

MATRIX ALGEBRA

CHAPTER THREE

3.1 Introduction

J. J. Sylvester was the first man who introduced the word 'matrix' in 1850 and later on in 1858 **Arthur Cayley** developed the theory of matrices in a systematic way. Matrix is a powerful tool of modern mathematics which is originated in the study of linear equations, and it has wide applications in every branch of science and especially in Physics, Chemistry, Mathematics, Statistics, Economics, and Engineerings.

3.2 Definition of matrix

A **matrix** is a rectangular array of numbers (real or complex) enclosed by a pair of brackets (or double vertical rolls) and the numbers in the array are called the **entries** or the **elements** of the matrix. that is, a rectangular array of (real or complex) numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}$$

is called a **matrix**. The numbers $a_{11}, a_{12}, \dots, a_{mn}$ are called the **entries** or the **elements** of the matrix. The above matrix has m rows and n columns and is called an **(m x n) matrix** (read "m by n matrix"). The matrix of m rows and n columns is said to be of **order** "m by n" or $m \times n$. The above matrix is also denoted by $[a_{ij}]$, $i = 1, 2, \dots, m$
 $j = 1, 2, \dots, n$

The m horizontal n -tuples $(a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{m1}, a_{m2}, \dots, a_{mn})$ are the m rows of the matrix and the n vertical m -tuples

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \dots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

are its n columns. The element a_{ij} , called the **ij -entry** or **ij -component**, appears in the i th row and j th column.

A matrix consisting of a single row is called a **row matrix** (or **row vector**) and a matrix consisting of a single column is called a **column matrix** (or **column vector**).

Matrices are generally denoted by capital letters A, B, X, Y etc. Square brackets $[]$, or, curved brackets $()$, or, Two pairs of parallel lines $|| \ ||$ are used for the mathematical notations of matrices. In this book we will use the notation $[]$.

Examples of matrices

Example 1. $A = \begin{bmatrix} 1 & 0 & -5 \\ 2 & -3 & 7 \end{bmatrix}$ is a matrix of order 2×3 over the real field $\mathbb{R}$ and also over the complex field $\mathbb{C}$.

The rows of A are $(1, 0, -5)$ and $(2, -3, 7)$ and its columns are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

Example 2. $B = \begin{bmatrix} 2 & 0 & i \\ -i & 1 & 4 \\ 1+i & -5 & 3 \end{bmatrix}$ is a matrix of order 3×3 over the complex field $\mathbb{C}$.

The rows of B are $(2, 0, i)$, $(-i, 1, 4)$ and $(1+i, -5, 3)$ and its columns are $\begin{pmatrix} 2 \\ -i \\ 1+i \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} i \\ 4 \\ 3 \end{pmatrix}$.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if and only if they are identical i.e if and only if they contain the same number of rows and the same number of columns and $a_{ij} = b_{ij}$ for all values of i and j .

3.3 Addition and scalar multiplication of matrices :

Addition of matrices is defined only for the matrices having same number of rows and the same number of columns. Let A and B be the two matrices having m rows and n columns.

$$\text{i. e. } A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and}$$

$$B = [b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

Then the sum of A and B is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

The multiplication of matrix by numbers (scalars) is defined as follows : The product of an $(m \times n)$ matrix A by a number k is denoted by kA or Ak and is the $(m \times n)$ matrix obtained by multiplying every element of A by k, that is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

We also define $-A = (-1)A$ and $A - B = A + (-B)$.

If the matrices A, B, C are conformable for addition and if k is any scalar, then we can state that

- (i) $A + B = B + A$ (**Commutative law**)
- (ii) $(A + B) + C = A + (B + C)$ (**Associative law**)
- (iii) $A + 0 = 0 + A = A$
- (iv) $k(A + B) = kA + kB = (A + B)k$

where 0 is the zero matrix of the same order.

For examples, if $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -5 & 4 \end{bmatrix}$

$$\text{then } A + B = \begin{bmatrix} 1+2 & (-2)+7 \\ 3+(-5) & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 9 \end{bmatrix},$$

$$2A = \begin{bmatrix} 2 \cdot 1 & 2 \cdot (-2) \\ 2 \cdot 3 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 6 & 10 \end{bmatrix}$$

$$\text{and } A - B = \begin{bmatrix} 1-2 & -2-7 \\ 3-(-5) & 5-4 \end{bmatrix} = \begin{bmatrix} -1 & -9 \\ 8 & 1 \end{bmatrix}.$$

3.4 Matrix Multiplication

Two matrices A and B are conformable for multiplication if the number of columns in A is equal to the number of rows in B.

$$\text{Let } A = [a_1 \ a_2 \ \dots \ a_n] \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\text{The } AB = [a_1 b_1 + a_2 b_2 + \dots + a_n b_n] = \left[\sum_{i=1}^n a_i b_i \right]$$

Again, let the $m \times p$ matrix $A = [a_{ij}]$ and the $p \times n$ matrix $B = [b_{kj}]$. then AB is the $m \times n$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj} = \sum_{k=1}^p a_{ik} b_{kj}$$

$$\begin{cases} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{cases}$$

If the matrices A, B, C are conformable for the indicated sums and products, we have the following properties :

(i) $(AB)C = A(BC)$ (**Associative law**)

(ii) $A(B + C) = AB + AC$

(iii) $(A + B)C = AC + BC$ } (**Distributive laws**)

(iv) $k(AB) = (kA)B = A(kB)$ where k is any scalar.

Remarks : In the matrix product AB , the matrix A is called the **pre-multiplier** (or **pre-factor**) and B is called the **postmultiplier** (or **post-factor**)

For examples, let $A = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix}$

$$\begin{aligned} \text{then } AB &= \begin{bmatrix} 1 \cdot 1 + (-3) \cdot (-2) + 5 \cdot 3 & 1 \cdot (-1) + (-3) \cdot 4 + 5 \cdot 0 \\ 2 \cdot 1 + 0 \cdot (-2) + (-1) \cdot 3 & 2 \cdot (-1) + 0 \cdot 4 + (-1) \cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + 6 + 15 & -1 - 12 + 0 \\ 2 + 0 - 3 & -2 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 22 & -13 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 2 & 1 \cdot (-3) + (-1) \cdot 0 & 1 \cdot 5 + (-1) \cdot (-1) \\ (-2) \cdot 1 + 4 \cdot 2 & (-2) \cdot (-3) + 4 \cdot 0 & (-2) \cdot 5 + 4 \cdot (-1) \\ 3 \cdot 1 + 0 \cdot 2 & 3 \cdot (-3) + 0 \cdot 0 & 3 \cdot 5 + 0 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & 6 \\ 6 & 6 & -14 \\ 3 & -9 & 15 \end{bmatrix} \therefore AB \neq BA.$$

3.5 Transpose of a matrix

If A is an $m \times n$ matrix over the real field $\mathbb{R}$, then the $n \times m$ matrix obtained from the matrix A by writing its rows as columns and its columns as rows is called the **transpose** of A and is denoted by the symbol A^T . That is, if $A = [a_{ij}]$ is an $m \times n$ matrix then $A^T = [a_{ji}]$ is an $n \times m$ matrix.

For examples, let $A = \begin{bmatrix} 1 & 0 & 5 & -7 \\ 2 & 3 & -1 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 5 & -1 \\ -7 & 6 \end{bmatrix}$.

3.6 Complex conjugate (or conjugate) of a matrix

The **conjugate** of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$. If A is an $m \times n$ matrix over the complex field, then we say that the **conjugate** of a matrix A is the matrix $\bar{A}$ whose elements are respectively the conjugates of the elements of A . That is, if $A = [a_{ij}]$, then $\bar{A} = [\bar{a}_{ij}]$.

