Mathematical Problems & Solution
গাণিতিক সমস্যাবলি এবং সমাধান

Example-1 Find the rank of each of the following matrices. [নিচের প্রত্যেকটি ম্যাট্রিক্সের Rank বের কর :]

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Solution

(i) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$. Then,

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

So the rank of the matrix A is 1 (one) since $|A| = 0$, but not every element of A is zero say $|2| \neq 0$

(ii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ then,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = 1\,(21 - 20) - 2(14 - 12) + 3(10 - 9)$$

$$= 1 - 4 + 3 = 0$$

So the rank of the matrix A is less then 3. Now let us take two-rowed minor of A, say

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

since $|A| = 0$, but $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0$

Therefore, the rank of the given matrix is 2 (Ans.)

Example-2 Find the rank of the matrix. [ম্যাট্রিক্সটির র‍্যাঙ্ক বের কর :]

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Solution Here, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$

$= 6 \times 0 = 0$ [since three rows are equal.]

So the rank of the matrix A is less than 3. Now let us consider the two-rowed minors of A.

Say, $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36 - 36 = 0$, $\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 18 - 18 = 0$, $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

$\begin{vmatrix} 2 & 6 \\ 3 & 9 \end{vmatrix} = 18 - 18 = 0$, $\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 9 - 9 = 0$, $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$

$\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 12 - 12 = 0$, $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$, $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

Thus every two rowed minor of A is zero, So the rank of A is less than 2, But $|4| = 4 \neq 0$. Hence the rank of A is 1 (one). (Ans.)

Example-3 Find the rank of the matrix. [ম্যাট্রিক্সটির rank বের কর :]

$$A = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{bmatrix}$$

Solution Since the given matrix A is of order 3×4, the rank of the given matrix A can not be greater than 3. Now we observe that the matrix A has the following largest square submatrices of order 3×3.

$$A_1 = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix}$$

Now $|A_1| = \begin{vmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{vmatrix} = 6(-6+3) - 2(-12+3) + 0$
$= -18 + 18 = 0$

$|A_2| = \begin{vmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{vmatrix} = 6(-10+4) - 2(-20+4) + 4(2-1)$
$= -36 + 32 + 4 = 0$

$|A_3| = \begin{vmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ -1 & 6 & 10 \end{vmatrix} = 6(30-24) + 0 + 4(-12+3)$
$= 36 - 36 = 0$

$$|A_4| = \begin{vmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{vmatrix} \begin{aligned} &= 2(30-24)+0+4(-6+3) \\ &= 12-12=0 \end{aligned}$$

So the rank of the matrix A can not be 3. Let us consider the square submatrices of order 2×2. Then we can at once show that $\begin{vmatrix} 6 & 2 \\ -2 & -1 \end{vmatrix} = -6+4=-2 \neq 0$

Therefore, the rank of the given matrix is 2. (Ans.)

Example-4 Reduce $A = \begin{bmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{bmatrix}$ to Echelon from. Hence find the rank of the matrix.

Solution *Step-I* : Apply $R_1 \leftrightarrow R_3$ to get, $\begin{bmatrix} 1 & -5 & 2 \\ -1 & 12 & -2 \\ 3 & -10 & 5 \end{bmatrix}$

Step-II : Apply $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 3R_1$ to get, $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 7 & 0 \\ 0 & 5 & -1 \end{bmatrix}$

Step-III : Apply $R_2 \to \frac{1}{7} R_2$ to get, $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$

Step-IV : Apply $R_3 \to R_3 - 5R_2$ to get, $\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ which is a matrix in Echelon form. (Ans.)

The above echelon matrix has three non-zero rows. Hence the rank of the matrix is 3. (Ans.)

Example-5 Reduce the following matrix to the normal (or canonical) form and hence obtain its rank. [নিম্নের ম্যাট্রিক্সকে নরমাল (বা ক্যানোনিক্যাল) আকারে পরিণত কর এবং এর থেকে র‍্যাঙ্ক নির্ণয় কর :] *[JNUH 2015]*

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 2 & 5 \end{bmatrix}$$

Solution We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix} \begin{bmatrix} C_2' = C_2 - 2C_1 \\ C_4' = C_4 + C_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix} \begin{bmatrix} C_2' = C_2 + 2C_3 \\ C_4' = C_4 - 5C_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix} \begin{bmatrix} C_1' = C_1 + C_3 \\ C_4' = C_4 + \frac{7}{11}C_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix} [R_2' = R_2 - 4R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix} [R_3' = R_3 - 2R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix} [C_2 \leftrightarrow C_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} [C_3' = \tfrac{1}{11}C_3]$$

$$\sim [I_3 : 0], \text{ where } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the rank of A is 3. (Ans.)

Example-6 Reduce the following matrix to the normal (or canonical) form and hence obtain its rank. [নিম্নের ম্যাট্রিক্সকে নরমাল (বা ক্যানোনিক্যাল) আকারে পরিণত কর এবং তা থেকে র‍্যাঙ্ক নির্ণয় কর :]

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

Solution Given, $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 2 & 8 & 13 & 12 \end{bmatrix} \left[C_1' = \frac{1}{2}C_1\right]$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} [R_3' = R_3 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} [R_3' = R_3 - R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_2' = C_2 - 3C_1 \\ C_3' = C_3 - 5C_1 \\ C_4' = C_4 - 4C_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left[C_2' = \frac{1}{2}C_2\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_3' = C_3 - 3C_2 \\ C_4' = C_4 - 4C_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(v) Given, $A = \begin{bmatrix} 2 & 1 & 0 & -1 & 3 \\ 1 & 2 & 1 & 2 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ -1 & -5 & -3 & -7 & 3 \end{bmatrix}$

We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 1 & 0 & -1 & 3 \\ 0 & 3 & 1 & 1 & 1 \\ -1 & -5 & -3 & -7 & 3 \end{bmatrix} \quad [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -3 & -2 & -5 & 3 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & -3 & -2 & -5 & 3 \end{bmatrix} \quad \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_4' = R_4 + R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -3 & -2 & -5 & 3 \\ 0 & 0 & -1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} R_3' = R_3 + R_2 \\ R_4' = R_4 - R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & -3 & -2 & -5 & 3 \\ 0 & 0 & -1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} C_2' = C_2 - 2C_1 \\ C_3' = C_3 - C_1 \\ C_4' = C_4 - 2C_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & 3 & -5 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} C_4' = C_4 - 4C_3 \\ C_5' = C_5 + 4C_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} C_2' = \left(\frac{1}{-3}\right) C_2 \\ C_3' = (-1)\, C_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} C_3' = C_3 - 2C_2 \\ C_4' = C_4 - 3C_2 \\ C_5' = C_5 - 5C_2 \end{bmatrix}$$

Hence the rank of A is 3. (Ans.)

Example-9 Find the rank of the following matrix. [নিম্নের ম্যাট্রিক্সের র‍্যাঙ্ক বের কর :]

$$A=\begin{bmatrix}0 & 1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 1 & 1 & -2 & 0\end{bmatrix}$$

[DUS '87]

Solution Reduce the given matrix to echelon form by means of elementary row transformation.

$$\sim\begin{bmatrix}1 & 1 & -2 & 0\\ 1 & 0 & 1 & 1\\ 3 & 1 & 0 & 2\\ 0 & 1 & -3 & -1\end{bmatrix}\quad [R_1 \leftrightarrow R_4]$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 0\\ 0 & -1 & 3 & 1\\ 0 & -2 & 6 & 2\\ 0 & 1 & -3 & -1\end{bmatrix}\quad \begin{bmatrix}R_2' = R_2 - R_1\\ R_3' = R_3 - 3R_1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 0\\ 0 & -1 & 3 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\quad \begin{bmatrix}R_3' = R_3 - 2R_2\\ R_4' = R_4 + R_2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -2 & 0\\ 0 & 1 & -3 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\quad [R_2' = (-1)\,R_2]$$

This matrix is row equivalent to the given matrix A and is in the row echelon form. Since the echelon matrix has two non-zero rows, the rank of the given matrix A is 2. (Ans.)

