3$$

Example-22 Show that the vectors (1, – 1, 2), (3, – 5, 1), (2, 7, 8) and (– 1, 1, 1) in $\mathbb{R}^3$ are linearly dependent. [দেখাও যে (1, – 1, 2), (3, – 5, 1), (2, 7, 8) এবং (– 1, 1, 1) ভেক্টরসমূহ $\mathbb{R}^3$ এ যোগাশ্রয়ীভাবে নির্ভরশীল ।] *[NUH 2004]*

Solution For any unknown scalars $x, y, z, t \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$x(1, -1, 2) + y(3, -5, 1) + z(2, 7, 8) + t(-1, 1, 1) = (0, 0, 0)$$

$$\Rightarrow (x + 3y + 2z - t, -x - 5y + 7z + t, 2x + y + 8z + t) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations, we get

$$\begin{cases} x + 3y + 2z - t = 0 \\ -x - 5y + 7z + t = 0 \\ 2x + y + 8z + t = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} x + 3y + 2z - t = 0 \\ -2y + 9z = 0 \\ -9y + 22z + 3t = 0 \end{cases} \quad \begin{bmatrix} L_2' = L_2 + L_1 \\ L_3' = L_3 + 2L_2 \end{bmatrix}$$

$$\sim \begin{cases} x + 3y + 2z - t = 0 \\ -2y + 9z = 0 \\ -37z + 6t = 0 \end{cases} \quad [L_3' = 2L_3 - 9L_2]$$

This system is in echelon form and has 3 equations in 4 unknowns. So it has (4 – 3) = 1 free variables and hence the system has at least one non-zero scalar (solution). Thus the given vectors are linearly dependent. [Showed]

Example-23 Show that the vectors (1, 1, – 1), (1, 0, 2) and (1, 1, 1) in $\mathbb{R}^3$ are linearly independent. [দেখাও যে (1, 1, – 1), (1, 0, 2) এবং (1, 1, 1) ভেক্টরসমূহ $\mathbb{R}^3$ এ যোগাশ্রয়ীভাবে অনির্ভরশীল ।] *[NUH '94, '98, '11]*

Solution For any unknown scalars $x, y, z \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$x(1, 1, -1) + y(1, 0, 2) + z(1, 1, 1) = (0, 0, 0)$$

$$\Rightarrow (x + y + z,\ x + z,\ -x + 2y + z) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations, we get

$$\begin{cases} x + y + z = 0 \\ x + z = 0 \\ -x + 2y + z = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} x + y + z = 0 \\ -y = 0 \\ 3y + 2z = 0 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_3 + L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + y + z = 0 \\ -y = 0 \\ 2z = 0 \end{cases} \quad [L_3' = L_3 + 2L_2]$$

This system is in echelon form and has 3 equations in 3 unknowns. So the system has a unique solution.

The solution of the above system is $z = 0, y = 0, x = 0$.

That is, all the scalars are zero.

Hence, the given vectors are linearly independent. [Showed]

Example-24 Show that the vectors $(1, 1, -1)$, $(1, 2, 3)$ and $(4, 5, -3)$ in $\mathbb{R}^3$ are linearly independent. [দেখাও যে $(1, 1, -1)$, $(1, 2, 3)$ এবং $(4, 5, -3)$ ভেক্টরসমূহ $\mathbb{R}^3$ যোগাশ্রয়ীভাবে অনির্ভরশীল ।] *[DUS '81; DUP '81; JNUH 2016*

Solution For any unknown scalars $x, y, z \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$x(1, 1, -1) + y(1, 2, 3) + z(4, 5, -3) = (0, 0, 0)$$

$$\Rightarrow (x + y + 4z, x + 2y + 5z, -x + 3y - 3z) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations we get

$$\begin{cases} x + y + 4z = 0 \\ x + 2y + 5z = 0 \\ -x + 3y - 3z = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} x + y + 4z = 0 \\ y + z = 0 \\ 5y + 2z = 0 \end{cases} \quad [L_2' = L_2' - L_1, L_3' = L_3 + L_2]$$

$$\sim \begin{cases} x + 2y - z = 0 \\ y + z = 0 \\ -3z = 0 \end{cases} \quad [L_3' = L_3 - 5L_2]$$

This system is in echelon form and has 3 equations in 3 unknowns. So the system has a unique solution.

The solution of the above system is $z = 0, y = 0, x = 0$.

That is, all the scalars are zero.

Hence, the given vectors are linearly independent. [Showed]

Example-25 Test whether the following sets are linearly dependent or independent : [নিম্নোক্ত সেটগুলোর যোগাশ্রয়ীভাবে নির্ভরশীলতা অথবা অনির্ভরশীলতা যাচাই কর :]

(i) $\{(1, 1, 2), (1, 0, 1), (2, 1, 3)\}$ *[NUH 2015, 2004; NU (Prel) 2007, 2009]*

(ii) $\{(2, -3, 1), (3, -5, -2), (4, -5, 1)\}$ *[NUH (NM) 2006]*

(iii) $\{(1, 0, 1), (-3, 2, 6), (4, 5, -2)\}$ *[NUH (NM) 2008]*

(iv) $\{(1, 1, -1), (2, 1, 0), (-1, 1, 2)\}$ *[NUH (NM) 2006, NU(Prel) 2006]*

(v) $\{(2, 1, 1), (3, -4, 6), (4, -9, 11)\}$ *[DUH 1983]*

(vi) $\{(1, 3, 2), (1, -7, -8), (2, 1, -1)\}$ *[DU (Prel) 1983]*

(vii) $\{(1, 2, -3), (2, 0, -1), (7, 6, -11)\}$ *[NUH 2002]*

(viii) $\{(1, -2, 3), (5, 6, -1)\ (3, 2, 1)\}$

Solution

(i) For any unknown scalars $x, y, z \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$x(1, 1, 2) + y(1, 0, 1) + z(2, 1, 3) = (0, 0, 0)$$

$$\Rightarrow (x + y + 2z, x + z, 2x + y + 3z) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations, we get

$$\begin{cases} x + y + 2z = 0 \\ x \quad\quad + z = 0 \\ 2x + y + 3z = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} x + y + 2z = 0 \\ \quad - y - z = 0 \\ \quad\quad y + z = 0 \end{cases} \quad [L_2' = L_2 - L_1, L_3' = L_3 - 2L_2]$$

$$\sim \begin{cases} x + y + 2z = 0 \\ \quad\quad y + z = 0 \end{cases}$$

This system is in echelon form and having 2 equations in 3 unknowns. So it has $(3 - 2) = 1$ free variable and hence the system has at least one non-zero scalar (solution). Thus the given vectors are linearly dependent. (Ans.)