For examples, if $A = \begin{bmatrix} 1 & 1 & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

$$\text{then } \bar{A} = \begin{bmatrix} 1 & 1 & 1-i \\ i & 2 & 2-3i \\ 5 & 1-2i & -5 \end{bmatrix}$$

Definition Real matrix

A matrix A is called **real** provided it satisfies the relation

$$A = \bar{A}$$

Definition Imaginary matrix

A matrix A is called **imaginary** provided it satisfies the relation $A = -\bar{A}$

3.7 Conjugate transpose of a complex matrix

The conjugate of the transpose of a given complex matrix A is said to be the **conjugate transpose** of A and is generally denoted by the symbol A^* . That is,

if $A = [a_{ij}]$ is a complex matrix, then

$$A^* = (\bar{A}^T) = [\bar{a}_{ji}]$$

For examples, $A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 2+3i \\ 5 & 1+2i & -5 \end{bmatrix}$

$$\text{then } A^* = \begin{bmatrix} 1 & i & 5 \\ -i & 2 & 1-2i \\ 1-i & 2-3i & -5 \end{bmatrix}$$

$$\text{and if } B = \begin{bmatrix} 5+7i & -3i & 2-5i \\ 4i & 1+6i & 3+2i \end{bmatrix}$$

$$\text{then } B^* = \begin{bmatrix} 5-7i & -4i \\ 3i & 1-6i \\ 2+5i & 3-2i \end{bmatrix}$$

3.8 Special types of matrices with examples

Square matrix : A matrix with the same number of rows and columns is called a **square matrix**.

For examples $\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \\ 7 & -2 & 6 \end{bmatrix}$ are square matrices.

Rectangular matrix : The number of rows and columns of a matrix need not be equal.

When $m \neq n$ i.e the number of rows and columns of the array are not equal, then the matrix is known as the **rectangular matrix :**

For examples, $\begin{bmatrix} 1 & -1 & 2 & 7 \\ -2 & 3 & 5 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 2 & 0 & 5 \end{bmatrix}$

are rectangular matrices.

Diagonal matrix : A square matrix whose elements $a_{ij} = 0$ when $i \neq j$ is called a **diagonal matrix**.

For examples, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

A diagonal matrix whose diagonal elements are all equal is called a **scalar matrix**.

Identity matrix (or Unit matrix) :

A square matrix whose elements $a_{ij} = 0$, if $i \neq j$ and $a_{ij} = 1$, if $i = j$ is called the **identity matrix** or **unit matrix**.

For examples, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are identity

matrices of order 3 and 4 respectively.

Zero matrix or null matrix : A matrix in which every element is zero is called a **null matrix** or a **zero matrix**.

For examples, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

Upper and lower triangular matrices :

A square matrix whose elements $a_{ij} = 0$ for $i > j$ is called an **upper triangular matrix** and a square matrix whose elements $a_{ij} = 0$ for $i < j$ is called a **lower triangular matrix**.

For examples, $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 7 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

are upper triangular matrices.

and $\begin{bmatrix} 5 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 7 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 3 & 7 & -1 & 6 \end{bmatrix}$ are lower triangular matrices.

Symmetric matrix : A matrix equal to its transpose i.e. a square matrix such that $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$ is said to be **symmetric**. In short we can say a square matrix A will be symmetric if $A^T = A$.

For examples, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \\ -3 & 7 & 3 \end{bmatrix}$

are symmetric matrices.

λA and λB are also symmetric if λ is a scalar.

Skew-symmetric matrix : A matrix equal to the negative of its transpose i.e. a square matrix such that $a_{ij} = -a_{ji}$ and in which therefore, $a_{ii} = 0$ is said to be **skew-symmetric**.

For examples, $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

are skew-symmetric matrices.

Hermitian matrix : If $A = [a_{ij}]$ is a square matrix over the complex field and $A^* = (\bar{A}^T) = A$ i.e. $a_{ij} = \bar{a}_{ji}$ for $i, j = 1, 2, \dots, n$; then A is called a **Hermitian matrix**.

For examples, $A = \begin{bmatrix} 2 & 2-3i & 3 \\ 2+3i & 5 & 1+i \\ 3 & 1-i & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & i & -1-2i \\ -i & 3 & 5+3i \\ -1+2i & 5-3i & -1 \end{bmatrix}$ are Hermitian matrices.

In this case diagonal elements of the matrices will be real numbers.

Skew-Hermitian matrix : If $A = [a_{ij}]$ is a square matrix over the complex field, and $A^* = (\bar{A}^T) = -A$ i.e. $a_{ij} = -\bar{a}_{ji}$ for $i, j = 1, 2, \dots, n$ then A is called a **skew-Hermitian matrix**.

For examples, $A = \begin{bmatrix} 2i & 2-3i & 3 \\ -2-3i & 5i & 1+i \\ -3 & -1+i & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} i & 1-i & 5 \\ -1-i & 2i & i \\ -5 & i & 0 \end{bmatrix}$ are skew-Hermitian matrices.

In this case diagonal elements of the matrix will be either zero or wholly complex number.

Orthogonal matrix : A real square matrix A is said to be **orthogonal** if $AA^T = A^T A = I$.

that is, if $A^T = A^{-1}$ (the inverse of the matrix A)

For examples, $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & \frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$ and

$B = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ are orthogonal matrices.

Idempotent matrix : A square matrix A is called an **idempotent matrix** if $A^2 = A$.

For examples, $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ are idempotent matrices.

Nilpotent matrix : A square matrix A is called a **nilpotent matrix** of order n if $A^n = 0$ and $A^{n-1} \neq 0$ where n is a positive integer and 0 is the null matrix.

For examples, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

are nilpotent matrices of order 2.

Periodic matrix : A square matrix A is called **periodic** if $A^{m+1} = A$ where m is a positive integer.

If m is the least positive integer for which $A^{m+1} = A$, then A is said to be of **period** m .

For example, $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is a periodic matrix of period 2.

Involutory matrix : A square matrix A is called an **involutory matrix** if $A^2 = I$.

For example, $A = \begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$ is an involutory matrix.

Unitary matrix : Let A be a complex square matrix, then A is called a **unitary matrix** if $AA^* = A^*A = I$ or equivalently $A^* = A^{-1}$ where $A^* = (\bar{A})^T = (\bar{A}^T)$.

For example, $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ is a unitary matrix.

Normal matrix : Let A be a complex square matrix, then A is called a **normal matrix** if $A^*A = AA^*$ where A^* is the conjugate transpose of A .

For example, $A = \begin{bmatrix} 2+3i & 1 \\ i & 1+2i \end{bmatrix}$ is a normal matrix.

Example: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$$\text{then } AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = BA.$$

Therefore A and B are invertible matrices.

1.8.2 Inverse matrix

বিপরীত ম্যাট্রিক্স

[DUH 2016]

Let A be a square matrix of order n . If A is invertible i.e., there exists a unique matrix B such that $AB = BA = I$, where I is the unit matrix, then the matrix B is called inverse of A and is generally denoted by A^{-1} and we write $B = A^{-1}$. In case of inverse matrix if B is the inverse of A , then A is also the inverse of B . [মনে করি A একটি n ক্রমের বর্গ ম্যাট্রিক্স। যদি A বিপরীতযোগ্য হয় অর্থাৎ, যদি একটি অনন্য ম্যাট্রিক্স B থাকে যেন $AB = BA = I$ হয়, যেখানে I অভেদ ম্যাট্রিক্স। তবে ম্যাট্রিক্স B কে A এর বিপরীত ম্যাট্রিক্স বলে এবং একে সাধারণত A^{-1} দ্বারা প্রকাশ করা হয় এবং লেখা হয় $B = A^{-1}$ । বিপরীত ম্যাট্রিক্সের ক্ষেত্রে যদি A এর বিপরীত ম্যাট্রিক্স B হয়, তবে B এর বিপরীত ম্যাট্রিক্স A হবে।]

Example: Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$

$$\text{Then } AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -5+6 & 3-3 \\ -10+10 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{and } BA = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -5+6 & -\frac{15}{2}+\frac{15}{2} \\ 4-4 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{i.e., } AB = BA = I$$

Therefore, the matrices A and B are invertible and they are inverses of each other. So that, $A^{-1} = B$ and $B^{-1} = A$.