Reduce the above matrix to row echelon form by means of elementary row transformations.

$$\sim \begin{bmatrix} 2 & 3 & 5 & -3 & -2 \\ 1 & 1 & -2 & 2 & -1 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix} [R_2' = R_2 - R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 2 & 3 & 5 & -3 & -2 \\ 5 & 6 & -1 & 3 & -5 \end{bmatrix} [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & -1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 1 & 9 & -7 & 0 \end{bmatrix} \begin{bmatrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 5R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 2 & 1 \\ 0 & 1 & 9 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} [R_3' = R_3 - R_2]$$

This matrix is row equivalent to the given matrix and is in the row echelon from. Since the echelon matrix has two non-zero rows, the rank of the given matrix is 2. (Ans.)

Example-11 Find the rank of the following matrix. [নিম্নের ম্যাট্রিক্সের র‍্যাঙ্ক বের কর :]

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$ *[DUH '87]*

Solution Given, $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

Reduce the given matrix to row echelon form by means of the elementary row transformations.

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - 2R_1 \\ R_4' = R_4 - 3R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_3' = R_3 + 3R_2 \\ R_4' = R_4 + R_2 \end{bmatrix}$$

সরল সমীকরণ

An equation of the form

$$ax + by = c \quad \text{..............} \quad (1)$$

in xy-plane is called linear equation of two variables x and y. In xy-plane the above equation represents a straight line, where a, b and c are real constants and both a, b are not zero. [$ax + by = c$ .... (1) আকারের সমীকরণকে xy-সমতলে x ও y দুইটি চলকের সরল সমীকরণ বলে। xy-সমতলে উপরের সমীকরণটি একটি সরলরেখা প্রকাশ করে, যেখানে a, b এবং c বাস্তব ধ্রুবক এবং a ও b উভয়ই শূন্য নয়।]

More generally, a linear equation in n variables $x_1, x_2, \ldots\ldots, x_n$ has the form [সাধারণভাবে n সংখ্যক চলক বিশিষ্ট $x_1, x_2, \ldots\ldots, x_n$ এর সরল সমীকরণের আকার]

$$a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = b \quad \text{..............} \quad (2)$$

Using sigma notation it can also be written as [সিগ্‌মা প্রতীক ব্যবহার করে একে লেখা যেতে পারে]

$$\sum_{i=1}^{n} a_ix_i = b \quad \text{..} \quad (3)$$

- $a_1, a_2, \ldots\ldots\ldots, a_n$ are called the coefficients of the equation; they are all real numbers. a_1 is called the leading coefficient. [$a_1, a_2, \ldots\ldots\ldots, a_n$ কে সমীকরণটির সহগ বলে; তারা সকলেই বাস্তব সংখ্যা। a_1 কে মুখ্য সহগ বলে।]

- $x_1, x_2, \ldots\ldots, x_n$ are the variables of the equation. The variables in a linear equation are sometimes called unknowns. x_1 is called the leading variable. [$x_1, x_2, \ldots\ldots, x_n$ হলো সমীকরণটির চলক। সরল সমীকরণের চলকগুলোকে অনেক সময় অজানা রাশিও বলে। x_1 কে বলা হয় মুখ্য চলক।]

- b is called the constant term; it is also a real number. [b কে বলা হয় ধ্রুব পদ; এটিও একটি বাস্তব সংখ্যা।]

Linear equations have no products or roots of variables. No variable is involved in trigometric, exponential or logarithmic functions. Variables appear only to the first power. [সরল সমীকরণে চলকগুলির গুণফল বা বর্গমূল থাকে না। কোন চলক ত্রিকোণমিতিক, সূচক বা লগারিদমিক ফাংশনে যুক্ত থাকে না। চলক শুধুমাত্র প্রথম ঘাতের আকারে থাকে।]

Example : The equations below are linear

(1) $x + y - z = 20$ (2) $\sqrt{2}x + \pi y = \sin 2$ (5) $2^k x + y = 5$ *[JNU 2015]*

(3) $x_1 - 2x_2 + 3x_3 - 4x_4 = 5$ (4) $x + 2y = 5$ (6) $kx - \frac{1}{k}y = \sin k$

Example : The equations below are not linear. Explain the reason.

(1) $\sqrt{2}x + y - z = 1$ (2) $e^x + y = 1$ (3) $\sin x_1 + \cos x_2 + x_4 = 5$

(4) $\frac{1}{x_1} + \frac{1}{x_2} = 1$ (5) $y = \sin x$

Solutions of system of two linear equations in two variables
দুই চলক বিশিষ্ট দুইটি সমীকরণ জোটের সমাধান

Let us consider a system of two linear equations in two variables as follows : [দুই চলক বিশিষ্ট দুইটি সমীকরণ জোট বিবেচনা করা যাক :]

$$\begin{cases} L_1 : ax + by = h \\ L_2 : cx + dy = k \end{cases} \quad \text{.......................} (1)$$

The graph of each equation in a system is a straight line in the xy plane, so that geometrically the solution to the system is the point of intersection of the two straight lines L_1 and L_2 represented by the first and second equation of the system. [সমীকরণ জোটের প্রত্যেকটির লেখচিত্র xy-সমতলে একটি সরলরেখা প্রকাশ করে। সুতরাং জ্যামিতিকভাবে সমীকরণ জোটের প্রথম ও দ্বিতীয় সমীকরণ L_1 ও L_2 দ্বারা নির্দেশিত সরলরেখার ছেদবিন্দু (গুলি) হবে সমীকরণদ্বয়ের সমাধান।]

Given two lines L_1 and L_2, one and only one of the following may occur : [প্রদত্ত দুইটি রেখা L_1 এবং L_2 এর জন্য নিম্নোক্ত একটি এবং কেবলমাত্র একটি ঘটবে]

(ক) L_1 and L_2 intersect at exactly one point. [L_1 এবং L_2 ঠিক একটি বিন্দুতে ছেদ করবে।]

(খ) L_1 and L_2 are parallel and coincident. [L_1 এবং L_2 সমান্তরাল এবং সমপাতিত হবে।]

(গ) L_1 and L_2 are parallel and distinct. [L_1 এবং L_2 সমান্তরাল এবং ভিন্ন হবে।]

(See Figure 4.1) In the first case, the system has a unique solution corresponding to the single point of intersection of the two lines. In the second case, the two lines are identical and the single line has infinitely many solutions corresponding to the points lying on the same line. Finally, in the third case, the system has no solution because the two lines do not intersect. [প্রথমক্ষেত্রে, জোটটির

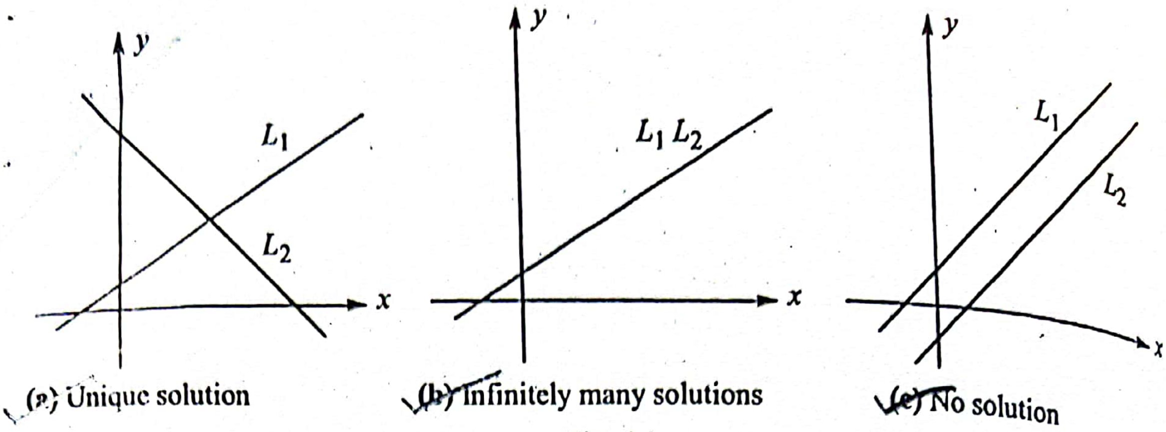


Fig 4.1

Let's illustrate each of these possibilities by considering some specific examples : [এ ধরনের প্রত্যেক সম্ভাবনাকে ব্যাখ্যার জন্য কিছু নির্দিষ্ট উদাহরণ বিবেচনা করা যাক :]

