Mathematical Problems & Solution
গাণিতিক সমস্যাবলি এবং সমাধান

Example-6 Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. *[NUH 2012]*

Solution Let λ be a scalar and I be the identity matrix of order 3×3.

The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -1 & 0 \\ -3 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 4 \end{bmatrix}$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ -3 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 2)(\lambda - 4) - 3(\lambda - 4)$$

$$= \{(\lambda - 2)^2 - 3\}$$

So the characteristic equation of A is

$$(\lambda - 4)\{(\lambda - 2)^2 - 3\} = 0$$

Either $\lambda - 4 = 0$

$\therefore\ \lambda = 4$

or, $(\lambda - 2)^2 - 3 = 0$

$\Rightarrow (\lambda - 2)^2 = 3$

$\Rightarrow \lambda - 2 = \pm\sqrt{3}$

$\therefore\ \lambda = 2 + \sqrt{3},\ 2 - \sqrt{3}$

Thus the eigenvalues of A are $4,\ 2 + \sqrt{3}$ and $2 - \sqrt{3}$ (Ans.)

Example-7 Find all the eigenvalues of the matrix $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$ in the field $\mathbb{R}$. [$\mathbb{R}$ ফিল্ডে $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$ ম্যাট্রিক্সটির সকল আইগেন মান নির্ণয় কর।]

[NUH (NM) 2002]

Solution Given matrix $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

Let λ be a scalar and I be the identity matrix of order 3×3.

Now the characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-3 & 1 & -1\\-7 & 5 & -1\\-6 & 6 & -2\end{bmatrix} = \begin{bmatrix}\lambda+3 & -1 & 1\\7 & \lambda-5 & 1\\6 & -6 & \lambda+2\end{bmatrix}$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix}\lambda+3 & -1 & 1\\7 & \lambda-5 & 1\\6 & -6 & \lambda+2\end{vmatrix}$$

$$= \begin{vmatrix}\lambda+2 & 0 & 1\\\lambda+2 & \lambda-4 & 1\\0 & \lambda-4 & \lambda+2\end{vmatrix} \quad [C'_1 = C_1 + C_2;\ C'_2 = C_2 + C_3]$$

$$= (\lambda+2)(\lambda-4)\begin{vmatrix}1 & 0 & 1\\1 & 1 & 1\\0 & 1 & \lambda+2\end{vmatrix}$$

$$= (\lambda+2)(\lambda-4)\{(\lambda+2-1)+1\}$$

$$= (\lambda+2)(\lambda-4)(\lambda+2) = (\lambda+2)^2(\lambda-4)$$

Therefore, the characteristic equation of A is $(\lambda+2)^2(\lambda-4) = 0$

$$\therefore \lambda = -2, -2, 4$$

which are the eigenvalues of the matrix A and distinct eigenvalues of the given matrix A are $\lambda = -2$ and $\lambda = 4$. (Ans.)

Example-8 Show that eigenvalues of the following matrices are real. [দেখাও যে, নিম্নের ম্যাট্রিক্সসমূহের আইগেন মান বাস্তব :]

Example-9 Find all the eigenvalues and associated eigenvectors of the matrix $A=\begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}$ in the field ℝ. [ℝ ফিল্ডে $A=\begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}$ ম্যাট্রিক্সটির সমস্ত আইগেন মান ও সংশ্লিষ্ট আইগেন ভেক্টর নির্ণয় কর।]

[NUH '09; NUH(NM) '05]

Solution Given matrix $A=\begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}$

Let, λ be a scalar and I be an 2×2 identity matrix.

The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}=\begin{bmatrix}\lambda-3 & 1\\-1 & \lambda-1\end{bmatrix}$$

The characteristic polynomial of matrix A is

$$|\lambda I - A| = \begin{vmatrix}\lambda-3 & 1\\-1 & \lambda-1\end{vmatrix}=(\lambda-3)(\lambda-1)+1$$

$$=\lambda^2-4\lambda+3+1=(\lambda-2)^2$$

So the characteristic equation of A is $(\lambda-2)^2=0$

$$\therefore \lambda = 2, 2$$

Hence, the eigenvalues of matrix A is $\lambda = 2$

Let $\mathrm{v}=\begin{bmatrix}x\\y\end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue λ if and on[ly]

if v is a non-trivial solution of $(\lambda I - A)\,\mathrm{v}=0$

i.e., $\begin{bmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} (\lambda-3)x + y = 0 \\ -x + (\lambda-1)y = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

When $\lambda = 2$, we get from (1)

$$\begin{cases} -x + y = 0 \\ -x + y = 0 \end{cases} \quad \Rightarrow x - y = 0 \ldots\ldots\ldots\ldots\ldots (2)$$

This system has a nonzero solution. Here y is a free variable and let $y = t$, then $x = t$.