1.8.3 Minors of a Matrix

ম্যাট্রিক্সের অনুরাশি

Consider a square matrix A of order 3, where [একটি 3 ক্রমের বর্গ ম্যাট্রিক্স A বিবেচনা করা যাক, যেখানে]

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

If we delete any one row and one column of A , then we get a 2×2 submatrix of A . For example, if we delete the first row and first column, we get the submatrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ of A এর একটি 2×2 কলাম মুছে ফেলি বা বাদ দিই তবে আমরা কলাম মুছে ফেলি তবে আমরা উপ-ম্যাট্রিক্স পাই। উদাহরণস্বরূপ, যদি আমরা প্রথম সারি এবং প্রথম কলাম মুছে ফেলি তবে আমরা উপ-ম্যাট্রিক্স পাই।

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

(The determinant of any such submatrix is called a minor of determinant A). [এ ধরনের যে কোন উপ-ম্যাট্রিক্সের নির্ণায়ককে A নির্ণায়কের অনুরাশি বলে।]

Thus $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ is a minor of determinant of A .

[সুতরাং $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ নির্ণায়ক A এর একটি অনুরাশি।]

The minors of $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ and a_{33} in $|A|$ are respectively, $[|A|$ এর $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}$ এবং a_{33} এর অনুরাশি যথাক্রমে,]

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

In general, the minor obtained by striking off the i -th row and j -th column of a matrix $A = [a_{ij}]_{n \times n}$ is called the minor of a_{ij} in $|A|$ and the minor of element a_{ij} is denoted by M_{ij} .

[সাধারণত, ম্যাট্রিক্স $A = [a_{ij}]_{n \times n}$ এর i -তম সারি ও j -তম কলাম বাদ দিলে যে অনুরাশি পাওয়া যায়, তাকে $|A|$ এর a_{ij} এর অনুরাশি বলে এবং a_{ij} এর অনুরাশিকে M_{ij} দ্বারা প্রকাশ করা হয়।]

1.8.4 Co-factors of a Matrix

ম্যাট্রিক্সের সহগুণক

If we multiply the minor of the element in the i -th row and j -th column of the determinant of the matrix by $(-1)^{i+j}$ then the product is called the co-factor of the element. Co-factor of an element usually denoted by the corresponding capital letters, symbolically

$$A_{ij} = (-1)^{i+j} \times \text{minor of } a_{ij} \text{ in } |A| = (-1)^{i+j} |M_{ij}|$$

[যদি আমরা ম্যাট্রিক্সের নির্ণায়কের i -তম সারি ও j -তম কলামের উপাদানের অনুরাশিকে $(-1)^{i+j}$ দ্বারা গুণ করি, তবে গুণফলকে ঐ উপাদানের সহগুণক বলা হয়। একটি উপাদানের সহগুণককে সচরাচর সংশ্লিষ্ট বড় হাতের অক্ষর দ্বারা প্রকাশ করা হয়। সাংকেতিকভাবে,

$$A_{ij} = (-1)^{i+j} \times |A| \text{ এ } a_{ij} \text{ এর অনুরাশি} = (-1)^{i+j} |M_{ij}|$$

For example, if $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{and } A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example-4 If $A = \begin{bmatrix} 3 & 4 & 7 \\ -2 & 5 & 6 \\ 7 & 3 & -9 \end{bmatrix}$ find the co-factors of elements 6, -9. [6, -9]

উপাদানের সহগণক নির্ণয় কর।

Solution The co-factor of element a_{23} , i.e., 6 is

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 4 \\ 7 & 3 \end{vmatrix} = -(9 - 28) = 19 \text{ (Ans.)}$$

The co-factor of the element A_{33} , i.e., -9 is

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} = (15 + 8) = 23 \text{ (Ans.)}$$

1.8.5 Adjoint or Adjugate of a square matrix

বর্গাকার ম্যাট্রিক্সের সংযুক্ত ম্যাট্রিক্স

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n , then transpose of the matrix $[A_{ij}]$ is called adjoint of A , where A_{ij} is co-factor of a_{ij} in $|A|$. [মনে করি, $A = [a_{ij}]_{n \times n}$ একটি n ক্রমের বর্গ ম্যাট্রিক্স, তাহলে ম্যাট্রিক্স $[A_{ij}]_{n \times n}$ এর রূপান্তরিত ম্যাট্রিক্সকে A এর সংযুক্ত ম্যাট্রিক্স বলা হয়।]

$$= \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix} \text{ (Ans.)}$$

Example-10 If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix}$ and $k_1 = i, k_2 = 2$, verify $(k_1 + k_2)A = k_1A + k_2A$.

Solution Now $k_1A = i \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & i & 2i \\ 2i & 3i & 4i \\ 4i & 5i & 6i \end{bmatrix}$

and $k_2A = 2 \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 6 & 8 \\ 8 & 10 & 12 \end{bmatrix}$

So, $k_1A + k_2A = \begin{bmatrix} 0 & 2+i & 4+2i \\ 4+2i & 6+3i & 8+4i \\ 8+4i & 10+5i & 12+6i \end{bmatrix}$

Also, $(k_1 + k_2)A = (i + 2) \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 2+i & 4+2i \\ 4+2i & 6+3i & 8+4i \\ 8+4i & 10+5i & 12+6i \end{bmatrix}$

Therefore, $(k_1 + k_2)A = k_1A + k_2A$ [verified]

Example-11 If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$, find AB and BA . Is $AB = BA$?

[যদি $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ এবং $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$ হয়, তবে AB এবং BA নির্ণয় কর। $AB = BA$ কি সত্যিকার?

Solution Here,

$$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 5 \times (-3) & 2 \times (-1) + 5 \times 2 \\ 1 \times 1 + 3 \times (-3) & 1 \times (-1) + 3 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix} \text{ (Ans.)}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + (-1) \times 1 & 1 \times 5 + (-1) \times 3 \\ (-3) \times 2 + 2 \times 1 & (-3) \times 5 + 2 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -4 & -9 \end{bmatrix} \text{ (Ans.)}$$

Thus, $AB \neq BA$.

Example-12 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, then show that $A(B + C) = AB + AC$.

Solution Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$\text{Here, } B + C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\text{L.H.S.} = A(B + C)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & -1+4 & 1+10 \\ 6+0 & -3+8 & 3+20 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 11 \\ 6 & 5 & 23 \end{bmatrix}$$

$$\text{Again } AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+2 & 1+4 \\ 3+0 & 0+4 & 3+8 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 11 \end{bmatrix}$$

$$\text{and } AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+2 & 0+6 \\ 3+0 & -3+4 & 0+12 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 6 \\ 3 & 1 & 12 \end{bmatrix}$$

$$\text{R.H.S.} = AB + AC$$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 11 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 6 \\ 3 & 1 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 11 \\ 6 & 5 & 23 \end{bmatrix}$$

Hence, $A(B + C) = AB + AC$ [Showed]

Example-13 If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix}$ then, prove

that $(AB)C = A(BC)$.