1. A system of linear equations with exactly one solution :

Consider the system

$$2x - y = 1$$
$$3x + 2y = 12$$

Solving the first equation for y in terms of x, we obtain the equation

$$y = 2x - 1$$

Substituting this expression for y into the second equation yields

$$3x + 2(2x - 1) = 12$$
$$\Rightarrow 3x + 4x - 2 = 12$$
$$\Rightarrow 7x = 14$$
$$\therefore x = 2$$

Finally, substituting this value of x into the expression for y gives

$$y = 2(2) - 1 = 3$$

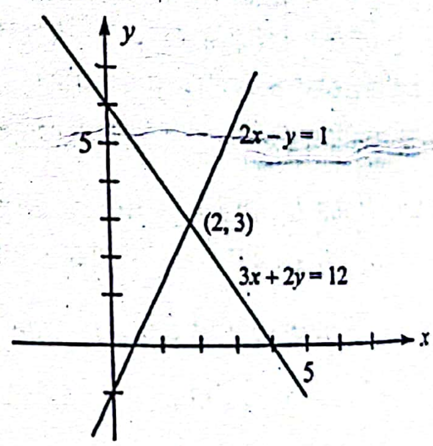


Fig 4.2

Therefore, the unique solution of the system is given by $x = 2$ and $y = 3$. Geometrically, the two lines represented by the two equations that make up the system intersect at the point (2, 3) (Figure 4.2).

2. A system of linear equations which are coixcident has infinitely many solutions :

Consider the system

$$2x - y = 1$$
$$6x - 3y = 3$$

Solving the first equation for y in terms of x, we obtain the equation

$$y = 2x - 1$$

Substituting this expression for y into the second equation gives

$$6x - 3(2x - 1) = 3$$
$$6x - 6x + 3 = 3$$
$$0 = 0$$

which is a true statement. This result follows from the fact that the second equation is equivalent to the first. (To see this, just multiply both sides of the first equation by 3.) Our computations have revealed that the system of two equations is equivalent to the single equation $2x - y = 1$. Thus, any ordered pair of numbers (x, y) satisfying the equation $2x - y = 1$ or $y = 2x - 1$ constitutes a solution to the system.

In particular, by assigning the value t to x, where t is any real number, we find that $y = 2t - 1$ and so the ordered pair $(t, 2t - 1)$ is a solution of the system. The variable t is called a parameter. For example, setting $t = 0$ gives the point $(0, -1)$ as a solution of the system, and setting $t = 1$ gives the point $(1, 1)$ as another solution. Since t represents any real number, there are infinitely many solutions of the system. Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line (Figure 4.3). Such a system is said to be ***dependent***.

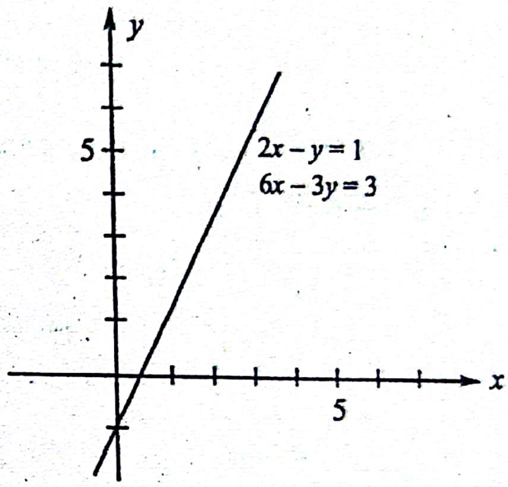


Fig 4.3

3. **A system of linear equations that has no solution :**

Consider the system

$$2x - y = 1$$
$$6x - 3y = 12$$

The first equation is equivalent to $y = 2x - 1$. Substituting this expression for y into the second equation gives

$$6x - 3(2x - 1) = 12$$
$$6x - 6x + 3 = 12$$
$$0 = 9$$

which is clearly untrue. Thus, there is no solution to the system of equations. To interpret this situation geometrically, cast both equations in the slope-intercept form, obtaining

$$y = 2x - 1$$
$$y = 2x - 4$$

We see at once that the lines represented by these equations are parallel (each has slope 2) and distinct since the first has y-intercept -1 and the second has y-intercept -4 (Figure 4.4). Systems having no solutions, such as this one, are said to be *inconsistent*.

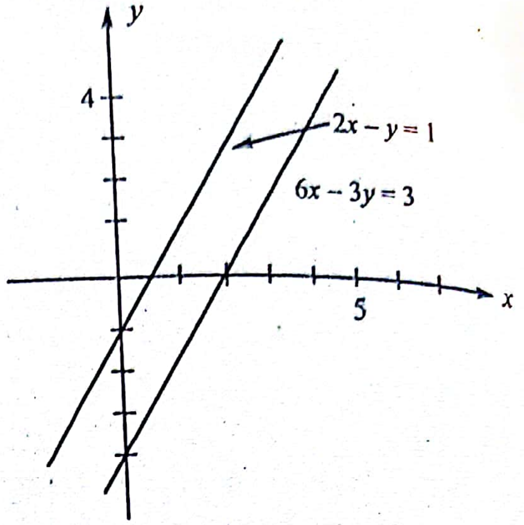


Fig 4.4

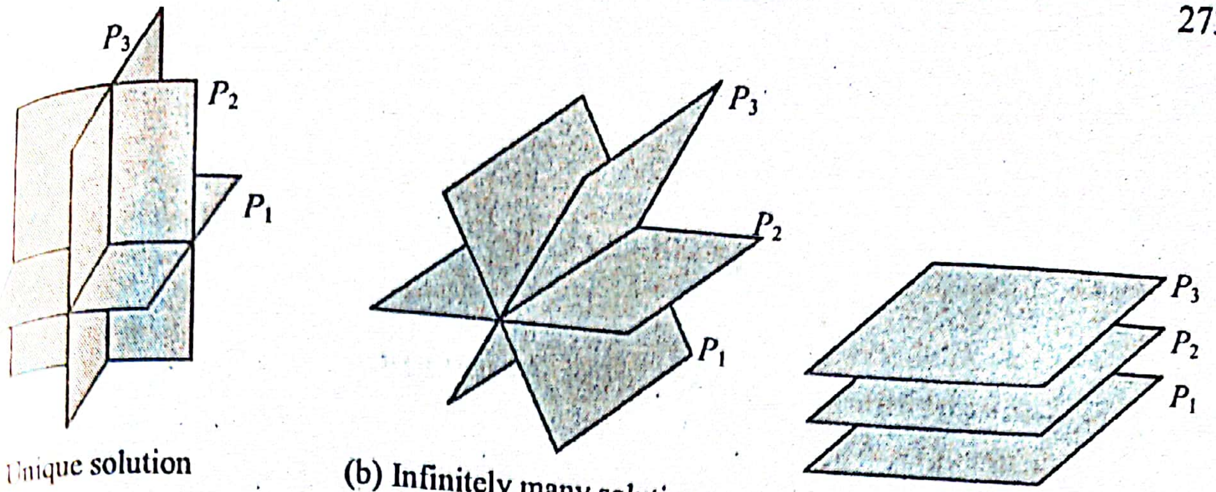
4.6 Solutions of system of three linear equations in three variables
তিন চলক বিশিষ্ট তিনটি সমীকরণ জোটের সমাধান

A linear system consists of three linear equations in three variables x, y and z has the general form

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \cdots\cdots\cdots (1)$$

Just as a linear equation in two variables that represents a straight line in the plane, it can be shown that a linear equation $ax + by + cz = d$ (a, b, and c not all equal to zero) in three variables represents a plane in three-dimensional space. Thus, each equation in system (1) represents a plane in three-dimensional space, and the solution of the system is precisely the point of intersection of the three planes defined by the three linear equations that make up the system. As before, the system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another. Figure 4.5 illustrates each of these possibilities.