(ii) For any unknown scalars $x, y, z \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$\Rightarrow x(2, -3, 1) + y(3, -5, -2) + z(4, -5, 1) = (0, 0, 0)$$

$$\Rightarrow (2x + 3y + 4z, -3x - 5y - 5z, x - 2y + z) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations, we get

$$\begin{cases} 2x + 3y + 4z = 0 \\ -3x - 5y - 5z = 0 \\ x - 2y + z = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} 2x + 3y + 4z = 0 \\ -y + 2z = 0 \\ -11y + 2z = 0 \end{cases} \quad \begin{bmatrix} L_2' = 2L_2 + 3L_1 \\ L_3' = 3L_3 + L_2 \end{bmatrix}$$

$$\sim \begin{cases} 2x + 3y + 4z = 0 \\ -y + 2z = 0 \\ -20z = 0 \end{cases} \quad [L_3' = L_3 - 11L_2]$$

This system is in echelon form and has 3 equations in 3 unknowns. So the system has a unique solution.

The solution of the above system is $z = 0, y = 0, x = 0$.

That is, all the scalars are zero.

Hence, the given vectors are linearly independent. (Ans.)

(iii) For any unknown scalars $x, y, z \in \mathbb{R}$ set a linear combination of the given vectors equal to zero, we get

$$x(1, 0, 1) + y(-3, 2, 6) + z(4, 5, -2) = (0, 0, 0)$$

$$\Rightarrow (x - 3y + 4z, 2y + 5z, x + 6y - 2z) = (0, 0, 0)$$

Equating corresponding components and forming the system of linear equations, we get

$$\begin{cases} x - 3y + 4z = 0 \\ 2y + 5z = 0 \\ x + 6y - 2z = 0 \end{cases}$$

Reducing the above system to echelon form by using elementary operations

$$\sim \begin{cases} x - 3y + 4z = 0 \\ 2y + 5z = 0 \\ 9y - 6z = 0 \end{cases} \quad [L_3' = L_3 - L_1]$$

$$\sim \begin{cases} x - 3y + 4z = 0 \\ 2y + 5z = 0 \\ -57z = 0 \end{cases} \quad [L_3' = 2L_3 - 9L_2]$$

This system is in echelon form and has 3 equations in 3 unknowns. So the system has a unique solution.

The solution of the above system is $z = 0, y = 0, x = 0$.

That is, all the scalars are zero.

Hence, the given vectors are linearly independent. (Ans.)

generator or span of $V(F)$, and it is denoted by $V(F) = \langle S \rangle$. $V(F)$ is called generated or spaned by S. [মনে করি, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n\}$, $V(F)$ ভেক্টর জগতের একটি সসীম উপসেট। যদি কোন ভেক্টর $\mathbf{u} \in V(F)$ এর জন্য $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots\ldots + \alpha_n\mathbf{v}_n$ $\forall\, \mathbf{u} \in V$ এবং $\forall\, \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ হয়, তবে S কে $V(F)$ এর সৃজক বা জাতক বলে এবং একে $V(F) = \langle S \rangle$ দ্বারা প্রকাশ করা হয়। $V(F)$ কে S দ্বারা সৃজিত বলা হয়।]

or, Any vector $\mathbf{v} \in V(F)$ is said to be generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, if can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n$. [অথবা, কোন ভেক্টর $\in V(F)$ কে $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ এর জাতক বলা হবে যদি $\mathbf{v}$ কে $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ এর যোগশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়।]

Spanning Sets [স্প্যানিং সেট] : If every vector in a vector space can be written as a linear combination of vectors in a set then it is called a **spanning** set of the vector space. [যদি ভেক্টর জগতের প্রত্যেক ভেক্টরকে একটি সেটের ভেক্টরসমূহের যোগশ্রয়ী সমাবেশরূপে প্রকাশ করা যায় তবে সেটটিকে ভেক্টর জগতের স্প্যানিং সেট বলে।]

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called **spanning set** of V if every vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V. [মনে করি, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ ভেক্টর জগত V এর উপজগত। সেট S কে V এর স্প্যানিং সেট বলা হবে যদি V এর প্রত্যেক ভেক্টরকে S এর ভেক্টরসমূহের যোগশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়। এ ধরনের ক্ষেত্রে একে কহে হয় S, V এর জাতক।]

Example :

(a) The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3)$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

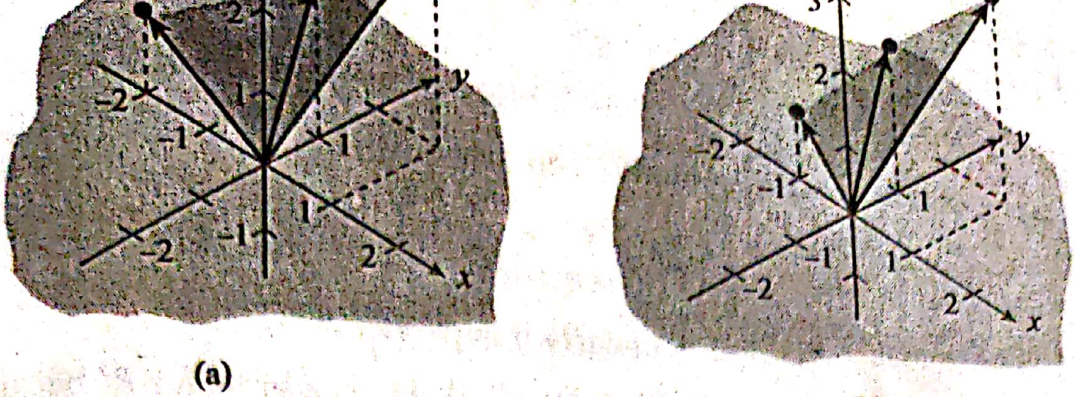
$$p(x) = a(1) + b(x) + c(x^2)$$
$$= a + bx + cx^2$$

The spanning sets in the above example are called the **standard spanning sets** of R^3 and P_2 respectively.