Solution Here, $AB = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1+0+1 & 2+2+0 \\ 0+0+4 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = (AB)C = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2-4 & 10-8 \\ 4-1 & 20-2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 3 & 18 \end{bmatrix}$$

$$\text{Again } BC = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1-2 & 5-4 \\ 0-1 & 0-2 \\ 1-0 & 5+0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \\ 1 & 5 \end{bmatrix}$$

$$\begin{aligned} \text{R.H.S.} = A(BC) &= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -2 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -1-2+1 & 1-4+5 \\ 0-1+4 & 0-2+20 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 3 & 18 \end{bmatrix} \end{aligned}$$

Hence, $(AB)C = A(BC)$ [Proved]

Example-14 Obtain the product : [गुणफल निर्णय कर :]

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$$

Solution Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3 \times 3}$, $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}_{3 \times 4}$

Since A is 3×3 and B is 3×4 , product AB is valid and AB is 3×4 .

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2+2+0 & 4+0+0 & 6+1+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 9+2+0 & 12+4+5 \\ 1+0+3 & 2+0+1 & 3+0+0 & 4+0+5 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix} \text{ (Ans.)} \end{aligned}$$

Example-15 Find (a) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (b) $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Solution (a) $\begin{bmatrix} x & y \end{bmatrix} \left\{ \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\}$

$$\begin{aligned} &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + hy \\ hx + by \end{bmatrix} \\ &= [x(ax + hy) + y(hx + by)] \\ &= ax^2 + hxy + hxy + by^2 = ax^2 + 2hxy + by^2. \text{ (Ans.)} \end{aligned}$$

(b) $\begin{bmatrix} x & y & z \end{bmatrix} \left\{ \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}$$

$$\begin{aligned}
 &= [x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)] \\
 &= ax^2 + hxy + gxz + hxy + by^2 + fyz + gzx + fyz + cz^2 \\
 &= ax^2 + by^2 + cz^2 + 2(hxy + fyz + gzx). \text{ (Ans.)}
 \end{aligned}$$

Example-16 If $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, then find the matrix X such that $3A + 5B + 2X = 0$. [যদি $A = \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$ হয়, তবে X এর মান বের কর যেন, $3A + 5B + 2X = 0$ হয়।]

Solution $3A + 5B + 2X = 0 \Rightarrow X = -\frac{1}{2}[3A + 5B]$

$$\begin{aligned}
 \Rightarrow X &= -\frac{1}{2} \left\{ 3 \begin{bmatrix} 9 & 1 \\ 4 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} \right\} \\
 &= -\frac{1}{2} \left\{ \begin{bmatrix} 27 & 3 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 25 \\ 35 & 60 \end{bmatrix} \right\} \\
 &= -\frac{1}{2} \begin{bmatrix} 27+5 & 3+25 \\ 12+35 & 9+60 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{32}{2} & -\frac{28}{2} \\ -\frac{47}{2} & -\frac{69}{2} \end{bmatrix} = \begin{bmatrix} -16 & -14 \\ -\frac{47}{2} & -\frac{69}{2} \end{bmatrix} \text{ (Ans.)}
 \end{aligned}$$

Example-17 If $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$

(a) Find a 2×4 matrix X such that $A - X = 3B$. [2×4 ক্রমের X ম্যাট্রিক্স বের কর যেন, $A - X = 3B$ হয়।]

(b) Find a 2×4 matrix Y such that $A + 2Y = 4B$. [2×4 ক্রমের Y ম্যাট্রিক্স বের কর যেন, $A + 2Y = 4B$ হয়।]

Solution (a) $A - X = 3B \Rightarrow X = A - 3B$

$$\begin{aligned}
 \Rightarrow X &= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 0 & 9 \\ 3 & -3 & 6 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} 1-6 & 2-3 & 0-0 & 4-9 \\ 2-3 & 4+3 & -1-6 & 3-9 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{bmatrix} \text{ (Ans.)}
 \end{aligned}$$

(b) $A + 2Y = 4B \Rightarrow Y = 2B - \frac{1}{2}A$

$$\begin{aligned}
 \Rightarrow Y &= 2 \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 2 & 0 & 6 \\ 2 & -2 & 4 & 6 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{2}{2} & 0 & \frac{4}{2} \\ \frac{2}{2} & \frac{4}{2} & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \\
 &= \begin{bmatrix} 4 - \frac{1}{2} & 2 - 1 & 0 - 0 & 6 - 2 \\ 2 - 1 & -2 - 2 & 4 + \frac{1}{2} & 6 - \frac{3}{2} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{7}{2} & 1 & 0 & 4 \\ 1 & -4 & \frac{9}{2} & \frac{9}{2} \end{bmatrix} \text{ (Ans.)}
 \end{aligned}$$

Example-18 Find the values of x and y , where $[x \text{ ও } y \text{ এর মান নির্ণয় কর, যেখানে}]$

$$x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \text{ and } x - y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution Given, $x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ (1)

and $x - y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (2)

Adding (1) & (2), we get $2x = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}$

$$\Rightarrow x = \frac{1}{2} \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Subtract (2) from (1), we get $2y = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix}$

$$\Rightarrow y = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

(Ans.) $x = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

~~plz~~ 2 If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then, show that, $A^3 = A^2A = AA^2 = I$

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on Here, $A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0-2+2 & 0+1+0 & 0+0+0 \\ 1-2+1 & -1+1+0 & 1+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & 0+1-1 & 1-2+1 \\ 0+0+0 & 0+1+0 & 2-2+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore A^3 = A^2A = AA^2 = I. \text{ [Showed]}$$

Example-45 If $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ then, find the value of $A^3 - 2A^2 - 9A$. [CUH '83]

Solution $A^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 11 \\ 2 & 9 & 1 \\ 5 & -1 & 8 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 8 & 6 & 11 \\ 2 & 9 & 1 \\ 5 & -1 & 8 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & 18 & 47 \\ 14 & -14 & 25 \\ 17 & 23 & 21 \end{bmatrix}$

Now, $A^3 - 2A^2 - 9A = \begin{bmatrix} 33 & 18 & 47 \\ 14 & -14 & 25 \\ 17 & 23 & 21 \end{bmatrix} - 2 \begin{bmatrix} 8 & 6 & 11 \\ 2 & 9 & 1 \\ 5 & -1 & 8 \end{bmatrix} - 9 \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 33 - 16 - 18 & 18 - 12 - 9 & 47 - 22 - 27 \\ 14 - 4 - 9 & -14 - 18 + 18 & 25 - 2 - 18 \\ 17 - 10 - 9 & 23 + 2 - 18 & 21 - 16 - 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & -2 \\ 1 & -14 & 5 \\ -2 & 7 & -4 \end{bmatrix} \text{ (Ans.)}$$

Example-46 If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ then find the value of $A^2 - 3A + 9I_3$.

[NUH 2010]

Solution $A^2 = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 - 4 - 9 & -2 - 6 + 3 & 3 + 2 + 6 \\ 2 + 6 + 3 & -4 + 9 - 1 & 6 - 3 - 2 \\ -3 + 2 - 6 & 6 + 3 + 2 & -9 - 1 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$\therefore A^2 - 3A + 9I_3 = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - 3 \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 - 3 + 9 & -5 + 6 + 0 & 11 - 9 + 0 \\ 11 - 6 + 0 & 4 - 9 + 9 & 1 + 3 + 0 \\ -7 + 9 + 0 & 11 - 3 + 0 & -6 - 6 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix} \text{ (Ans.)}$$

Example-47 If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then show that $A^3 - 3A^2 - A + 9I = 0$.