In Figure 4.5(a) the three planes intersect at a point corresponding to the situation in which system (1) has a unique solution. Figure 4.5(b) depicts a situation in which there are infinitely many solutions to the system. Here, the three planes intersect along a line, and the solutions are represented by the infinitely many points lying on this line. In Figure 4.5(c), the three planes are parallel and distinct, so there is no point in common to all three planes. System (1) has no solution in this case.



(a) Unique solution (b) Infinitely many solutions (c) No solution

Fig 4.5

4.5 System of linear equations
সরল সমীকরণ জোট

[NUH (NM) 2008; KUH 2005]

An arbitrary system of m linear equations in n variables is a set of m equations, each equation being linear in n variables. Such a system may be written in the general form [একটি n চলক বিশিষ্ট m সংখ্যক সরল সমীকরণের ইচ্ছামূলক জোট হলো m সংখ্যক সমীকরণের সেট, প্রত্যেক সমীকরণ n চলক বিশিষ্ট সরল সমীকরণ। এ ধরনের জোটকে সাধারণ আকারে লেখা যায়]

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots (1)$$

where a_{ij}, b_i are constants and x_j are n variables, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. [যেখানে a_{ij}, b_i হলো ধ্রুবক এবং x_j হলো n সংখ্যক চলক, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$]

Example :

$$\begin{aligned} 2x_1 + \ x_2 - 2x_3 &= 13 \\ 3x_1 + 2x_2 + 2x_3 &= 1 \\ 5x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

is a system of three linear equations in three variables.

4.6 Non-homogeneous system of linear equations
অসমমাত্রিক সরল সমীকরণ জোট

[KUH 2006]

We consider the above system (1). If at least one element of the set $\{b_i\}$, $i = 1, 2, \ldots, m$ is no zero, then the system (1) is called a non-homogeneous system of linear equations. [উপরের জোট (1) বিবেচনা করি। যদি $\{b_i\}$, $i = 1, 2, \ldots, m$ সেটের অন্তত পক্ষে একটি উপাদান অশূন্য হয়, তবে জোট (1) -কে অসমমাত্রিক সরল সমীকরণ জোট বলে।]

Example : $x_1 + 2x_2 + 2x_3 = 2$

$$x_1 + 4x_2 + 6x_3 = 0$$

is a non-homogeneous system of 4 linear equations in four variables.

4.7.2 Homogeneous system of linear equations
সমমাত্রিক সরল সমীকরণ জোট

[KUH 2006]

We consider the above system (1). If all the elements of the set $\{b_i\}$, $i = 1, 2, ..., m$ is zero, then the system (1) is called a homogeneous system of linear equations.

Example :

$$\begin{aligned} x_1 + x_2 + 6x_3 + 4x_4 &= 0 \\ x_1 + x_2 + 4x_3 + 6x_4 &= 0 \\ 4x_1 + x_2 + x_3 + 4x_4 &= 0 \\ 6x_1 + 4x_2 + x_3 + x_4 &= 0 \end{aligned}$$

is a homogeneous system of 4 linear equations in four variables.

4.8 Particular solution and general solution of a system of linear equations
সরল সমীকরণ জোটের বিশেষ সমাধান এবং সাধারণ সমাধান

Let us consider the following system of m linear equations : [নিম্নের m সংখ্যক সরল সমীকরণ জোট বিবেচনা করা যাক :]

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \cdots\cdots\cdots (1)$$

If the above system is satisfied by $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ then it is called a particular solution. The set of all particular solutions of (1) is called the general solution of the given system. [যদি উপরের জোটটি $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$ দ্বারা সিদ্ধ হয়, তবে একে বিশেষ সমাধান বলা হয়। (1) নং এর সকল বিশেষ সমাধানের সেটকে প্রদত্ত জোটের সাধারণ সমাধান বলা হয়।]

Example : The system of linear equations

$$\left.\begin{aligned} x + 2y - z &= 2 \\ y - z &= 1 \end{aligned}\right\}$$

satisfied by $x = -1$, $y = 2$ and $z = 1$. So the set $(x, y, z) = (-1, 2, 1)$ is the particular solution of the above system. Also $(-2, 3, 2)$ is the particular solution of the above system.

...above system has two equations in three variables. So it has 3 – 2 = 1 free ...iable which is z.

...general solution let $z = t$ then $y = 1 + t$ and $x = -t$.

...us the general solution of the above system is $(-1, 1 + t, t)$

...Trivial solution or zero solution
...গুরুত্বহীন সমাধান বা শূন্য সমাধান

...t us consider the following homogeneous linear equations :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right\} \cdots\cdots\cdots (1)$$

...e system (1) is satisfied by $x_1 = 0, x_2 = 0, \ldots\ldots, x_n = 0$. Generally every ...mogeneous system of linear equations always is satisfy by the solution (0, 0, ..., 0). [সমীকরণ জোট (1), $x_1 = 0, x_2 = 0, \ldots\ldots, x_n = 0$ দ্বারা সিদ্ধ হয়। সাধারণভাবে ...তাক সমমাত্রিক সরল সমীকরণ জোট সর্বদাই (0, 0, ..., 0) সমাধান দ্বারা সিদ্ধ হয়।]

...any other solution exists except (0, 0, ..., 0) then the solution (0, 0,......., 0) ...called the trivial solution or zero solution. [যদি (0, 0, ..., 0) ব্যতিত অন্য সমাধান ...মান থাকে, তবে (0, 0, ..., 0)-কে গুরুত্বহীন বা শূন্য সমাধান বলে।]

...ample : (0, 0, 0) is the trivial or zero solution of the system of homogeneous ...ear equations :

$$\left.\begin{array}{r} 2x + 3y - 8z = 0 \\ 3y - 2z = 0 \end{array}\right\}$$

...ain, the above system is satisfied by $x = 1, y = 2$ and $z = 3$. Thus (1, 2, 3) is ...e non-trivial solution of the above system.

... Consistent and inconsistent system of linear equations
সঙ্গত এবং অসঙ্গত সরল সমীকরণ জোট *[JNUH 2015, 2014, 2012]*

...onsider the following system of linear equations. [নিম্নোক্ত সরল সমীকরণ জোটটি ...বেচনা করি :]

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots (1)$$

The system (1) is said to be consistent if no equation of the form $0 = a$ exists. Again, the system (1) is called inconsistent if the equation of the form $0 = a$ exists. [(1) নং জোটকে সঙ্গত বলা হবে যদি $0 = a$ আকারের কোন সমীকরণ বিদ্যমান না থাকে। আবার জোট (1) -কে অসঙ্গত বলা হবে যদি $0 = a$ আকারের সমীকরণ বিদ্যমান থাকে।]

In other words, a system of equations that has no solution is said to be inconsistent. If there is at least one solution of the system, it is called consistent. [অন্যকথায়, একটি সমীকরণ জোট যার কোন সমাধান বিদ্যমান নেই, তাকে অসঙ্গত বলা হয়। যদি কমপক্ষে একটি সমাধান থাকে, তবে একে সঙ্গত বলা হয়।]

Example : (i) The system of linear equations

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 10 \\ x_2 + 10x_3 &= -28 \\ -14x_3 &= 42 \end{aligned}$$

is consistent.

(ii) The system of linear equations

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -1 \\ -7x_2 + 11x_3 &= 10 \\ 0 &= -3 \end{aligned}$$

is inconsistent.

1.12 Gauss-Jordan elimination method

গাউস-জর্ডান এলিমিনেশন পদ্ধতি

The ***Gauss-Jordan elimination method*** is a suitable technique for solving systems of linear equations of any size. One advantage of this technique is its adaptability to the computer. This method involves a sequence of operations on a system of linear equations to obtain at each stage an ***equivalent system*** that is, a system having the same solution as the original system. The reduction is complete when the original system has been transformed so that it is in a certain standard form from which the solution can be easily read.