Example :

(a) The set $S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.

(b) The set $S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$ does not span R^3 because $\mathbf{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S_2.



(a) (b)

Fig 8.1

Example : Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ and let $\mathbf{v} = (a, b, c)$ then $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$. Here $\mathbf{v}$ is spanned or generated by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the generators of $\mathbf{v}$. [এখানে $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ সেটকে $\mathbf{v}$ এর জাতক বলে এবং $\mathbf{v}$ কে $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ দ্বারা সৃষ্ট বা সৃজিত বলে ।]

Example : Let $\mathbf{v} = (2, 7, 8)$ and $\mathbf{u}_1 = (1, 2, 3)$, $\mathbf{u}_2 = (1, 3, 5)$, $\mathbf{u}_3 = (1, 5, 9)$.

$\mathbf{v}$ can not be expressd as a linear combination of u_1, u_2, u_3. So $\{u_1, u_2, u_3\}$ do not span or generated $\mathbf{v}$. [মনে করি, $\mathbf{v} = (2, 7, 8)$ এবং $\mathbf{u}_1 = (1, 2, 3)$, $\mathbf{u}_2 = (1, 3, 5)$, $\mathbf{u}_3 = (1, 5, 9)$ $\mathbf{v}$ কে u_1, u_2, u_3 এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায় না । অতএব $\{u_1, u_2, u_3\}$ সেটটি $\mathbf{v}$ এর জাতক নয়, কিংবা $\mathbf{v}$ ভেক্টরটি $\{u_1, u_2, u_3\}$ দ্বারা সৃজিত বা সৃষ্ট নয় ।]

Remark :

1. Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V, then for any vector $\mathbf{v}$, the set $\{v, u_1, u_2, \ldots\ldots, u_n\}$ also spans V.
2. Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots\ldots, \mathbf{u}_n$ span V and suppose $\mathbf{u}_k$ is a linear combination of some of the other $\mathbf{u}$'s. Then $\mathbf{u}$'s without $\mathbf{u}_k$ are also spans V.
3. Suppose $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V and suppose one of these $\mathbf{u}$'s is zero vector then $\mathbf{u}$'s without zero are also span V.

Definition of the span of a set : If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span** of S is the set of all linear combinations of the vectors in S.

span $(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k : c_1, c_2, \ldots\ldots, c_k$ are real numbers$\}$.

The span of S is denoted by span (S) or span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_k\}$. If span $(S) = V$, it is said that V is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_k\}$ or the S **spans** V.

8.3 Basis
ভিত্তি

[DUH 2015; JNU 2016, 2015, 2014, 2013, 2012, 2011; NUH '12, '94; NUH(NM) '09, '12; RUH '03, '06, '08]

Let V be a vector space over the field F and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n\}$ be the finite set of vectors in V, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\ldots, \mathbf{v}_n\}$ is called the basis of V if and only if

(i) $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_n\}$ is linearly independent

(ii) $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_n\}$ spans V.

[যদি $V(F)$ ভেক্টর জগতের কোন সসীম সেট $\{v_1, v_2, \ldots\ldots, v_n\}$ হয়, তবে $\{v_1, v_2, \ldots$ কে $V(F)$ এর ভিত্তি বলা হবে যদি ও কেবল যদি

(i) $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_n\}$ যোগাশ্রয়ীভাবে অনির্ভরশীল হয়।

(ii) $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_n\}$, $V(F)$ এর জাতক হয়।]

Definition: A set of vectors $S = \{v_1, v_2, \ldots\ldots, v_n\}$ in a vector space V is call **basis** for V if the following conditions are true.

1. S spans V. 2. S is linearly independent.

Example : The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for R^3 because linearly independent and S spans $\mathbb{R}^3$.

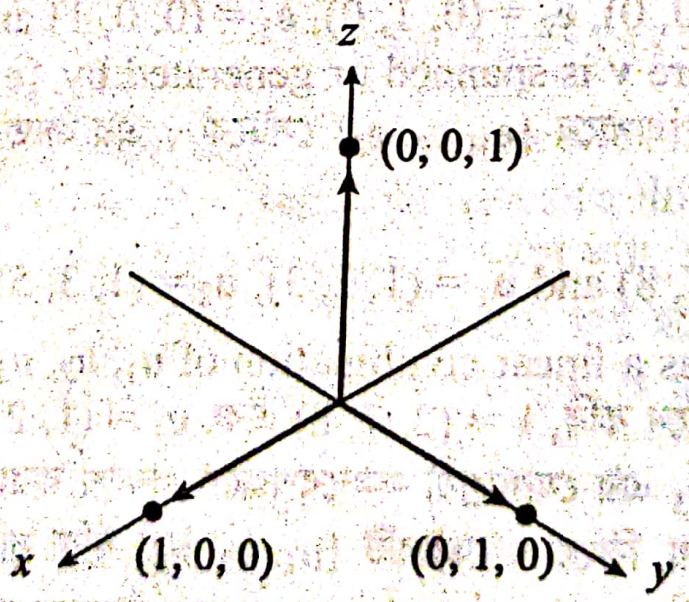


Fig 8.2

The basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is called the **standard basis** for This result can be generalized to n-space. That is, the vectors

$e_1 = (1, 0, \ldots\ldots, 0)$

$e_2 = (0, 1, \ldots\ldots, 0)$

Example : A Nonstandard Basis for $\mathbb{R}^3$: The set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans $\mathbb{R}^3$ and is linearly independent. So, S is a basis for $\mathbb{R}^3$.

Example : A Basis for Polynomials : The vector space P_3 has the basis $S = \{1, x, x^2, x^3\}$ since S spans P_3 and S is linearly independent.