Thus the eigenvectors of A corresponding to the eigenvalue $\lambda = 2$ are nonzero vectors of the form $\mathbf{v} = \begin{bmatrix} t \\ t \end{bmatrix}$.

In particular, let $t = 1$, then $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$. (Ans.)

Example-10 Find all the eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ in the field ℝ. [ℝ ফিল্ডে $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ ম্যাট্রিক্সটির সমস্ত আইগেন মান ও সংশ্লিষ্ট আইগেন ভেক্টর নির্ণয় কর।] *[RUH 2006]*

Solution The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix}$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3) - 8$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

∴ The characteristic equation of A is $(\lambda-5)(\lambda+1) = 0$

$$\Rightarrow \lambda = 5, -1$$

Hence, the eigenvalues of A are $\lambda = 5$ and $\lambda = -1$.

Let $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector of A corresponding to λ if and only if $\mathbf{v}$ is a non-trivial solution of $(\lambda I - A)\mathbf{v} = 0$.

i.e., $\begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} (\lambda-1)x - 4y = 0 \\ -2x + (\lambda-3)y = 0 \end{cases} \qquad \text{.....................} (1)$$

When $\lambda = 5$, equation (1) becomes

$$\begin{cases} 4x - 4y = 0 \\ -2x + 2y = 0 \end{cases} \Rightarrow x - y = 0 \qquad \text{.................} (2)$$

This system is in echelon form having one equation in two unknowns. So the system has a nonzero solution.

Let y be the free variable and let $y = 1$, then $x = 1$.

Thus a nonzero eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$.

When $\lambda = -1$, equation (1) becomes

$$\begin{cases} -2x - 4y = 0 \\ -2x - 4y = 0 \end{cases} \Rightarrow x + 2y = 0 \qquad \text{.................} (3)$$

This system is in echelon form having one equation in two unknowns. So the system has a nonzero solution.

Let y be the free variable and let $y = -1$, then $x = 2$.

Hence the nonzero eigenvector $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = -1$.

Example-11 Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$. [$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ ম্যাট্রিক্সের আইগেন এবং সংশ্লিষ্ট আইগেন ভেক্টর নির্ণয় কর ।]

Solution The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -2 & \lambda-1 \end{bmatrix}$$

The characteristic polynomial of the matrix A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -3 \\ -2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-2)(\lambda-1) - 6$$

$$= \lambda^2 - 2\lambda - \lambda + 2 - 6$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda-4)(\lambda+1)$$

$\therefore$ The characteristic equation of A is $|\lambda I - A| = 0$

i.e. $(\lambda + 1)(\lambda - 4) = 0$

$\therefore \lambda = -1, 4$

Hence, the eigenvalues of the matrix A are -1 and 4.

Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue λ if and only if v is a non-trivial solution of $(\lambda I - A)\, v = 0$

i.e., $\begin{bmatrix} \lambda-2 & -3 \\ -2 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or, $\begin{cases} (\lambda-2)x - 3y = 0 \\ -2x - (\lambda-1)y = 0 \end{cases}$ .... (1)

For $\lambda = 4$ equations (1) becomes,

$\begin{cases} -3x - 3y = 0 \\ -2x - 2y = 0 \end{cases} \sim x + y = 0$

This system has a nonzero solution. Here y is a free variable and let $y = 1$, then $x = -1$.

Thus the eigenvector of A corresponding to the eigenvalue $\lambda = -1$ is $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = -1$ equations (1) becomes,

$\begin{cases} 2x - 3y = 0 \\ -2x + 3y = 0 \end{cases} \sim 2x - 3y = 0$

This system has a nonzero solution. Here y is a free variable and let $y = 2$, then $x = 3$.

Thus the eigenvector of A corresponding to the eigenvalue $\lambda = 4$ is $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Hence, the required eigenvalues are -1, 4 and corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ respectively. (Ans.)

Example-12 Find bases for the eigenspaces of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ *[JNUH '15]*

Solution The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2)$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2) = (\lambda-2)(\lambda-1)(\lambda-2) = (\lambda-1)(\lambda-2)^2$$

So the characteristic equation of matrix A is $(\lambda - 1)(\lambda - 2)^2 = 0$

$$\therefore \lambda = 1, 2, 2$$

which are the eigenvalues of A and there are two eigenspaces of A.