Solution

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 1 + 2 \times 0 + 1 \times 3 & 1 \times 2 + 2 \times 1 + 1 \times (-1) & 1 \times 1 + 2 \times (-1) + 1 \times 1 \\ 0 \times 1 + 1 \times 0 + (-1) \times 3 & 0 \times 2 + 1 \times 1 + (-1) \times (-1) & 0 \times 1 + 1 \times (-1) + (-1) \times 1 \\ 3 \times 1 + (-1) \times 0 + 1 \times 3 & 3 \times 2 + (-1) \times 1 + 1 \times (-1) & 3 \times 1 + (-1) \times (-1) + 1 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} \end{aligned}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix}$$

$$\begin{aligned} \text{Now } A^3 - 3A^2 - A + 9I &= \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix} - \begin{bmatrix} 12 & 9 & 0 \\ -9 & 6 & -6 \\ 18 & 12 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ [Showed]} \end{aligned}$$

Solution Given, $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 0 \end{bmatrix}$

$$\therefore A^T = \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 6 & 0 \end{bmatrix}$$

$$\text{Now, } A + B^T = \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 5 & 5 \\ 7 & 0 \end{bmatrix}$$

$$\text{L.H.S.} = A^T + B$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 7 \\ 7 & 5 & 0 \end{bmatrix}$$

$$\text{and R.H.S.} = (A + B^T)^T$$

$$= \begin{bmatrix} 2 & 7 \\ 5 & 5 \\ 7 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 5 & 7 \\ 7 & 5 & 0 \end{bmatrix}$$

$$\text{Hence, } A^T + B = (A + B^T)^T \text{ [Proved]}$$

Example-50 If $A = \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}$ then prove that $(A + B)^T = A^T + B^T$

Solution Given that,

$$A = \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, B = \begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}$$

$$\text{Now } A + B = \begin{bmatrix} 3+7 & 4+8 \\ 6+4 & 2+3 \end{bmatrix} = \begin{bmatrix} 10 & 12 \\ 10 & 5 \end{bmatrix}$$

$$\therefore (A + B)^T = \begin{bmatrix} 10 & 10 \\ 12 & 5 \end{bmatrix}$$

$$\text{Again } A^T = \begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 7 & 4 \\ 8 & 3 \end{bmatrix}$$

$$\therefore A^T + B^T = \begin{bmatrix} 3+7 & 6+4 \\ 4+8 & 2+3 \end{bmatrix} = \begin{bmatrix} 10 & 10 \\ 12 & 5 \end{bmatrix}$$

$$\therefore (A + B)^T = A^T + B^T \text{ [Proved]}$$

Example-51 Let, $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 3 & -5 \end{bmatrix}$
 Show that $[A + B]^T = A^T + B^T$ [দেখাও যে, $[A + B]^T = A^T + B^T$].

Solution $A + B = \begin{bmatrix} 2+3 & (-3)+(-2) & 1+4 \\ 4+1 & 2+3 & 3+(-5) \end{bmatrix} = \begin{bmatrix} 5 & -5 & 5 \\ 5 & 5 & -2 \end{bmatrix}$
 $\therefore [A + B]^T = \begin{bmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{bmatrix} \dots\dots\dots (1)$

Now $A^T = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 3 & 1 \\ -2 & 3 \\ 4 & -5 \end{bmatrix}$
 $\therefore A^T + B^T = \begin{bmatrix} 2+3 & 4+1 \\ (-3)+(-2) & 2+3 \\ 1+4 & 3+(-5) \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{bmatrix} \dots\dots\dots (2)$

Hence from (1) and (2), we have $[A + B]^T = A^T + B^T$. [Showed]

Example-52 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$. Prove that, $(AB)^T = B^T A^T$. [NUH 2011]

Solution Given, $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$ and $B^T = \begin{bmatrix} 0 & 1 \\ 5 & 7 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 0+2 & 5+14 \\ 0-4 & 15-28 \end{bmatrix} = \begin{bmatrix} 2 & 19 \\ -4 & -13 \end{bmatrix}$

$\therefore (AB)^T = \begin{bmatrix} 2 & -4 \\ 19 & -13 \end{bmatrix} \dots\dots\dots (1)$

Again, $B^T A^T = \begin{bmatrix} 0 & 1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 0+2 & 0-4 \\ 5+14 & 15-28 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 19 & -13 \end{bmatrix} \dots\dots\dots (2)$

From (1) & (2), we get $(AB)^T = B^T A^T$ [Proved]

$$\therefore \text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Hence } A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \text{ (Ans.)}$$

Example-55 Find the inverse of the matrix. [ম্যাট্রিক্সটির বিপরীত ম্যাট্রিক্স নির্ণয় কর:]

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution Let D be the determinant of the matrix; then $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$\neq 0$. So the matrix A is non-singular and hence A^{-1} exists.

Now the cofactors of D are

$$A_{11} = 4, \quad A_{12} = -3$$

$$A_{21} = -2, \quad A_{22} = 1$$

$$\text{Then } \text{Adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj } A = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \text{ (Ans.)}$$

Example-56 Find the adjoint of the matrix [ম্যাট্রিক্সটির সংযুক্ত ম্যাট্রিক্স বের কর:]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

and verify the theorem. [এবং উপপাদ্যটির সত্যতা প্রমাণ কর:]

$$A (\text{Adj } A) = (\text{Adj } A) A = |A| I_3$$

Solution $\text{Adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

where, $A_{11} = \text{co-factor of } a_{11} \text{ in } |A| = (-1)^{1+1} \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = 3$

$$A_{12} = \text{co-factor of } a_{12} \text{ in } |A| = (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = -9$$

$$A_{13} = \text{co-factor of } a_{13} \text{ in } |A| = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$$

$$\text{Similarly, } A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = -4, A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3, A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = 4, A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$\text{Therefore, } \text{Adj } A = \begin{bmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$\begin{aligned} \text{Also } |A| &= 1(6-3) - 1(3+6) + 1(-1-4) \\ &= 3 - 9 - 5 \\ &= -11 \end{aligned}$$

$$\begin{aligned} \text{Now } A(\text{Adj } A) &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3-9-5 & -4+1+3 & -5+4+1 \\ 3-18+15 & -4+2-9 & -5+8-3 \\ 6+9-15 & -8-1+9 & -10-4+3 \end{bmatrix} \\ &= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} = -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= |A| I_3 \dots\dots\dots (1) \end{aligned}$$

$$\begin{aligned} \text{Also } (\text{Adj } A) A &= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3-4-10 & 3-8+5 & 3+12-15 \\ -9+1+8 & -9+2-4 & -9-3+12 \\ -5+3+2 & -5+6-1 & -5-9+3 \end{bmatrix} \\ &= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} = |A| I_3 \dots\dots\dots (2) \end{aligned}$$

From (1) and (2), we get

$$A (\text{Adj } A) = (\text{Adj } A) A = |A| I_3. \text{ [Proved]}$$

Example-57 Compute the inverse of the matrix [ম্যাট্রিক্সটির বিপরীত ম্যাট্রিক্স নির্ণয় কর:]

$$A = \begin{bmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

Solution We know, $A^{-1} = \frac{\text{Adj } A}{|A|}$

$$|A| = \begin{vmatrix} 1 & 0 & -4 \\ -2 & 2 & 5 \\ 3 & -1 & 2 \end{vmatrix} = 1(4 + 5) - 0(-4 - 15) - 4(2 - 6) = 25 \neq 0$$

So the matrix is non-singular and hence A^{-1} exists.