> ✪ *Note* : The basis $S = \{1, x, x^2, x^3\}$ is called the standard basis for P_3. Similarly, the standard basis for P_n is $S = \{1, x, x^2, \ldots\ldots, x^n\}$

8.4 Dimension

মাত্রা

[DUH 2015; JNUH 2016, 2015, 2014, 2013, 2012, 2011; NUH '94, '12; NU(NM) '09, '05; RUH '03, '06, '08, NU(prel) '09]

Let V be a vector space. Then V is said to be finite dimension n or n-dimensional (written as dim $V = n$) if V has a basis with n-elements. [মনে করি, V একটি ভেক্টর জগত। তাহলে V ভেক্টর জগতকে সসীম মাত্রা n বা n-মাত্রিক বলা হবে যদি V এর একটি ভিত্তিতে n-সংখ্যক উপাদান থাকে।]

V is said to be finite dimensional if any finite set of vectors $\{v_1, v_2, \ldots\ldots\ldots, v_n\}$ form a basis of V [V ভেক্টর জগতকে সসীম মাত্রিক বলা হবে যদি কোন সসীম সেট $\{v_1, v_2, \ldots\ldots, v_n\}$, V এর ভিত্তি গঠন করে।]

If V does not have any finite basis then V is said to be infinite dimension. [V এর কোন সসীম ভিত্তি না থাকলে V-কে অসীম মাত্রিক বলা হবে।]

The dimension of any vector space is the number of vector in any basis of it. [কোন ভেক্টর জগতের মাত্রা বলতে ঐ ভেক্টরের ভিত্তির সংখ্যাকে বুঝানো হয়।]

The dimension of a vector space is the maximum number of linearly independent vectors containes in it. [ভেক্টর জগতের মাত্রা বলতে ঐ ভেক্টরে উপস্থিত যোগাশ্রয়ী অনির্ভরশীল ভেক্টরের সর্বোচ্চ সংখ্যা বুঝায়।]

Definition: If a vector space V has a basis consisting of n vectors, then the number n is called the **dimension** of V, denoted by dim $(V) = n$. If V consists of the zero vector alone, the dimension of V is defined as zero.

> ✪ *Note* : The above definition allows you to observe the characteristics of the dimensions of the familiar vector spaces listed below. In each case, the dimension is determined by simply counting the number of vectors in the standard basis.
>
> 1. The dimension of R^n with the standard operations is n.
> 2. The dimension of P_n with the standard operations is $n + 1$.
> 3. The dimension of $M_{m,n}$ with the standard operations is mn.

Example-1 Find the basis and dimension of the subspace S generated (spanned) by the set of vectors $\{(1, 2, 0), (1, 3, 4), (2, 5, 4)\}$. [$\{(1, 2, 0), (1, 3, 4), (2, 5, 4)\}$ ভেক্টরসমূহের সেট এর জাতক S উপজগতের ভিত্তি ও মাত্রা নির্ণয় কর।]

Solution Consider a matrix whose rows are the given set of vectors i.e., [একটি ম্যাট্রিক্স বিবেচনা করি যার সারিগুলো হলো প্রদত্ত ভেক্টরসমূহ। অর্থাৎ]

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \\ 2 & 5 & 4 \end{bmatrix}$$

Reducing the matrix to echelon form by the elementary row operations. [প্রাথমিক সারি অপারেশনের মাধ্যমে ম্যাট্রিক্সটিকে ইচালন আকারে প্রকাশ করে পাই]

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_3 - 2L_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad [L_3' = L_3 - L_2]$$

This system is in echelon form having two non-zero rows. [এই জোটটি ইচালন আকারের যার দুইটি অশূন্য সারি বিদ্যমান ।]

$\therefore$ Hence the basis is [ভিত্তি] = $\{(1, 2, 0), (0, 1, 4)\}$

and dimension [মাত্রা] = 2 (Ans.)

* Now we can generate new definition of Basis and Dimension as follows: [এখন আমরা ভিত্তি এবং মাত্রার নতুন সংজ্ঞা নিম্নরূপে দিতে পারি:]

Definition : If the vectors are reduce to echelon form then the non-vanishing vectors are called the basis and the number of non-vanishing rows are the dimension of the vectors. [ইচালন আকারে প্রকাশিত ভেক্টরসমূহের অশূন্য ভেক্টরগুলোকে ভিত্তি এবং অশূন্য সারি সংখ্যাকে মাত্রা বলে ।]

[মনে করি, $S = \{u_1, u_2, \ldots\ldots, u_n\}$, $V(F)$ এর ভিত্তি এবং $V(F)$ এর একটি উপসেট। T $\{v_1, v_2, \ldots\ldots, v_r\}$ যা যোগাশ্রয়ী অনির্ভরশীল। তাহলে বিনিময় প্রতীজ্ঞা হতে আমরা জানি সেট $A = \{v_1, v_2, \ldots\ldots, v_r, u_{i1}, u_{i2}, \ldots\ldots, u_{in-r}\}$, $V(F)$ এর জাতক হবে। কিন্তু আমরা জানি, কোন সসীম সেট A যদি $V(F)$ এর জাতক হয়, তবে তা কোন উপসেট $V(F)$ এর ভিত্তি হবে। কিন্তু A এর n সংখ্যক উপাদান আছে এবং A, $V(F)$ এর ভিত্তি হবে।

যেখানে $A = T \cup \{u_{i1}, u_{i2}, \ldots\ldots\ldots\ldots, u_{in-r}\}$ অর্থাৎ $T \subset A$

সুতারং T, $V(F)$ এর ভিত্তি এবং এটা A এর ভিত্তির অংশ।]

Mathematical Problems & Solution
গাণিতিক সমস্যাবলি এবং সমাধান

Example-4 Determine whether or not the vectors $u_1 = (1, 1, 2)$, $u_2 = (1, -1, 2)$ and $u_3 = (1, 0, 1)$ span $\mathbb{R}^3$. [(1, 1, 2), (1, – 1, 2) এবং (1, 0, 1) ভেক্টরগুলো $\mathbb{R}^3$ এর জাতক কি-না নির্ধারণ কর।] *[NUH (old) '10]*