By definition, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = 0$ that is of

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{.....................} (1)$$

If $\lambda = 2$, then (1) becomes $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here x_2 and x_3 are free variables.

Let $x_2 = t$ and $x_3 = s$, then $x_1 = -s$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent, these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then (1) becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{cases} x_1 + 2x_3 = 0 \\ -x_1 - x_2 - x_3 = 0 \\ -x_1 - 2x_3 = 0 \end{cases}$$

$$\sim \begin{cases} x_1 + 2x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases} \quad \begin{bmatrix} L_2' = L_2 + L_1 \\ L_3' = L_3 + L_1 \end{bmatrix}$$

Here x_3 is a free variable. Let $x_3 = s$, then $x_1 = -2s, x_2 = s$

Thus the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form $\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

So that $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

Example-13 Find the eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. [ম্যাট্রিক্স $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ এর সকল আইগেন মান ও সংশ্লিষ্ট আইগেন ভেক্টরসমূহ নির্ণয় কর।] *[NUH '94, '11; DUH '92; JNUH 2013]*

Solution The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-3)(\lambda^2 - 3\lambda - 4) + 2(-2\lambda + 6 - 8) - 4(4 + 4\lambda)$$

$$= \lambda^3 - 3\lambda^2 - 4\lambda - 3\lambda^2 + 9\lambda + 12 - 4\lambda - 4 - 16 - 16\lambda$$

$$= \lambda^3 - 6\lambda^2 - 15\lambda - 8$$

$$= \lambda^3 - 8\lambda^2 + 2\lambda^2 - 16\lambda + \lambda - 8$$

$$= \lambda^2(\lambda - 8) + 2\lambda(\lambda - 8) + 1(\lambda - 8)$$

$$= (\lambda - 8)(\lambda^2 + 2\lambda + 1)$$

$$= (\lambda - 8)(\lambda + 1)^2$$

So, the characteristic equation of A is $(\lambda - 8)(\lambda + 1)^2 = 0$

$$\therefore \lambda = 8, -1, -1$$

Thus the eigenvalues of the matrix A are $\lambda = 8$ and $\lambda = -1$.

Now, to find the eigenvectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to λ, solve the homogeneous linear system represented by $(\lambda I - A)\mathbf{v} = 0$

i.e., $$\begin{bmatrix} \lambda-3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} (\lambda-3)x - 2y - 4z = 0 \\ -2x + \lambda y - 2z = 0 \quad \text{.......................... (1)} \\ -4x - 2y + (\lambda-3)z = 0 \end{cases}$$

For $\lambda = -1$, we get from (1)

$$\begin{cases} -4x - 2y - 4z = 0 \\ -2x - y - 2z = 0 \\ -4x - 2y - 4z = 0 \end{cases}$$

$$\Rightarrow 2x + y + 2z = 0 \text{ .... (2)}$$

This system is in echelon form having one equation in three unknowns. So the system has nonzero solutions and two free variables exists.

Let y and z be the two free variables also let $y = 2s$, $z = t$; $s, t \in \mathbb{R}$ $(s \neq 0, t \neq 0)$.

Then $x = -s - t$.

Thus the solution of the system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s-t \\ 2s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Hence, the eigenvectors corresponding to the eigenvalue $\lambda = -1$ are $v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = 8$, equation (1) becomes,

$$\begin{cases} 5x - 2y - 4z = 0 \\ -2x + 8y - 2z = 0 \\ -4x - 2y + 5z = 0 \end{cases}$$

$$\sim \begin{cases} 5x - 2y - 4z = 0 \\ 36y - 18z = 0 \\ -18y + 9z = 0 \end{cases} \quad \begin{bmatrix} L_2' = 5L_2 + 2L_1 \\ L_3' = L_3 - 2L_2 \end{bmatrix}$$

$$\sim \begin{cases} 5x - 2y - 4z = 0 \\ 2y - z = 0 \end{cases}$$

This system is in echelon form having two equations in three unknowns. So the system has nonzero solutions and one free variable exists.

Let z be the free variable also let $z = 2t$; $t \in \mathbb{R}$ $(t \neq 0)$. Then $y = t$ and $x = 2t$.

Thus the eigenvector corresponding to the eigenvalue $\lambda = 8$ is of the form

$t \in \mathbb{R}$ $(t \neq 0)$.