The operations of the Gauss-Jordan elimination method are :

1. Interchange any two equations.
2. Replace an equation by a nonzero constant multiple of itself.
3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

To illustrate the Gauss-Jordan elimination method for solving systems of linear equations, let's apply it to the solution of the following examples :

Example-7 Solve the following system of linear equations. [নিম্নের একঘাত বিশিষ্ট সমীকরণ জোটের সমাধান কর :]

[NUH 2006]

$$\begin{cases} x_1 + 2x_2 - 3x_3 = -1 \\ 3x_1 - x_2 + 2x_3 = 7 \\ 5x_1 + 3x_2 - 4x_3 = 2 \end{cases}$$

Solution The given system of linear equation is

$$\begin{cases} L_1 : x_1 + 2x_2 - 3x_3 = -1 \\ L_2 : 3x_1 - x_2 + 2x_3 = 7 \\ L_3 : 5x_1 + 3x_2 - 4x_3 = 2 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x_1 + 2x_2 - 3x_3 = -1 \\ -7x_2 + 11x_3 = 10 \\ -7x_2 + 11x_3 = 7 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 3L_1 \\ L_3' = L_3 - 5L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + 2x_2 - 3x_3 = -1 \\ -7x_2 + 11x_3 = 10 \\ 0 = -3 \end{cases} \quad [L_3' = L_3 - L_2]$$

From last equation of the above system we have $0 = -3$, which is not true. Thus the system is inconsistent. Hence the given system of linear equations has no solution. (Ans.)

Example-8 Solve the following system of linear equations. [নিম্নের একঘাত বিশিষ্ট সমীকরণ জোটের সমাধান কর :]

(i) $2x_1 + x_2 - 2x_3 = 10$
$3x_1 + 2x_2 + 2x_3 = 1$
$5x_1 + 4x_2 + 3x_3 = 4$

(ii) $x_1 + 2x_2 + 3x_3 = 4$
$2x_1 + 5x_2 + 3x_3 = 5$
$x_1 + 8x_3 = 9$ *[NU (prel) 2008]*

Solution

(i) The given system of linear equation is

$$\begin{cases} L_1 : 2x_1 + x_2 - 2x_3 = 10 \\ L_2 : 3x_1 + 2x_2 + 2x_3 = 1 \\ L_3 : 5x_1 + 4x_2 + 3x_3 = 4 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} 2x_1 + x_2 - 2x_3 = 10 \\ x_2 + 10x_3 = -28 \\ 3x_2 + 16x_3 = -42 \end{cases} \quad \begin{bmatrix} L_2' = 2L_2 - 3L_1 \\ L_3' = 2L_3 - 5L_1 \end{bmatrix}$$

$$\sim \begin{cases} 2x_1 + x_2 - 2x_3 = 10 \\ x_2 + 10x_3 = -28 \\ -14x_3 = 42 \end{cases} \quad [L_3' = L_3 - 3L_2] \ldots\ldots\ldots\ldots (1)$$

Here the system (1) is in echelon form and equivalent to the given system. There are three equations in three unknowns. So the given system has a unique solution.

From third equation of (1), we get $-14x_3 = 42 \Rightarrow x_3 = -3$

From second equation : $x_2 + 10.(-3) = -28 \Rightarrow x_2 = -28 + 30 = 2$

From first equation : $2x_1 + 2 + 6 = 10 \Rightarrow 2x_1 = 2 \Rightarrow x_1 = 1$

Therefore, the required solution is : $(x_1, x_2, x_3) = (1, 2, -3)$ (Ans.)

(ii) The given system of linear equation is

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 5x_2 + 3x_3 = 5 \\ x_1 + 8x_3 = 9 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 - 3x_3 = -3 \\ -2x_2 + 5x_3 = 5 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 - 3x_3 = -3 \\ -x_3 = -1 \end{cases} \quad [L_3' = L_3 + 2L_2] \ldots\ldots\ldots\ldots (1)$$

Here the system (1) is in echelon form. There are three equations in three variables. So the given system has a unique solution.

From third equation, we get $x_3 = 1$

From second equation, $x_2 - 3(1) = -3 \Rightarrow x_2 = 0$

From first equation, $x_1 + 2(0) + 3(1) = 4 \Rightarrow x_1 = 1$

Therefore, the required solution is : $(x_1, x_2, x_3) = (1, 0, 1)$ (Ans.)

4.13 Consistency of a system of linear equations

সরল সমীকরণ জোটের সামঞ্জস্যতা

(a) Let us consider the following system of non-homogeneous linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots (1)$$

The above system can be written in the following matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$$

or, $Ax = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix}$.

So that the matrix $C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \cdots & \cdots & \cdots & \cdots & : & \cdots \\ \cdots & \cdots & \cdots & \cdots & : & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_n \end{bmatrix}$ ....(2)

is called the augmented matrix.

Now after reducing the system (2) to row-echelon form, we have the following two cases:

Case-I : Consistent equations : If Rank (A) = Rank (C), then the system of equations is consistent and there are two possibilities:

(i) Unique solution : If Rank (A) = Rank $(C) = n$

(ii) Infinite solution : Rank (A) = Rank $(C) = r,\ r < n$

Case-II : Inconsistent equations : If Rank $(A) \neq$ Rank (C), then the system of equations is inconsistent and the system has no solution.

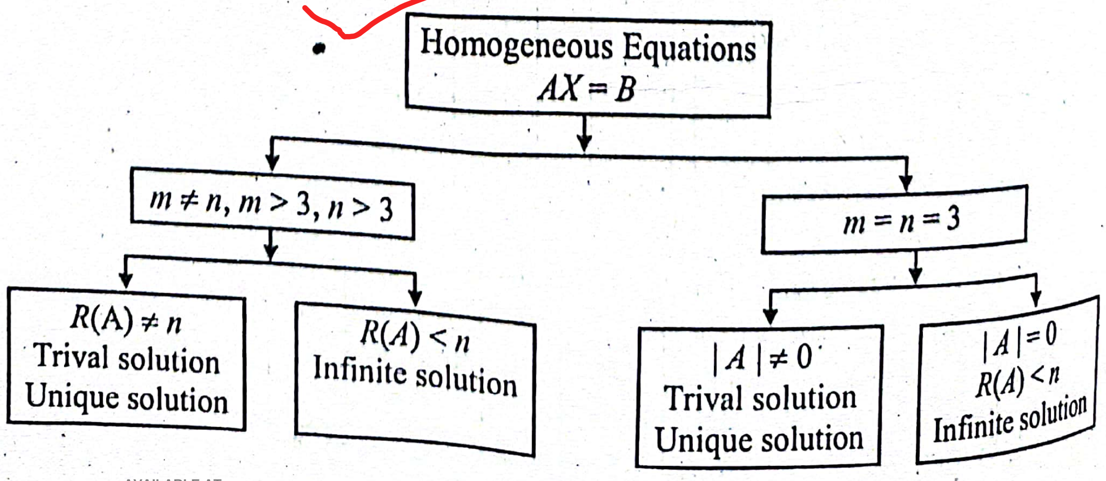
(b) Let us consider the following system of linear homogeneous equations:

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{matrix}\right\} \qquad \text{.............(3)}$$

Here the coefficient matrix of the above system can be written below:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In this case we have the possibilities in following diagram :



Solution Given system of linear equations

$$\begin{cases} x - 2y + 4z = p \\ 2x + 3y - z = q \\ 3x + y + 2z = r \end{cases}$$

Reducing the system to row echelon form by the elementary row operations.

$$\sim \begin{cases} x - 2y + 4z = p \\ 7y - 9z = q - 2p \\ 7y - 10z = r - 3p \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{bmatrix}$$

$$\sim \begin{cases} x - 2y + 4z = p \\ 7y - 9z = q - 2p \\ -z = r - p - q \end{cases} \quad [L_3' = L_3 - L_2] \text{ .... (1)}$$

The above system is in echelon form and it has three equations in three unknowns. So the given system has a unique solution.