Solution The given set of vectors spans $\mathbb{R}^3$, if for any arbitrary vector $\mathbf{v} = (a, b, c)$ and for any unknown scalars $x, y, z \in \mathbb{R}$, the given vectors can be expressed as a linear combination of $\mathbf{v}$. [প্রদত্ত ভেক্টরগুলোর সেট $\mathbb{R}^3$ এর জাতক হবে যদি যে কোন ইচ্ছামূলক ভেক্টর $\mathbf{v} = (a, b, c)$ এবং যে কোন অজানা স্কেলার $x, y, z \in \mathbb{R}$ এর জন্য প্রদত্ত ভেক্টরসমূহকে $\mathbf{v}$ এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়।]

$$\text{i.e, } \mathbf{v} = (a, b, c) = xu_1 + yu_2 + zu_3$$

$$\Rightarrow \mathbf{v} = (a, b, c) = x(1, 1, 2) + y(1, -1, 2) + z(1, 0, 1)$$

$$\Rightarrow \mathbf{v} = (a, b, c) = (x + y + z, x - y, 2x + 2y + z)$$

Equating corresponding components yields the system of linear equations below. [অনুরূপ উপাদানসমূহকে সমীকৃত করে নিম্নের সরল সমীকরণ জোট পাই]

$$\begin{cases} x + y + z = a \\ x - y = b \\ 2x + 2y + z = c \end{cases}$$

Reducing the system to echelon form by the elementary row operations.

$$\sim \begin{cases} x + y + z = a \\ -2y - z = b - a \\ -z = c - 2a \end{cases} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_3 - 2L_1 \end{bmatrix}$$

By back substitution method, we get $z = 2a - c$,

$$y = \frac{1}{2}(a - b - z) = \frac{1}{2}\{a - b - 2a + c\} = \frac{1}{2}(c - a - b)$$

and $x = a - y - z$

$$= a - \frac{1}{2}(c - a - b) - 2a + c$$

$$= \frac{1}{2}(2a - c + a + b - 4a + 2c)$$

$$= \frac{1}{2}(-a + b + c)$$

Thus $\mathbf{v} = (a, b, c) = \frac{1}{2}(b + c - a)(1, 1, 2) + \frac{1}{2}(c - a - b)(1, -1, 2)$

$$+ (2a - c)(1, 0, 1)$$

Since the given vectors can be expressed as a linear combination of **v**. So the given set of vectors spans $\mathbb{R}^3$. [যেহেতু প্রদত্ত ভেক্টরগুলোকে **v** এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়। অতএব প্রদত্ত ভেক্টরের সেট $\mathbb{R}^3$ এর জাতক।]

Example-5 Test whether the following vectors span $\mathbb{R}^3$ or not. [নিম্নোক্ত ভেক্টরগুলো $\mathbb{R}^3$ এর জাতক কি-না পরীক্ষা কর :]

(i) $(1, 2, 1), (2, 1, 0), (1, -1, 2)$ *[NUH 1992]*

(ii) $(1, -2, 1), (2, -1, 1), (2, 3, 1)$ *[NUH 1994]*

(iii) $(1, 2, 3), (0, 1, 2), (0, 0, 1)$ *[NUH 2011]*

Solution

(i) The given set of vectors spans $\mathbb{R}^3$, if for any arbitrary vector $\mathbf{v} = (a, b, c)$ and for any unknown scalars $x, y, z \in \mathbb{R}$, the given vectors can be expressed as a linear combination of **v**.

i.e., $\mathbf{v} = (a, b, c) = x(1, 2, 1) + y(2, 1, 0) + z(1, -1, 2)$

$$\Rightarrow v = (a, b, c) = (x + 2y + z, 2x + y - z, x + 2z)$$

Equating corresponding components and forming system of linear equations

$$\begin{cases} x + 2y + z = a \\ 2x + y - z = b \\ x + 2z = c \end{cases}$$

Reducing the system to echelon form by the elementary row operations.

$$\sim \begin{cases} x + 2y + z = a \\ -3y - 3z = b - 2a \\ -2y + z = c - a \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + 2y + z = a \\ -3y - 3z = b - 2a \\ 9z = a - 2b + 3c \end{cases} \quad [L_3' = 3L_3 - 2L_2]$$

Now by back substitution method, we get

$$z = \frac{1}{9}(a - 2b + 3c)$$

$$y=\frac{1}{3}(2a-b-3z)=\frac{1}{3}[2a-b-\frac{1}{3}(a-2b+3c)]$$

$$\therefore\ y=\frac{1}{9}(5a-b-3c)$$

and $x=a-z-2y=a-\frac{1}{9}(a-2b+3c)-\frac{2}{9}(5a-b-3c)$

$$\therefore\ x=\frac{1}{9}(3c+4b-2a)$$

Thus $\mathbf{v}(a,b,c)=\frac{1}{9}(3c+4b-2a)(1,2,1)+\frac{1}{9}(5a-b-3c)(2,1,0)$

$$+\frac{1}{9}(a-2b+3c)(1,-1,$$

Since the given vectors can be expressed as a linear combination of v, S the given set of vectors sapns $\mathbf{R}^3$.

(ii) The given set of vectors spans $\mathbf{R}^3$, if for any arbitrary vector $\mathbf{v}=(a,b,$ and for any unknown scalars $x, y, z \in \mathbf{R}$, the given vectors can be expresse as a linear combination of v.

i.e., $\mathbf{v}=(a,b,c)=x(1,-2,1)+y(2,-1,1)+z(2,3,1)$

$\Rightarrow\ \mathbf{v}=(a,b,c)=(x+2y+2z,-2x-y+3z,x+y+z)$

Equating corresponding components and forming system of linear equatior

$$\begin{cases} x+2y+2z=a \\ -2x-y+3z=b \\ x+y+z=c \end{cases}$$

Reducing the system to echelon form by the elementary row operations.

$$\sim \begin{cases} x+2y+2z=a \\ 3y+7z=b+2a \\ -y-z=c-a \end{cases} \quad \begin{bmatrix} L_2'=L_2+2L_1 \\ L_3'=L_3-L_1 \end{bmatrix}$$

$$\sim \begin{cases} x+2y+2z=a \\ 3y+7z=b+2a \\ 4z=3c+b-a \end{cases} \quad [L_3'=3L_3+L_2]$$