In particular, if $t = 1$ then $v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.

Example-14 Find all the eigenvalues and associated eigenvectors of the $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$ in the field $\mathbb{R}$. [$\mathbb{R}$ ফিল্ডে $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$ ম্যাট্রিক্সটির আইগেন মান ও সংশ্লিষ্ট আইগেন ভেক্টর নির্ণয় কর ।]

[NUH '02; DU...

Solution The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda-4 \end{vmatrix}$$

$$= (\lambda - 1)\,\lambda(\lambda - 4) + 2(0 - 2\lambda)$$

$$= \lambda(\lambda^2 - 4\lambda - \lambda + 4) - 4\lambda$$

$$= \lambda\,(\lambda^2 - 5\lambda + 4 - 4)$$

$$= \lambda^2(\lambda - 5)$$

So, the characteristic equation of A is $\lambda^2(\lambda - 5) = 0$

$$\therefore \lambda = 0, 0, 5$$

which are the eigenvalues of A.

Now to find the nonzero eigenvectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ corresponding to λ, solve homogeneous linear system represented by $(\lambda I - A)\mathbf{v} = 0$

i.e., $\begin{bmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda-4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} (\lambda-1)x + 2z = 0 \\ \lambda y = 0 \\ 2x + (\lambda-4)z = 0 \end{cases} \quad \text{.................. (1)}$$

When $\lambda = 0$, we get from (1)

$$\begin{cases} -x + 2z = 0 \\ 2x - 4z = 0 \end{cases} \Rightarrow x - 2z = 0$$

This system is in echelon form having one equation in two unknowns. So the system has nonzero solutions and two free variables exists.

Let y, z be the free variables, also let $y = s$ and $z = t$; $s, t \in \mathbb{R}$ $(s \neq 0, t \neq 0)$.

Then $x = 2t$.

Thus the solution of the system is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Thus the nonzero eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 0$.

When $\lambda = 5$, we get from (1)

$$\begin{cases} 4x + 2z = 0 \\ 5y = 0 \\ 2x + z = 0 \end{cases} \sim \begin{cases} 2x + z = 0 \\ y = 0 \end{cases}$$

There are two equations in three variables. So the system has a free variable.

Let z be the free variable and $z = -2$, then $y = 0$ and $x = 1$.

Hence the nonzero eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$.

Example-15 Find all eigenvalues and the corresponding eigenvectors of the following matrix. [নিম্নলিখিত ম্যাট্রিক্সটির আইগেন মান ও সংশ্লিষ্ট আইগেন ভেক্টরসমূহ নির্ণয় কর :]

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$ ***[DUP 1991]***

Solution The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)\begin{vmatrix} \lambda + 2 & 0 \\ 5 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda + 2)(\lambda - 2)$$

11.5 Eigenvalues and Eigenvectors of a Square Matrix

বর্গাকার ম্যাট্রিক্সের আইগেন মান এবং আইগেন ভেক্টর [NUH '02; NUH(NM)'08 DUH '80, '83, '13, '15; JNUH 2015, 2014, '12, '11; JUH '89, 91; CUH '81, 83, 84]

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an eigenvector corresponding to λ.

or, Let A be a square matrix of order n. A nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an eigenvector of A if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$ i.e. $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called an eigenvalue of A and $\mathbf{x}$ is called an eigenvecor of A corresponding to eigenvalue λ. [মনে করি, A একটি n ক্রমের বর্গ ম্যাট্রিক্স। $\mathbb{R}^n$ এ একটি অশূন্য ভেক্টর $\mathbf{x}$ কে A এর আইগেন ভেক্টর বলা হবে যদি $A\mathbf{x}$ কিছু স্কেলার λ এর জন্য ভেক্টর $\mathbf{x}$ এর একটি স্কেলার গুণিতক হয় অর্থাৎ $A\mathbf{x} = \lambda\mathbf{x}$। স্কেলার λ কে A এর আইগেন মান বলা হয় এবং $\mathbf{x}$ কে A ভেক্টরের আইগেনমান λ এর সংশ্লিষ্ট আইগেন ভেক্টর বলা হয়।]

Let A be an $n \times n$ matrix.

(i) An eigenvalue of A is a scalar such that $|\lambda I - A| = 0$

(ii) An eigenvectors of A corresponding to λ are the nonzero solutions of $(|I - A|)\mathbf{x} = 0$

Example-2 Find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.