Now by Back substitution method, from 3rd equation, we get $z = p + q - r$

From 2nd equation, we get $7y = 9p + 9q - 9r + q - 2p \Rightarrow y = p + \frac{10}{7}q - \frac{9}{7}r$

and from 1st equation, we get $x = p + 2\left(p + \frac{10}{7}q - \frac{9}{7}r\right) - 4(p + q - r)$

$$\Rightarrow x = -p - \frac{8}{7}q + \frac{10}{7}r$$

So the solution set of the given system is

$$(x, y, z) = \left(-p - \frac{8}{7}q + \frac{10}{7}r,\ p + \frac{10}{7}q - \frac{9}{7}r, p + q - r\right) \text{ (Ans.)}$$

Example-11 Examine for consistency the following systems of linear equations and solve the one which is consistent. [নিম্নোক্ত সমীকরণ জোটদ্বয়ের সমাধান যোগ্যতা যাচাই কর এবং সমাধানযোগ্য জোটটির সমাধান কর :]

(i) $2x - 3y + 5z = 1$
$3x + y - z = 2$
$x + 4y - 6z = 1$ *[DUH 1994]*

(ii) $x - 4y + 5z = 8$
$3x + 7y - z = 3$
$x + 15y - 11z = -14$ *[DUH 1994]*

Solution

(i) The given system of linear equation is

$$\begin{cases} 2x - 3y + 5z = 1 \\ 3x + y - z = 2 \\ x + 4y - 6z = 1 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} 2x - 3y + 5z = 1 \\ 11y - 17z = 1 \\ 11y - 17z = 1 \end{cases} \quad \begin{bmatrix} L_2' = 2L_2 - 3L_1 \\ L_3' = 2L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} 2x - 3y + 5z = 1 \\ 11y - 17z = 1 \end{cases}$$

This system is in echelon form having two eqations in three unknowns. So the system has $3 - 2 = 1$ free variable, which is z.

Let $z = t$, where t is any real number.

Now from 2nd equation : $11y - 17t = 1 \Rightarrow y = \dfrac{1+17t}{11}$

From 1st equation : $2x = 1 + \dfrac{3+51t}{11} - 5t = \dfrac{14-4t}{11} \Rightarrow x = \dfrac{7-2t}{11}$

Therefore, the required solution of given system of linear equations is

$x = \dfrac{7-2t}{11}$, $y = \dfrac{1+17t}{11}$, $z = t$, where t is any real number. (Ans.)

ii) The given system of linear equation is

$$\begin{cases} x - 4y + 5z = 8 \\ 3x + 7y - z = 3 \\ x + 15y - 11z = -14 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x - 4y + 5z = 8 \\ 19y - 16z = -21 \\ 19y - 16z = -22 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 3L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x - 4y + 5z = 8 \\ 19y - 16z = -21 \\ 0 = -1 \end{cases} \quad [L_3' = L_3 - L_2]$$

Thus the given system has been reduced to an echelon form and from 3rd equation we have $0 = -1$, which is not true. So the given system is inconsistent.

Hence the given system has no solution. (Ans.)

Example-12 Determine the relationship among the constants a, b and c under which the following system has a solution. [a, b এবং c ধ্রুবকসমূহের মধ্যে যে সম্পর্কের জন্য নিম্নের সমীকরণ জোটের সমাধান বিদ্যমান তা নির্ণয় কর :]

$$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$ [NUH (Old) '10; NUH (NM) '06; DUH '06]

Solution The given system of linear equation is

$$\begin{cases} x + 2y - 3z = a \\ 2x + 6y - 11z = b \\ x - 2y + 7z = c \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x + 2y - 3z = a \\ 2y - 5z = b - 2a \\ -4y + 10z = c - a \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + 2y - 3z = a \\ 2y - 5z = b - 2a \\ 0 = c + 2b - 5a \end{cases} \quad [L_3' = L_3 + 2L_2] \text{ .... (1)}$$

This system is in echelon form. From 3rd equation of (1), if $c + 2b - 5a = 0$ then the system has solution.

Thus the required condition is $5a - 2b - c = 0$ (Ans.)

Example-13 When a system of non-homogeneous linear equations is said to be consistent? Ascertain whether the following system is consistent; if it is, find all solutions. [অসমমাত্রিক যোগাশ্রয়ী সমীকরণসমূহের শ্রেণীকে কখন সংগত বলা হয়? নিম্নোক্ত যোগাশ্রয়ী সমীকরণসমূহের শ্রেণীটি সংগত কি-না নিরূপণ কর; সংগত হলে সকল সমাধান নির্ণয় কর।]

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - x_2 + 2x_3 + 2x_4 = 2 \\ 3x_1 + 2x_2 - 4x_3 - 3x_4 = 3 \end{cases}$$ [NUH '98; DUH '92]

Solution A system of linear equation is said to be consistent if no equation is of the form $0 = a$ exists.

Given system of linear equations is

$$\begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - x_2 + 2x_3 + 2x_4 = 2 \\ 3x_1 + 2x_2 - 4x_3 - 3x_4 = 3 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1 \\ -3x_2 + 6x_3 = 0 \\ -x_2 + 2x_3 - 6x_4 = 0 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1 \\ -3x_2 + 6x_3 = 0 \\ -18x_4 = 0 \end{cases} \quad [L_3' = 3L_3 - L_2]$$

$$\sim \begin{cases} x_1 + x_2 - 2x_3 + x_4 = 1 \\ x_2 - 2x_3 = 0 \\ x_4 = 0 \end{cases} \quad \begin{bmatrix} L_2' = -\frac{1}{3}L_2 \\ L_3' = -\frac{1}{18}L_3 \end{bmatrix} \quad \text{.......................} (1)$$

The above system is in echelon form and has three equations in four unknowns and so it has $4 - 3 = 1$ free variable, which is x_3. Let $x_3 = t$ (parameter), where t is any real number.

Now by back substitution method from 3rd, we have $x_4 = 0$.

From 2nd equation, we get $x_3 = t, x_2 = 2t$

and from 1st equation, we get $x_1 = 1 - 2t + 2t = 1$

So the solution set of the given system is : $(x_1, x_2, x_3, x_4) = \{(1, 2t, t, 0) : t \in \mathbb{R}\}$.(Ans.)

Example-14 When a system of linear equations said to be consistent? Ascertain wheather the system below is consistent; if it is, find all solutions. [কখন সরল সমীকরণ জোট সমাধান যোগ্য হয়? নিচের জোটের সমাধান বিদ্যমান কি-না যাচাই কর; সমাধানযোগ্য হলে সকল সমাধান নির্ণয় কর :] *[NUH 1999]*

$$\begin{cases} x_1 - x_3 + 3x_4 + x_5 = -3 \\ 2x_1 + x_2 - 2x_4 - x_5 = 5 \\ x_1 + 2x_2 + 2x_3 + 4x_5 = 6 \\ x_2 + x_3 + 5x_4 + 6x_5 = -2 \end{cases}$$

Solution A system of linear equation is said to be consistent if no equation is of the form $0 = a$ exists.