Now by back substitution method, we get

$$z=\frac{1}{4}(3c+b-a),$$

$$y=\frac{1}{3}(b+2a-7z)=\frac{1}{3}[b+2a-\frac{7}{4}(3c+b-a)]$$

$$=\frac{1}{12}(-3b+15a-21c)$$

$\therefore \quad y = \frac{1}{4}(5a - b - 7c)$

and $\quad x = a - 2y - 2z \quad = a - \frac{1}{2}(5a - b - 7c) - \frac{1}{2}(3c + b - a)$

$= \frac{1}{2}(-2a + 4c)$

$\therefore \quad x = 2c - a$

Thus $\quad \mathbf{v} = (a, b, c) = (2c - a)(1, -2, 1) + \frac{1}{4}(5a - b - 7c)(2, -1, 1) +$

$\frac{1}{4}(3c + b - a)(2, 3, 1)$

Since the given vectors can be expressed as a linear combination of **v**. So the given set of vectors spans $\mathbb{R}^3$. (Ans.)

(iii) The given set of vectors spans $\mathbb{R}^3$, if for any arbitrary vector $\mathbf{v} = (a, b, c)$ and for any unknown scalars $x, y, z \in \mathbb{R}$, the given vectors can be expressed as a linear combination of **v**.

i.e., $\quad \mathbf{v} = (a, b, c) = x(1, 2, 3) + y(0, 1, 2) + z(0, 0, 1)$

$\Rightarrow \quad \mathbf{v} = (a, b, c) = (x, 2x + y, 3x + 2y + z)$

Equating corresponding components and forming system of linear equations

$$\begin{cases} x = a \\ 2x + y = b \\ 3x + 2y + z = c \end{cases}$$

Solving, we get $x = a, y = b - 2a, z = a - 2b + c$

Thus $\mathbf{v} = (a, b, c) = a(1, 2, 3) + (b - 2a)(0, 1, 2) + (a - 2b + c)(0, 0, 1)$.

Since the given vectors can be expressed as a linear combination of **v**. So the given set of vectors spans $\mathbb{R}^3$. (Ans.)

Example-6 Show that the set of vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ are basis of $\mathbb{R}^3$. [দেখাও যে, ভেক্টরসমূহের সেট $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $\mathbb{R}^3$ এর ভিত্তি ।]

[NUH 2015, 2011, 2003; KUH '06]

Solution The given vectors will form a basis of $\mathbb{R}^3$ if (i) They are linearly independent and (ii) the vectors spans $\mathbb{R}^3$. [প্রদত্ত ভেক্টরগুলো $\mathbb{R}^3$ এর ভিত্তি গঠন করবে যদি (i) ভেক্টসমূহ যোগাশ্রয়ী অনির্ভরশীল হয় এবং (ii) ভেক্টরগুলো $\mathbb{R}^3$ এর জাতক হয় ।]

(i) To show the given vectors are linearly independent, consider a matrix whose rows are given vectors. [প্রদত্ত ভেক্টরগুলো যোগাশ্রয়ীভাবে অনির্ভরশীল দেখানোর জন্য একটি ম্যাট্রিক্স বিবেচনা করি যার সারিগুলো হলো প্রদত্ত ভেক্টরসমূহ :]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is in row echelon form which has no any zero rows. So the given vectors are linearly independent. [এ ম্যাট্রিক্সটি সারি ইচালন আকারের যা কোন শূন্য সারি নেই। অতএব প্রদত্ত ভেক্টরসমূহ যোগাশ্রয়ীভাবে অনির্ভরশীল।]

(ii) To show that the given vectors spans $\mathbb{R}^3$, we must show that for any scalars $x, y, z \in \mathbb{R}$ an arbitrary vector $\mathbf{v} = (a, b, c)$ can be expressed as a linear combination of the given vectors. [প্রদত্ত ভেক্টরগুলো $\mathbb{R}^3$ এর জাতক দেখানোর জন্য আমাদের অবশ্যই দেখাতে হবে যে, যে কোন স্কেলার $x, y, z \in \mathbb{R}$ এর জন্য ইচ্ছামূলক ভেক্টর $\mathbf{v} = (a, b, c)$ কে প্রদত্ত ভেক্টরগুলোর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যাবে।]

$$\text{i.e.,}\quad \mathbf{v} = (a, b, c) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\Rightarrow \quad (a, b, c) = (x, y, z)$$

Equating corresponding components from both sides, we get [উভয়পক্ষ হতে অনুরূপ উপাদানসমূহ সমীকৃত করে পাই]

$$x = a, y = b, \text{ and } z = c$$

Thus $\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$

Since the vectors are linearly independent and can be expressed as a linear combination of an arbitrary vector, that is the vectors spans $\mathbb{R}^3$. Therefore, the given vectors form a basis of $\mathbb{R}^3$. [যেহেতু ভেক্টরগুলো যোগাশ্রয়ীভাবে অনির্ভরশীল এবং যে কোন ইচ্ছামূলক ভেক্টরের যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায়, অর্থাৎ ভেক্টরগুলো $\mathbb{R}^3$ এর জাতক। সুতরাং ভেক্টরগুলো $\mathbb{R}^3$ এর ভিত্তি গঠন করে।] [Showed]

Example-7 Show that the set of vectors $\{(1, 0), (0, 1)\}$ are basis of $\mathbb{R}^2$. [দেখাও যে, ভেক্টরসমূহের সেট $\{(1, 0), (0, 1)\}$, $\mathbb{R}^2$ এর ভিত্তি।] *[NUH (NM) '06]*

Solution The given vectors will form a basis of $\mathbb{R}^2$ if (i) the vectors are linearly independent and (ii) the vectors spans $\mathbb{R}^2$.

(i) To show the given vectors are linearly independent, consider a matrix whose rows are given vectors.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix is in row echelon form which has no any zero rows. So the given vectors are linearly independent.