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

The co-factors of the elements of A are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} = 9, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & 5 \\ 3 & 2 \end{vmatrix} = 19$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} -2 & 2 \\ 3 & -1 \end{vmatrix} = -4, \quad A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & -4 \\ -1 & 2 \end{vmatrix} = 4$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -4 \\ 3 & 2 \end{vmatrix} = 14, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = 1$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & -4 \\ 2 & 5 \end{vmatrix} = 8, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -4 \\ -2 & 5 \end{vmatrix} = 3$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} = 2$$

$$\therefore \text{Adj } A = \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}$$

10+12=22

$$\text{Thus } A^{-1} = \frac{1}{25} \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{25} & \frac{4}{25} & \frac{8}{25} \\ \frac{19}{25} & \frac{14}{25} & \frac{3}{25} \\ -\frac{4}{25} & \frac{1}{25} & \frac{2}{25} \end{bmatrix} \text{ (Ans.)}$$

Example-60 Is each matrix has inverse matrix? Give reason. [প্রত্যেক ম্যাট্রিক্সের বিপরীত ম্যাট্রিক্স আছে কি? কারণ দর্শাও।]

Find the inverse matrix of $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$ [ম্যাট্রিক্সটির বিপরীত ম্যাট্রিক্স নির্ণয় কর।]

[NUH '00]

Solution Every matrix has no inverse, we know inverse formula of a matrix A

$$\text{is } = \frac{1}{|A|} \text{Adj } A.$$

Therefore, if $|A| = 0$ i.e., A is singular then the matrix does not exists inverse.

$$\text{Given, } A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$$

$$\text{Now, } |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = (-1).5 + 2.(-10) - 3.(-8) = -1 \neq 0$$

So that the given matrix is non-singular and its inverse exists.

$$\text{Now, } \text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T \dots\dots\dots (1)$$

$$\text{Here, } A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5, \quad A_{12} = - \begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10, \quad A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$$

$$A_{21} = - \begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4, \quad A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7, \quad A_{23} = - \begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$$

$$A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3, \quad A_{32} = - \begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6, \quad A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$$

$$\therefore \text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|} = - \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix} \text{ (Ans.)}$$

$$\text{and } B^{-1}A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \dots\dots\dots (2)$$

From (1) & (2), $(AB)^{-1} = B^{-1}A^{-1}$ [Proved]

Example-63 If $A^2 = I$ then what is A^{-1} ? If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then find A^3 and

A^{-1} . [$A^2 = I$ হলে A^{-1} কি? যদি $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ হয়, তবে A^3 এবং A^{-1} নির্ণয় কর।]

Solution Given, $A^2 = I$

$$\Rightarrow A.A = I$$

$$\Rightarrow A.AA^{-1} = IA^{-1}$$

$$\Rightarrow AI = A^{-1}$$

$$\Rightarrow A = A^{-1}$$

Hence, A and A^{-1} are same.

Given matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Again, $A^3 = A^2 . A$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow A^{-1} A^3 = A^{-1} I$$

$$\Rightarrow A^2 = A^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

Therefore, if $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ (Ans.)

Thus A

$$= \begin{bmatrix} 8 & -5 & -10 & -7 & -18 \\ -10 & 0 & -15 & 10 & -14 & 12 \\ 20 & 0 & 30 & 8 & -12 & 10 \\ 16 & 0 & 25 & -8 & -6 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & -3 & 18 \\ 50 & 8 & -35 \\ 41 & 6 & -29 \end{bmatrix} \text{ (Ans.)}$$

Example-66 If $A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ then prove that $A^{-1} = A^T$, A^T being transpose of A.

Solution Given $A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, $\therefore A^T = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$$\text{Now, } AA^T = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \cdot \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81} \begin{bmatrix} 64 + 1 + 16 & -32 + 4 + 28 & -8 - 8 + 16 \\ -32 + 4 + 28 & 16 + 16 + 49 & 4 - 32 + 28 \\ -8 - 8 + 16 & 4 - 32 + 28 & 1 + 64 + 16 \end{bmatrix}$$

$$= \frac{1}{81} \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or, $AA^T = I$

$\therefore A^T = A^{-1}$ [Proved]

Example-67 If $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, then show that $A \times A^{-1} = I$ (Identity matrix) [যদি $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ হয়, তখন দেখাও যে, $A \times A^{-1} = I$ (আইডেনটিটি ম্যাট্রিক্স)]

Solution Given, $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

Here, $|A| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 - 4 = 2 \neq 0$

$\therefore A^{-1} = \frac{\text{Adj } A}{|A|}$ exists

Matrix Algebra

The co-factors of the elements of A are

$$A_{11} = (-1)^{1+1} \cdot 2 = 2, \quad A_{12} = (-1)^{1+2} \cdot 4 = -4$$

$$A_{21} = (-1)^{2+1} \cdot 1 = -1, \quad A_{22} = (-1)^{2+2} \cdot 3 = 3$$

$$\therefore \text{Adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

$$\text{Now } A \cdot A^{-1} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times 1 + 1 \times (-2) & 3 \times (-\frac{1}{2}) + 1 \times \frac{3}{2} \\ 4 \times 1 + 2 \times (-2) & 4 \times (-\frac{1}{2}) + 2 \times \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 2 & -\frac{3}{2} + \frac{3}{2} \\ 4 - 4 & -2 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \text{ [Identity matrix]}$$

$$\therefore A \cdot A^{-1} = I \text{ [Showed]}$$

Applying $R_2 \rightarrow 9R_2$, we have,

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{25} & \frac{18}{25} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

Applying $R_1 \rightarrow R_1 - 3R_2$, we have, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

So, the desired inverse is $\begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$ (Ans.)

Example-73 Find the inverse of the matrix [ম্যাট্রিক্সটির বিপরীত ম্যাট্রিক্স নির্ণয় কর:]

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

by using row canonical form. [row canonical পদ্ধতি ব্যবহারের দ্বারা।]

Solution Let $[A:I_2] = \begin{bmatrix} 2 & 5 & : & 1 & 0 \\ 1 & 3 & : & 0 & 1 \end{bmatrix}$

Reducing the system to row canonical form by using elementary row operation

$$\sim \begin{bmatrix} 1 & 3 & : & 0 & 1 \\ 2 & 5 & : & 1 & 0 \end{bmatrix} [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 3 & : & 0 & 1 \\ 0 & -1 & : & 1 & -2 \end{bmatrix} [R_2' = R_2 - 2R_1]$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 & -5 \\ 0 & -1 & : & 1 & -2 \end{bmatrix} [R_1' = R_1 + 3R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & : & 3 & -5 \\ 0 & 1 & : & -1 & 2 \end{bmatrix} [R_2' = (-1)R_2]$$

$$= [I_2 A^{-1}]$$

Hence A is invertible and $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ (Ans.)

Example-74 Find the inverse of the following matrix by using row canonical form. [row canonical পদ্ধতি ব্যবহারের মাধ্যমে নিচের ম্যাট্রিক্সটির বিপরীত ম্যাট্রিক্স নির্ণয় কর]

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution Let $[A: I_3] = \begin{bmatrix} 3 & 4 & -1 & : & 1 & 0 & 0 \\ 1 & 0 & 3 & : & 0 & 1 & 0 \\ 2 & 5 & -4 & : & 0 & 0 & 1 \end{bmatrix}$

Reducing the system to row canonical form by using elementary row operation

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 3 & 4 & -1 & : & 1 & 0 & 0 \\ 2 & 5 & -4 & : & 0 & 0 & 1 \end{bmatrix} [R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 0 & 4 & -10 & : & 1 & -3 & 0 \\ 0 & 5 & -10 & : & 0 & -2 & 1 \end{bmatrix} \begin{cases} [R_2' = R_2 - 3R_1] \\ [R_3' = R_3 - 2R_1] \end{cases}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 0 & -1 & 0 & : & 1 & -1 & -1 \\ 0 & 5 & -10 & : & 0 & -2 & 1 \end{bmatrix} [R_2' = R_2 - R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 \\ 0 & 5 & -10 & : & 0 & -2 & 1 \end{bmatrix} [R_2' = (-1)R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 \\ 0 & 0 & -10 & : & 5 & -7 & -4 \end{bmatrix} [R_3' = R_3 - 5R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & : & 0 & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 \\ 0 & 0 & 1 & : & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix} [R_3' = \left(-\frac{1}{10}\right)R_3]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & : & -1 & 1 & 1 \\ 0 & 0 & 1 & : & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix} [R_1' = R_1 - 3R_3]$$

$$= [I_3 A^{-1}]$$

The above system is in echelon form and it has three equations in four unknowns. So the system has $(4-3)=1$ free variable.