Given system of linear equations

$$\begin{cases} x_1 - x_3 + 3x_4 + x_5 = -3 \\ 2x_1 + x_2 - 2x_4 - x_5 = 5 \\ x_1 + 2x_2 + 2x_3 + 4x_5 = 6 \\ x_2 + x_3 + 5x_4 + 6x_5 = -2 \end{cases}$$

Reducing the system to row echelon form by the elementary row operations.

$$\sim \begin{cases} x_1 - x_3 + 3x_4 + x_5 = -3 \\ x_2 + 2x_3 - 8x_4 - 3x_5 = 11 \\ 2x_2 + 3x_3 - 3x_4 + 3x_5 = 9 \\ x_2 + x_3 + 5x_4 + 6x_5 = -2 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 \quad\quad - x_3 + 3x_4 + x_5 = -3 \\ x_2 + 2x_3 - 8x_4 - 3x_5 = 11 \\ \quad\quad - x_3 + 13x_4 + 9x_5 = -13 \\ \quad\quad - x_3 + 13x_4 + 9x_5 = -13 \end{cases} \quad \begin{bmatrix} L_3' = L_3 - 2L_2 \\ L_4' = L_4 - L_2 \end{bmatrix}$$

$$\sim \begin{cases} x_1 \quad\quad - x_3 + 3x_4 + x_5 = -3 \\ x_2 + 2x_3 - 8x_4 - 3x_5 = 11 \\ \quad\quad - x_3 + 13x_4 + 9x_5 = -13 \end{cases} \quad \text{.........................} (1)$$

The above system is in echelon form and it has three equations in five unknowns. So the system has $5 - 3 = 2$ free variables which are x_4 and x_5. Hence the system has infinite number of non-zero solutions.

Let $x_4 = s$ and $x_5 = t$, where s and t are any real numbers.

Then from 3rd equation, we get $x_3 = 13s + 9t + 13$

From 2nd equation, we get $x_2 = 11 - 2(13s + 9t + 13) + 8s + 3t$

$$= -18s - 15t - 15$$

and from 1st equation, we get $x_1 = -3 + 13s + 9t + 13 - 3s - t$

$$= 10s + 8t + 10$$

So the required solution of the given system of linear equations is

$x_1 = 10s + 8t + 10$, $x_2 = -18s - 15t - 15$, $x_3 = 13s + 9t + 13$, $x_4 = s$ and $x_5 = t$, where s and t are any real numbers. (Ans.)

Example-15 Solve the following system of linear equations by the elementary row operations. [নিম্নের যোগাশ্রয়ী সমীকরণজোটকে প্রাথমিক সারি-ক্রিয়ার সাহায্যে সমাধান কর :]

$$\begin{cases} x_1 + 2x_2 - 2x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - 3x_3 - x_4 = 1 \\ 3x_1 + 8x_2 - 4x_3 - x_4 = 2 \\ x_1 + 5x_2 + x_3 + 2x_4 = 3 \end{cases}$$

Solution Given system of linear equations

$$\begin{cases} x_1 + 2x_2 - 2x_3 - x_4 = 0 \\ 2x_1 + 5x_2 - 3x_3 - x_4 = 1 \\ 3x_1 + 8x_2 - 4x_3 - x_4 = 2 \\ x_1 + 5x_2 + x_3 + 2x_4 = 3 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x_1 + 2x_2 - 2x_3 - x_4 = 0 \\ x_2 + x_3 + x_4 = 1 \\ 2x_2 + 2x_3 + 2x_4 = 2 \\ 3x_2 + 3x_3 + 3x_4 = 3 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \\ L_4' = L_4 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + 2x_2 - 2x_3 - x_4 = 0 \\ \quad x_2 + x_3 + x_4 = 1 \end{cases} \quad \begin{bmatrix} L_3' = L_3 - 2L_2 \\ L_4' = L_4 - 3L_2 \end{bmatrix} \cdots\cdots\cdots (1)$$

...bove system is in echelon form and it has two equations in four ...owns. So, it has $4 - 2 = 2$ free variables, which are x_3 and x_4. Let $x_3 = s$ and ..., where s and t are any real numbers.

..., by back substitution method from 2nd equation, we have $x_2 = 1 - s - t$.

...om 1st equation, we get $x_1 = -2(1 - s - t) + 2s + t = 4s + 3t - 2$.

...he solution set of the given system is : (x_1, x_2, x_3, x_4)

$$= \{(4s + 3t - 2, 1 - s - t, s, t) : s, t \in \mathbb{R}\}. \text{ (Ans.)}$$

...ample-16 Solve the following system of linear equations by the Gauss-...rdan elimination method. [নিম্নের একঘাত বিশিষ্ট সমীকরণ জোটকে গাউস-জর্ডান ...লিমিনেশন পদ্ধতিতে সমাধান কর :]

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

...olution Given system of linear equations

$$\begin{cases} x + y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0 \end{cases}$$

...educing the system to row echelon form by elementary operations.

$$\begin{cases} x + y + 2z = 9 \\ \quad 2y - 7z = -17 \\ \quad 3y - 11z = -27 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 3L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + y + 2z = 9 \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \quad 3y - 11z = -27 \end{cases} \quad [L_2' = \tfrac{1}{2}L_2]$$

$$\sim \begin{cases} x + y + 2z = 9 \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \quad -\frac{1}{2}z = -\frac{3}{2} \end{cases} \quad [L_3' = L_3 - 3L_2]$$

$$\sim \begin{cases} x + y + 2z = 9 \\ \quad y - \frac{7}{2}z = -\frac{17}{2} \\ \quad z = 3 \end{cases} \quad [L_3' = (-2)L_3]$$

$$\sim \begin{cases} x \quad + \frac{11}{2}z = \frac{35}{2} \\ y - \frac{7}{2}z = -\frac{17}{2} \\ z = 3 \end{cases} \qquad [L_1' = L_1 - L_2]$$

$$\sim \begin{cases} x = 1 \\ y = 2 \\ z = 3 \end{cases} \qquad \begin{bmatrix} L_1' = L_1 - \frac{11}{2}L_3 \\ L_2' = L_2 + \frac{7}{2}L_3 \end{bmatrix}$$

Therefore, the required solution of the given system is $x = 1, y = 2, z = 3$ (Ans.)

Example-17 Solve the following system of linear equations. [নিম্নের একঘাত বিশিষ্ট সমীকরণ জোটের সমাধান কর :]

$$\begin{cases} x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + 5x_3 + x_4 = 5 \\ 3x_1 + 6x_2 - 2x_3 + x_4 = 8 \end{cases}$$

Solution Given system of linear equations

$$\begin{cases} x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 \\ 2x_1 + x_2 + 5x_3 + x_4 = 5 \\ 3x_1 + 6x_2 - 2x_3 + x_4 = 8 \end{cases}$$

Reducing the system to row echelon form by the elementary row operations.

$$\sim \begin{cases} x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 8x_3 + 5x_4 = 4 \\ -x_2 + 11x_3 + 9x_4 = 7 \\ 3x_2 + 7x_3 + 13x_4 = 11 \end{cases} \qquad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - 2L_1 \\ L_4' = L_4 - 3L_1 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + x_2 - 3x_3 - 4x_4 = -1 \\ 8x_3 + 5x_4 = 4 \\ -x_2 + 11x_3 + 9x_4 = 7 \\ 40x_3 + 40x_4 = 32 \end{cases} \qquad [L_4' = L_4 + 3L_3]$$

$$\begin{cases} x_1 + x_2 - 3x_3 - 4x_4 = -1 \end{cases}$$

Example-24 Determine the values of λ and μ such that the following system of linear equations has (i) no solution (ii) more than one solution (iii) an unique solution. [λ এবং μ -এর এরূপ মান নির্ণয় কর যার জন্য নিম্নলিখিত সমীকরণ জোটটির (i) সমাধান না থাকে (ii) একাধিক সমাধান থাকে (iii) একক সমাধান থাকে ।]

$$\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ x + 2y + \lambda z = \mu \end{cases}$$

[NUH '09; NU (Prel) '09; DUH '93; JNUH '13]

Solution The given system of linear equation is

$$\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ x + 2y + \lambda z = \mu \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x + y + z = 6 \\ \quad y + 2z = 4 \\ \quad (\lambda - 3)z = \mu - 10 \end{cases} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_3 - L_2 \end{bmatrix} \quad \text{..................} (1)$$

The above system is in echelon form. Now we consider the following three cases :

(i) If $\lambda = 3$ and $\mu \neq 10$ then third equation of (1) is of the form $0 = a$, where $a = \mu - 10 \neq 0$ which is not true. So the system is inconsistent. Thus the system has no solution for $\lambda = 3$ and $\mu \neq 10$.