(ii) To show that the given vectors spans $\mathbb{R}^2$, we must show that for any scalars $x, y \in \mathbb{R}$, an arbitrary vector $\mathbf{v} = (a, b)$ can be expressed as a linear combination of the given vectors.

$$\text{i.e.,}\quad \mathbf{v} = (a, b) = x(1, 0) + y(0, 1)$$

$$\Rightarrow \quad (a, b) = (x, y)$$

Equating corresponding components from both sides, we get

$$x = a, \text{ and } y = b$$

Thus $\mathbf{v} = (a, b) = a\,(1, 0) + b\,(0, 1)$

Since the vectors are linearly independent and can be expressed as a linear combination of an arbitrary vector, that is the vectors spans $\mathbb{R}^2$. Therefore, the given vectors form a basis of $\mathbb{R}^2$. [Showed]

Example-8 Show that the set of vectors $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$ are basis of $\mathbb{R}^4$. [দেখাও যে, ভেক্টরসমূহের সেট $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$, $\mathbb{R}^4$ এর ভিত্তি ।]

Solution

The given vectors will form a basis of $\mathbb{R}^4$ if (i) the vectors are linearly independent and (ii) the vectors spans $\mathbb{R}^4$.

(i) To show that the given vectors are linearly independent, consider a matrix whose rows are given vectors.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is in row echelon form which has no any zero rows. So the given vectors are linearly independent.

(ii) To show that the given vectors spans $\mathbb{R}^4$, we must show that for any scalars $x, y, z, t \in \mathbb{R}$, an arbitrary vector $\mathbf{v} = (a, b, c, d)$ can be expressed as a linear combination of the given vectors.

i.e., $\mathbf{v} = (a, b, c, d) = x(1, 1, 1, 1) + y(0, 1, 1, 1) + z(0, 0, 1, 1) + t(0, 0, 0, 1)$

Equating corresponding components and forming system of equations, we get

$$\begin{cases} x = a \\ x + y = b \\ x + y + z = c \\ x + y + z + t = d \end{cases}$$

The solution of the above system is $x = a, y = b - a, z = c - b$, and $t = d - c$.

Thus $\mathbf{v} = (a, b, c, d) = a(1, 1, 1, 1) + (b - a)(0, 1, 1, 1)$
$+ (c - b)(0, 0, 1, 1) + (d - c)(0, 0, 0, 1)$

Since the vectors are linearly independent and can be expressed as a linear combination of an arbitrary vector, that is the vectors spans $\mathbb{R}^4$. Therefore, the given vectors form a basis of $\mathbb{R}^4$. [Showed]

Example-9 Show that the set of vectors $\{(1, 2, 7), (-1, 3, 5)\}$ do not form basis of $\mathbb{R}^3$. [দেখাও যে, $\{(1, 2, 7), (-1, 3, 5)\}$ ভেক্টরসমূহের সেট $\mathbb{R}^3$ এর ভিত্তি গঠন করে না।]

Solution The given vectors will form a basis of $\mathbb{R}^3$ if (i) They are linearly independent and (ii) the vectors spans $\mathbb{R}^3$.

(i) To show that the given vectors are linearly independent, consider a matrix whose rows are given vectors.

$$\begin{bmatrix} 1 & 2 & 7 \\ -1 & 3 & 5 \end{bmatrix}$$

Reducing the matrix to echelon form by the elementary row operations

$$\sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 12 \end{bmatrix} \quad [R_2' = R_2 + R_1]$$

This matrix is in row echelon form which has no any zero rows. So the given vectors are linearly independent.

(ii) To show that the given vectors spans $\mathbb{R}^3$, we must show that for any scalar $x, y \in \mathbb{R}$ an arbitrary vector $\mathbf{v} = (a, b, c)$ can be expressed as a linear combination of the given vectors.

i.e., $\mathbf{v} = (a, b, c) = x(1, 2, 7) + y(-1, 3, 5)$

$\Rightarrow (a, b, c) = (x - y, 2x + 3y, 7x + 5y)$

Equating corresponding components and forming system of equation, we get

$$\begin{cases} x - y = a \\ 2x + 3y = b \\ 7x + 5y = c \end{cases}$$

Reducing the system to echelon form by the elementary row operations.

$$\sim \begin{cases} x - y = a \\ 5y = b - 2a \\ 12y = c - 7a \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 7L_1 \end{bmatrix}$$

$$\sim \begin{cases} x - y = a \\ 5y = b - 2a \\ 0 = 5c - 11a - 12b \end{cases} \quad [L_3' = 5L_3 - 12L_2]$$

This system is in echelon form and consisting a equation $0 = 5c - 11a - 12b$, which is not true. So, the system is inconsistent. Thus $\mathbf{v}$ can not be expressed as a linear combination of the given vectors. That is the vectors do not spans $\mathbb{R}^3$. Therefore, the given vectors do not form a basis of $\mathbb{R}^3$. [এ জোটটি ইচালন আকারের এবং এতে $0 = 5c - 11a - 12b$ আকারের সমীকরণ বিদ্যমান, সত্য নয়। অতএব জোটটি অসামঞ্জস্য। সুতরাং প্রদত্ত ভেক্টরসমূহকে $\mathbf{v}$ এর যোগাশ্রয়ী সমাবেশরূপে প্রকাশ করা যায় না। অর্থাৎ এরা $\mathbb{R}^3$ এর জাতক নয়। অতএব প্রদত্ত ভেক্টরসমূহ $\mathbb{R}^3$ এর ভিত্তি গঠন করে না।] [Showed]

Example-10 Show that the set of vectors $\{(1, 2, 3, 4), (2, 1, -1, 3), (3, 3, 2, 7), (1, -1, -1, -1)\}$ do not form basis of $\mathbb{R}^4$. [দেখাও যে $\{(1, 2, 3, 4), (2, 1, -1, 3), (3, 3, 2, 7), (1, -1, -1, -1)\}$ ভেক্টরসমূহের সেট $\mathbb{R}^4$ এর ভিত্তি গঠন করে না।]

Solution

The given vectors will form a basis of $\mathbb{R}^4$ if (i) the vectors are linearly independent and (ii) the vectors spans $\mathbb{R}^4$.