Let c be the free variable and $c=t, t \in \mathbb{R}$. Then, we obtain $b=7t, a=10t+1$ and $d=0$

$\therefore a=10t+1, b=7t, c=t$ and $d=0$, where $t \in \mathbb{R}$. (Ans.)

Example-76 Find the symmetric and skew-symmetric parts of the matrix
[ম্যাট্রিক্সটির প্রতিসম ও স্কিউ-প্রতিসম অংশ বের কর:]

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Solution Given $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} \therefore A^T = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$

The symmetric part of $A = \frac{1}{2}(A + A^T)$

$$\begin{aligned} &= \frac{1}{2} \left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} 2 & 8 & 7 \\ 8 & 16 & 6 \\ 7 & 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix} \text{ (Ans.)} \end{aligned}$$

The skew-symmetric part of $A = \frac{1}{2}(A - A^T)$

$$\begin{aligned} &= \frac{1}{2} \left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -4 \\ -1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix} \text{ (Ans.)} \end{aligned}$$

Example-77 Determine the value of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

On transposing A , we have $A^T = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

If A is orthogonal, then $AA^T = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$4\beta^2 + \gamma^2 = 1 \dots\dots\dots (1)$$

$$2\beta^2 - \gamma^2 = 0 \dots\dots\dots (2)$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \dots\dots\dots (3)$$

Adding (1) & (2), we get $\beta = \pm \frac{1}{\sqrt{6}}$

(1) - 2 × (2), we get $\gamma = \pm \frac{1}{\sqrt{3}}$

Also from (3), we get $\alpha^2 = 1 - \beta^2 - \gamma^2$

$$\Rightarrow \alpha^2 = 1 - \frac{1}{6} - \frac{1}{3}$$

$$\therefore \alpha = \pm \frac{1}{\sqrt{2}}$$

Hence, $\alpha = \pm \frac{1}{\sqrt{2}}$, $\beta = \pm \frac{1}{\sqrt{6}}$, $\gamma = \pm \frac{1}{\sqrt{3}}$ (Ans.)

Example-78 Prove that [প্রমাণ কর যে,]

$A = \frac{1}{6} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$ is an orthogonal matrix [একটি লম্ব ম্যাট্রিক্স।] [KUH 2006]

Solution Given $A = \frac{1}{6} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix} \therefore A^T = \frac{1}{6} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$

$$\begin{aligned} \text{Now } AA^T &= A^T A = \frac{1}{36} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix} \\ &= \frac{1}{36} \begin{bmatrix} 9+9+9+9 & 9-15+3+3 & 9+3+3-15 & 9+3-15+3 \\ 9-15+3+3 & 9+25+1+1 & 9-5+1-5 & 9-5-5+1 \\ 9+3+3-15 & 9-5+1-5 & 9+1+1+25 & 9+1-5-5 \\ 9+3-15+3 & 9-5-5+1 & 9+1-5-5 & 9+1+25+1 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{36} \begin{bmatrix} 36 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $AA^T = A^T A = I$

Hence A is orthogonal. [Proved]

Example-79 If $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$ and $B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$ then prove that

$$\overline{A+B} = \overline{A} + \overline{B}.$$

Solution Given, $A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}$ and $B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}$

$$\overline{A} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix} \text{ and } \overline{B} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$$

$$\therefore \overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \dots\dots\dots (1)$$

$$\text{Again } A+B = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

$$\therefore \overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix} \dots\dots\dots (2)$$

Thus from (1) and (2), we get

$$\overline{A+B} = \overline{A} + \overline{B} \text{ [Proved]}$$

Example-80 Show that $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ is an idempotent matrix.

Solution Here, $A^2 = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 4+3-5 & -6-12+15 & -10-15+20 \\ -2-4+5 & 3+16-15 & 5+20-20 \\ 2+3-4 & -3-12+12 & -5-15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A$$

That is $A^2 = A$.

Hence, A is an idempotent matrix [Shown]

Example-81 Prove that $A = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$ is Hermitian.

Solution Given, $A = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix}$

$$\therefore \overline{A} = \begin{bmatrix} 2 & 2+3i & 3-5i \\ 2-3i & 3 & -i \\ 3+5i & i & 5 \end{bmatrix}$$

$$A^* = \bar{A}^T = \begin{bmatrix} 2 & 2-3i & 3+5i \\ 2+3i & 3 & i \\ 3-5i & -i & 5 \end{bmatrix} = A$$

$$\therefore A^* = A.$$

Hence A is Hermitian matrix. [Proved]

Example-82 Prove that, $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is an unitary matrix. [প্রমাণ কর যে, A

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ একটি ঐকিক ম্যাট্রিক্স।]

Solution Given, $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$$\therefore \bar{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$$

$$\text{Thus } A^* = \bar{A}^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$\begin{aligned} \text{Now } AA^* &= \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+1 & i-i \\ -i+i & 1+1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Similarly, $A^*A = I$

Thus $AA^* = A^*A = I$

Hence A is an unitary matrix. [Proved]

Example-83 Verify that, $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Solution Given $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \therefore A^T = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$$\begin{aligned} AA^T &= \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

Hence A is an orthogonal matrix [Verified]

Example-84 Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory. [দেখাও যে,

ম্যাট্রিক্স $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ একটি অভেদঘাতী ম্যাট্রিক্স।]

$$\sim \begin{bmatrix} 1 & 3 & 5 & 6 \\ 0 & -11 & -22 & -20 \\ 0 & 6 & 13 & 13 \\ 0 & 0 & 11 & 23 \end{bmatrix} \begin{cases} R_2' = R_2 - 4R_1 \\ R_3' = R_3 + 2R_1 \end{cases}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 6 \\ 0 & -11 & -22 & -20 \\ 0 & 0 & 11 & 23 \end{bmatrix} [R_3' = 11R_3 + 6R_2]$$

Which is the echelon form of the given matrix.

Example-89 Solve the following equations by using inverse matrix [বিপরীত ম্যাট্রিক্সের ব্যবহার করে নিচের সমীকরণগুলোর সমাধান কর:]

$$2x - 3y = 3$$

$$4x - y = 11$$

Solution The above equations can be written in the matrix forms as

$$\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B \dots\dots\dots (1)$$

$$\text{Here, } A = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{we know } A^{-1} = \frac{\text{Adj } A}{|A|} \dots\dots\dots (2)$$

$$\text{Now, } |A| = \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix} = (-2 + 12) = 10$$

$$\text{Adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1}(-1) = -1, \quad A_{12} = (-1)^{1+2}4 = -4$$

$$A_{21} = (-1)^{2+1}(-3) = 3, \quad A_{22} = (-1)^{2+2}2 = 2$$

$$\therefore \text{Adj } A = \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ -\frac{4}{10} & \frac{2}{10} \end{bmatrix}$$

From (1), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{3}{10} \\ -\frac{4}{10} & \frac{2}{10} \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} + \frac{33}{10} \\ -\frac{12}{10} + \frac{22}{10} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = 3, y = 1 \text{ (Ans.)}$$