(ii) If $\lambda = 3$ and $\mu = 10$, then the third equation of (1) vanishes and the system will be in echelon form having two equations in three variables. So it has $3 - 2 = 1$ free variables which is z. Hence the given system has more than one solution for $\lambda = 3$ and $\mu = 10$.

(iii) For an unique solution the coefficient of z in the 3rd equation must be non-zero i.e. $\lambda \neq 3$ and μ may have any value. Therefore the given system has unique solution for $\lambda \neq 3$ and arbitrary values of μ.

Example-7 Determine the values of λ such that the following system of linear equations has (i) no solution (ii) more than one solution (iii) an unique solution. [λ -এর এরূপ মান নির্ণয় কর যার জন্য নিম্নলিখিত একঘাত বিশিষ্ট সমীকরণ জোটের (i) সমাধান নেই (ii) একাধিক সমাধান থাকে (iii) একক সমাধান থাকে।]

$$\begin{cases} x + y - z = 1 \\ 2x + 3y + \lambda z = 3 \\ x + \lambda y + 3z = 2 \end{cases}$$

[NUH 2016, '70; NUH (NM) '90; NU(Prel) '05; DUH '06; JUH '06]

Solution The given system of linear equations are

$$\begin{cases} x + y - z = 1 \\ 2x + 3y + \lambda z = 3 \\ x + \lambda y + 3z = 2 \end{cases}$$

Reduce the system to echelon form by the elementary operations.

$$\sim \begin{cases} x + y - z = 1 \\ y + (\lambda+2)z = 1 \\ (\lambda-1)y + 4z = 1 \end{cases} \qquad \begin{bmatrix} L_2' = L_2 - 2L_1 \\ L_3' = L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + y - z = 1 \\ y + (\lambda+2)z = 1 \\ \{4 - (\lambda-1)(\lambda+2)\}z = 2 - \lambda \end{cases} \qquad [L_3' = L_3 - (\lambda - 1)L_2]$$

$$\sim \begin{cases} x + y - z = 1 \\ y + (\lambda+2)z = 1 \\ (6 - \lambda - \lambda^2)z = 2 - \lambda \end{cases}$$

$$\sim \begin{cases} x + y - z = 1 \\ y + (\lambda+2)z = 1 \\ (3+\lambda)(2-\lambda)z = 2 - \lambda \end{cases} \quad \text{.................... (1)}$$

This system is in echelon form. Now we consider the following three cases :

(i) From third equation of (1), we see that if $\lambda + 3 = 0$ or $\lambda = -3$ then the equation becomes $0 = 5$, which is contradiction. Therefore, the system is inconsistent if $\lambda = -3$. Thus the system has no solution for $\lambda = -3$.

(ii) We know, if the number of variables is greater than the number of equations, then the system has more than one solution.

From third equation of (1), we see that if $\lambda = 2$ then it becomes $0 = 0$. In this case the system has three variables within two equations. So the given system has more than one solution for $\lambda = 2$.

We know, if the number of variables and the number of equations be equal, then the system has unique solution. The system (1) has an unique solution $(3+\lambda)(2-\lambda) \neq 0 \Rightarrow \lambda \neq -3, \lambda \neq 2$. (Ans.)

ample-26 Find the conditions on λ and μ so that the following system of ar equations will have (i) an unique solution (ii) more than one solution (iii) solution. [λ ও μ -এর উপর এরূপ শর্ত নির্ণয় কর যেন নিম্নলিখিত সরল সমীকরণ জোটের একটি অনন্য সমাধান থাকে (ii) একাধিক সমাধান থাকে (iii) আদৌ কোন সমাধান থাকে না ।]

$$\begin{cases} 2x + 3y + \ \ z = 5 \\ 3x - \ \ y + \lambda z = 2 \\ \ \ x + 7y - 6z = \mu \end{cases}$$

[NUH 2015, '14, '02; '01, '00; NUH(NM) '07]

lution The given system of linear equation is

$$\begin{cases} 2x + 3y + \ \ z = 5 \\ 3x - \ \ y + \lambda z = 2 \\ \ \ x + 7y - 6z = \mu \end{cases}$$

duce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} 2x + 3y + \qquad\qquad z = \quad 5 \\ \quad - 11y + (2\lambda - 3)z = \ -11 \\ \qquad 11y - \qquad 13z = 2\mu - 5 \end{cases} \qquad \begin{bmatrix} L_2' = 2L_2 - 3L_1 \\ L_3' = 2L_3 - L_1 \end{bmatrix}$$

$$\sim \begin{cases} 2x + 3y + \qquad\qquad z = \quad 5 \\ \quad - 11y + (2\lambda - 3)z = \ -11 \\ \qquad\qquad 2(\lambda - 8)z = 2(\mu - 8) \end{cases} \qquad [L_3' = L_3 + L_2] \ \ldots\ldots\ldots\ldots\ldots (1)$$

e above system is in echelon form. Now we consider the following three cases :

For an unique solution the coefficient of z in the 3rd equation must be non-zero i.e., $\lambda \neq 8$ and μ may have any value. Therefore, the given system has an unique solution for $\lambda \neq 8$ and arbitrary values of μ.

) If $\lambda = 8$ and $\mu = 8$, then the third equation of (1) vanishes and the system will be in echelon form having two equations in three variables. So it has $3 - 2 = 1$ free variable, which is z. Hence the given system has more than one solution for $\lambda = 8$ and $\mu = 8$.

i) If $\lambda = 8$ and $\mu \neq 8$ then third equation of (1) is of the form $0 = a$, where $a = \mu - 8 \neq 0$ which is not true. So the system is inconsistent. Thus the system has no solution for $\lambda = 8$ and $\mu \neq 8$. (Ans.)

Example-27 Determine the values of λ such that the following system of linear equations has (i) no solution (ii) more than one solution (iii) a unique solution.
[λ -এর এরূপ মান নির্ণয় কর যার জন্য নিম্নলিখিত একঘাত বিশিষ্ট সমীকরণ জোটের (i) সমাধান নেই (ii) একাধিক সমাধান থাকে (iii) একক সমাধান থাকে ।]

$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = 1 \\ \lambda x + y + z = 1 \end{cases}$$

[NUH 2013, 2010, '03, '00; JNUH '14; JUH '03]

Solution The given system of linear equation is

$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = 1 \\ \lambda x + y + z = 1 \end{cases}$$

Reduce the system to echelon form by means of elementary operations.

$$\sim \begin{cases} x + y + \lambda z = 1 \\ (\lambda - 1)y - (\lambda - 1)z = 0 \\ -(\lambda - 1)y - (\lambda^2 - 1)z = 1 - \lambda \end{cases} \quad \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_2 - \lambda L_1 \end{bmatrix}$$

$$\sim \begin{cases} x + y + \lambda z = 1 \\ (\lambda - 1)y - (\lambda - 1)z = 0 \\ -(\lambda^2 + \lambda - 2)z = 1 - \lambda \end{cases} \quad [L_3' = L_3 + L_2]$$

$$\sim \begin{cases} x + y + \lambda z = 1 \\ (\lambda - 1)y - (\lambda - 1)z = 0 \\ -(\lambda + 2)(\lambda - 1)z = 1 - \lambda \end{cases} \quad \text{.................. (1)}$$

This system is in echelon form. Now we consider the following three cases:

(i) From third equation of (1), we see that if $\lambda + 2 = 0$ or $\lambda = -2$ then the equation becomes $0 = 3$, which is contradiction. Therefore, the system is inconsistent if $\lambda = -2$. Thus the system has no solution for $\lambda = -2$.

(ii) We know, if the number of variables is greater than the number of equations, then the system has more than one solution.

From third equation of (1), we see that if $\lambda = 1$ then it becomes $0 = 0$. In this case the system has three variables in two equations. So the given system has more than one solution for $\lambda = 1$.

(iii) We know, if the number of variables and the number of equations be equal, then the system has unique solution. The system (1) has a unique solution if $(\lambda + 2)(\lambda - 1) \neq 0 \Rightarrow \lambda \neq -2, \lambda \neq 1$. (Ans.)