(i) To show that the given vectors are linearly independent, consider a matrix whose rows are given vectors.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 3 & 3 & 2 & 7 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

Reducing the matrix to echelon form by the elementary row operations.

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -7 & -5 \\ 0 & -3 & -7 & -5 \\ 0 & -3 & -4 & -5 \end{bmatrix} \quad \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \\ R_4' = R_3 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -7 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} R_3' = R_3 - R_2 \\ R_4' = R_4 - R_2 \end{bmatrix}$$

This matrix is in row echelon form having a zero row. So the given vectors are linearly dependent. [Showed]

(ii) To show that the given vectors spans $\mathbb{R}^4$, we must show that for any scalars $x, y, z, t \in \mathbb{R}$, an arbitrary vector $\mathbf{v} = (a, b, c, d)$ can be expressed as a linear combination of the given vectors.

i.e., $\mathbf{v} = (a, b, c, d) = x(1, 2, 3, 4) + y(2, 1, -1, 3) + z(3, 3, 2, 7) + t(1, -1, -1, -1)$

Equating corresponding components and forming system of equations, we get

$$\begin{cases} x + 2y + 3z + t = a \\ 2x + y + 3z - t = b \\ 3x - y + 2z - t = c \\ 4x + 3y + 7z - t = d \end{cases}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{cases} x + 2y + 3z + t = a \\ -3y - 3z - 3t = b - 2a \\ -7y - 7z - 4t = c - 3a \\ -5y - 5z - 5t = d - 4a \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \\ L_4' = L_3 - 4L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + 2y + 3z + t = a \\ -3y - 3z - 3t = b - 2a \\ 9t = 5a - 7b + 3c \\ 0 = -2a - 5b + 3d \end{cases} \quad \begin{bmatrix} L_3' = 3L_3 - 7L_2 \\ L_4' = 3L_4 - 5L_2 \end{bmatrix}$$

This system is in echelon form and consisting a equation $0 = -2a - 5b + 3$ which is not true. So, the system is inconsistent. Thus v can not be expressed a linear combination of the given vectors. Hence the vectors do not spans $\mathbb{R}^4$. Therefore, the given vectors do not form a basis of $\mathbb{R}^4$. [Showed]

Example-11 Show whether the following vectors are a basis of $\mathbb{R}^3$ or no [নিম্নের ভেক্টরগুলো $\mathbb{R}^3$ এর ভিত্তি কি না যাচাই করে দেখাও :]

(a) $\{(1, 2, 0), (0, 5, 7), (-1, 1, 3)\}$ *[B.Sc (Pass) '05*

(b) $\{(1, 1, -1), (1, 0, 2), (1, 1, 1)\}$ *[NUH '98, '94, NU(Prel) '09 DUH '86*

(c) $\{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$ *[B.Sc (Pass) '06*

(d) $\{(5, 7, 9), (1, 2, 0), (4, 5, 9)\}$

(e) $\{(1, 3, -4), (1, 4, -3), (2, 3, -11)\}$ *[JUH '94*

(f) $\{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ *[NUH 2012; CUH '79; DUH '80*

(g) $\{(1, 1, 2), (1, 2, 5), (5, 3, 4)\}$ *[CUH '86, '8*

Solution

(a) The given vectors will form a basis of $\mathbb{R}^3$ if (i) They are linear independent and (ii) the vectors spans $\mathbb{R}^3$.

(i) To show the given vectors are linearly independent, consider a matr whose rows are given vectors.

i.e., $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 7 \\ -1 & 1 & 3 \end{bmatrix}$

Reducing the matrix to echelon form by the elementary row operations.

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 7 \\ 0 & 3 & 3 \end{bmatrix} \quad [R_3' = R_3 + R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 7 \\ 0 & 0 & -6 \end{bmatrix} \quad [R_3' = 5R_3 - 3R_2]$$

This matrix is in row echelon form which has no any zero rows. So the given vectors are linearly independent.

(ii) To show that the given vectors spans $\mathbb{R}^3$, we must show that for any scalars $x, y, z \in \mathbb{R}$, an arbitrary vector $\mathbf{v} = (a, b, c)$ can be expressed as a linear combination of the given vectors.

i.e., $\mathbf{v} = (a, b, c) = x(1, 2, 0) + y(0, 5, 7) + z(-1, 1, 3)$

$\Rightarrow \quad (a, b, c) = (x - z, 2x + 5y + z, 7y + 3z)$

Equating corresponding components and forming system of equations, we get

$$\begin{cases} x \quad\quad - z = a \\ 2x + 5y + z = b \\ \quad\quad 7y + 3z = c \end{cases}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{cases} x \quad\quad - z = a \\ \quad 5y + 3z = b - 2a \\ \quad 7y + 3z = c \end{cases} \quad [L_2' = L_2 - 2L_1]$$

$$\sim \begin{cases} x \quad\quad - z = a \\ \quad 5y + 3z = b - 2a \\ \quad 2y \quad\quad = c - b + 2a \end{cases} \quad [L_3' = L_3 - L_2]$$

Now by back substitution method,

we get $y = \frac{1}{2}(c - b + 2a), z = \frac{1}{3}(b - 2a) - \frac{5}{6}(c - b + 2a) = \frac{1}{6}(7b - 5c - 14a)$

and $x = a + \frac{1}{6}(7b - 5c - 14a) = \frac{1}{6}(7b - 5c - 8a)$

Thus $\mathbf{v} = (a, b, c) = \frac{1}{6}(7b - 5c - 8a)(1, 2, 0) + \frac{1}{2}(c - b + 2a)(0, 5, 7)$

$$+ \frac{1}{6}(7b - 5c - 14a)(-1, 1, 3)$$

Since the vectors are linearly independent and can be expressed as a linear combination of an arbitrary vector, that is the vectors spans $\mathbb{R}^3$. Therefore, the given vectors form a basis of $\mathbb{R}^3$.

(b) The given vectors will form a basis of $\mathbb{R}^3$ if (i) They are linearly independent and (ii) the vectors spans $\mathbb{R}^3$.

(i) To show the given vectors are linearly independent, consider a matrix whose rows are given vectors.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$