Example 90

Solve the following equations (নিচের সমীকরণগুলোর সমাধান নিন)

$$5x - 6y + 4z = 15$$

$$7x + 4y - 3z = 19$$

$$2x + y + 6z = 46$$

Solution

The above system of linear equations can be written in the matrix form

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\Rightarrow X = A^{-1}B \dots\dots\dots (1)$$

$$\text{Now } A^{-1} = \frac{\text{Adj } A}{|A|}, \text{ where } |A| = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$$

$$= 5(24 + 3) + 6(42 + 6) + 4(7 - 8) \\ = 135 + 288 - 4 \\ = 419$$

$$\text{Adj } A = \text{transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\text{where, } A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 24 + 3 = 27$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -(42 + 6) = -48$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = (7 - 8) = -1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = -(-36 - 4) = 40$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = (30 - 8) = 22$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = -(5 + 12) = -17$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = (18 - 16) = 2$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = -(-15 - 28) = 43$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 5 & -6 \\ 7 & 4 \end{vmatrix} = (20 + 42) = 62$$

$$\therefore \text{Adj } A = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{419} \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

From (1) we get,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419} \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix} \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419} \begin{bmatrix} 27 \cdot 15 + 40 \cdot 19 + 2 \cdot 46 \\ -48 \cdot 15 + 22 \cdot 19 + 43 \cdot 46 \\ -1 \cdot 15 - 17 \cdot 19 + 62 \cdot 46 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{419} \begin{bmatrix} 1257 \\ 1676 \\ 2514 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\Rightarrow x = 3, y = 4 \text{ and } z = 6 \text{ (Ans.)}$$

Example-91 Show that the following equations has no solution. [দেখাও যে, নিচের সমীকরণগুলোর সমাধান নেই:]

$$2x + 6y + 11 = 0$$

$$6x + 20y - 6z - 3 = 0$$

$$6y - 18z + 1 = 0$$

Solution The above system of equations may be written as

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ -1 \end{bmatrix}$$

$$\Rightarrow AX = B$$

$$\Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B \quad [\because A^{-1}A = I \text{ \& } IX = X]$$

$$\Rightarrow X = A^{-1}B$$

But A^{-1} does not exist, since

$$|A| = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix} = 2(-360 + 36) - 6(-108 - 0) + 0(36 - 0) = -648 + 648 = 0$$

Hence the equations has no solution. [Showed]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 34 & -17 & -17 \\ 1365 & -1348 \\ 442 & -442 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 20 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 156 + 30 + 160 \\ 442 - 102 - 340 \end{bmatrix}$$

$$= \frac{1}{17} \begin{bmatrix} 1365 - 1348 \\ -156 + 190 \\ 442 - 442 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 17 \\ 34 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore x = 1, y = 2 \text{ and } z = 3$$

$\therefore$ The required solution is $x = 1, y = 2$ and $z = 3$ (Ans.)

Example-02

Solve the system of the following equations using matrix method:

ম্যাট্রিক্স পদ্ধতি ব্যবহার করে নিম্নের সমীকরণগুলোর সমাধান কর :

$$x + y + z = 7$$

$$x + 2y + 3z = 16$$

$$x + 3y + 4z = 22$$

Solution The given system of linear equations can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix} \dots\dots\dots (1)$$

$$\text{Let, } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\text{We know, } A^{-1} = \frac{1}{|A|} \text{Adj } A.$$

$$\text{Here, } |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1(8 - 9) - 1(4 - 3) + 1(3 - 2)$$

$$= 1(-1) - 1(1) + 1(1)$$

$$= -1 - 1 + 1$$

$$= -1 \neq 0$$

Here the cofactors of the elements of $|A|$ are :

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1(8 - 9) = -1$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1(4 - 3) = -1$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1(3 - 2) = 1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = -1(4 - 3) = -1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 1(4 - 1) = 3$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -1(3 - 1) = -2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1(3 - 2) = 1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -1(3 - 1) = -2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1(2 - 1) = 1$$

$$\text{Adj } A = \text{Transpose of } \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} (\text{Adj } A) = \frac{1}{-1} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{From (1), we get } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix} = \begin{bmatrix} 7 + 16 - 22 \\ 7 - 48 + 44 \\ -7 + 32 - 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$\therefore$ The required solution $x = 1, y = 3, z = 3$ (Ans.)

$$D_y = \begin{vmatrix} 2 & 5 \\ 3 & -3 \end{vmatrix} = 2 \cdot (-3) - 3 \cdot 5 = -6 - 15 = -21$$

$$x = \frac{D_x}{D} = \frac{7}{7} = 1 \text{ and } y = \frac{D_y}{D} = \frac{-21}{7} = -3$$

∴ The required solution, $x = 1, y = -3$ (Ans.)

Example-69 Solve the following equations by using Cramer's rule. [নিচের সমীকরণগুলোর Cramer's এর সূত্র ব্যবহার করে সমাধান কর:]

$$\begin{cases} x + 5y - z = 9 \\ 3x - 3y + 2z = 7 \\ 2x - 4y + 3z = 1 \end{cases}$$

Solution Using Cramer's rule, we have

$$\Delta = \begin{vmatrix} 1 & 5 & -1 \\ 3 & -3 & 2 \\ 2 & -4 & 3 \end{vmatrix} = 1(-9 + 8) - 5(9 - 4) - 1(-12 + 6)$$

$$= -1 - 25 + 6 = -20 \neq 0$$

$$\Delta_x = \begin{vmatrix} 9 & 5 & -1 \\ 7 & -3 & 2 \\ 1 & -4 & 3 \end{vmatrix} = 9(-9 + 8) - 5(21 - 2) - 1(-28 + 3)$$

$$= -9 - 95 + 25 = -79$$

$$\Delta_y = \begin{vmatrix} 1 & 9 & -1 \\ 3 & 7 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 1(21 - 2) - 9(9 - 4) - 1(3 - 14)$$

$$= 19 - 45 + 11 = -15$$

$$\Delta_z = \begin{vmatrix} 1 & 5 & 9 \\ 3 & -3 & 7 \\ 2 & -4 & 1 \end{vmatrix} = 1(-3 + 28) - 5(3 - 14) + 9(-12 + 6)$$

$$= 25 + 55 - 54 = 26$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-79}{-20} = \frac{79}{20}, y = \frac{\Delta_y}{\Delta} = \frac{-15}{-20} = \frac{3}{4} \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{26}{-20} = -\frac{13}{10}$$

$$\text{(Ans.) } x = \frac{79}{20}, y = \frac{3}{4} \text{ \& } z = -\frac{13}{10}$$

Example-70 Solve the following equations by using Cramer's rule. [Cramer's rule]
 এর নিয়ম ব্যবহার করে নিচের সমীকরণগুলোর সমাধান কর :] [RUH 1979]

$$\begin{cases} x - y + z = 1 \\ x + y - 2z = 0 \\ 2x - y - z = 0 \end{cases}$$

Solution Using Cramer's rule, we have

$$\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} = 1(-1-2) + 1(-1+4) + 1(-1-2) = -3 + 3 - 3 = -3 \neq 0$$

$$\Delta_x = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1 & -1 \end{vmatrix} = (-1-2) = -3$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 2 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -(-1+4) = -3$$

$$\Delta_z = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = (-1-2) = -3$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-3}{-3} = 1, y = \frac{\Delta_y}{\Delta} = \frac{-3}{-3} = 1, \text{ and } z = \frac{\Delta_z}{\Delta} = \frac{-3}{-3} = 1$$

(Ans.) $x = 1, y = 1$ & $z = 1$.

Example-71 Solve the following equations by using Cramer's rule. [Cramer's rule]
 এর নিয়ম ব্যবহার করে নিচের সমীকরণগুলোর সমাধান কর :]

