Differential Relations

Presented by:

Md.Nahidul Islam(20CSE020)
Md. Rashedul Islam(20CSS047)

Differential Motion and Statics

- Tool configuration and Joint Space velocity
- Jacobian
  - Tool configuration Jacobian
  - Manipulator Jacobian
- Singularity
  - Boundary Singularity
  - Interior Singularity
- Generalised Inverse
- Pseudo Inverse
- Statics
- Examples

Differential Relations

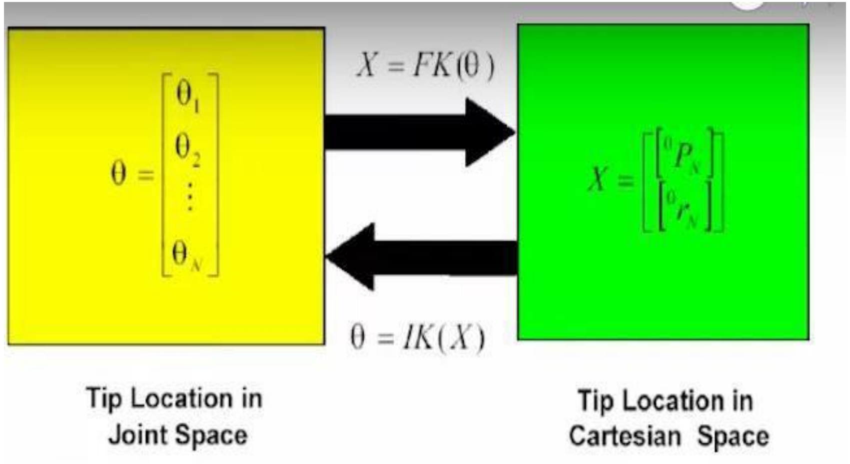
- Robot path planning problem is formulated in tool-configuration space
- Robot motion is controlled at the joint space

x= w(q); x= tool configuration vector,
w = tool-configuration function and q = joint variables

Differential relationship

$\dot{x} = J(q)\dot{q}$; $J(q)$ is a 6xn matrix and is called the Jacobian matrix or Jacobian

$$J_{kj}(q) = \frac{\partial w_k(q)}{\partial q_j} \quad 1 \le k \le 6, \quad 1 \le j \le n$$



X=wq ⟶

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} =
\begin{bmatrix}
\frac{\partial w_1}{\partial q_1} & \frac{\partial w_1}{\partial q_2} & \frac{\partial w_1}{\partial q_3} & . & \frac{\partial w_1}{\partial q_n} \\
\frac{\partial w_2}{\partial q_1} & \frac{\partial w_2}{\partial q_2} & \frac{\partial w_2}{\partial q_3} & . & \frac{\partial w_2}{\partial q_n} \\
\frac{\partial w_3}{\partial q_1} & \frac{\partial w_3}{\partial q_2} & \frac{\partial w_3}{\partial q_3} & . & \frac{\partial w_3}{\partial q_n} \\
\frac{\partial w_4}{\partial q_1} & \frac{\partial w_4}{\partial q_2} & \frac{\partial w_4}{\partial q_3} & . & \frac{\partial w_4}{\partial q_n} \\
\frac{\partial w_5}{\partial q_1} & \frac{\partial w_5}{\partial q_2} & \frac{\partial w_5}{\partial q_3} & . & \frac{\partial w_5}{\partial q_n} \\
\frac{\partial w_6}{\partial q_1} & \frac{\partial w_6}{\partial q_2} & \frac{\partial w_6}{\partial q_3} & . & \frac{\partial w_6}{\partial q_n}
\end{bmatrix}
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \cdot \\ \dot{q}_n \end{bmatrix}
$$

$$6 X 1 \qquad 6 X n \qquad n X 1$$

For a rotary manipulator,

$$\dot{x} = [J(\theta)]\,\dot{\theta}$$

Calculation of Jacobian

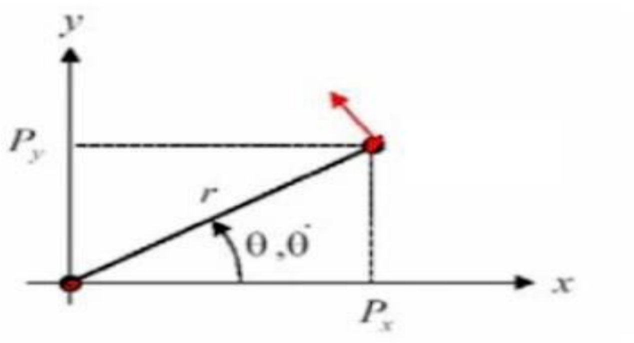
- Get the forward kinematics relationship, X=w(θ)
- Differentiate X wrt θ

Example: Planar 1R Robot

The end effector position is given by

$$P_x = r\cos\theta$$

$$P_y = r\sin\theta$$



$$\dot{P}_x = \dot{x} = -r\sin\theta.\dot{\theta}$$

$$\dot{P}_y = \dot{y} = r\cos\theta.\dot{\theta}$$

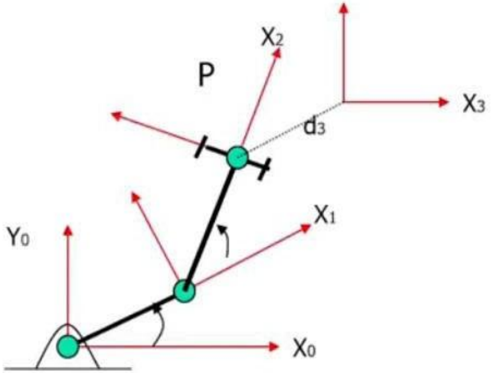
$$\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix}$$

Example: 3R Planar Manipulator

$$Px = l_1 C_1 + l_2 C_{12}$$

$$Py = l_1 S_1 + l_2 S_{12}$$

$$Pz = d_3$$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \frac{\partial p_x}{\partial \theta_3} \\ \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} & \frac{\partial p_y}{\partial \theta_3} \\ \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} & \frac{\partial p_z}{\partial \theta_3} \end{bmatrix} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

3 X 1   3 X 3   3 X 1

$$T_{base}^{tool} = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1 C_1 + l_2 C_{12} \\ S_{123} & C_{123} & 0 & l_1 S_1 + l_2 S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} & 0 \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Singularities

The joint space velocity is given as $\dot{\theta} = [J(\theta)]^{-1} \dot{x}$

One of the potential problems with solving for joint space velocity is the non-existence of inverse. The Jacobian may not be invertible for all the values of θ.

At certain points in joint space, Jacobian loses its rank; i.e. there is a reduction in no. of independent rows and columns. The points at which the Jacobian loses rank are called Joint space Singularities.

NOTE: The Jacobian Matrix J(q) is of full rank as long as q is not a joint space singularity.

Manipulator dexterity, $dex(q)=det[\mathbf{J}^T\mathbf{J}]$ $n<=6$

For the general case $n<=6$, the tool Jacobian matrix is less than full rank if and only if the nXn matrix $\mathbf{J}^T\mathbf{J}$ is singular.

For redundant manipulators ($n>6$), determinant of the 6X6 matrix, $\mathbf{J}\mathbf{J}^T$ must be used.

A manipulator is at joint space singularity if and only if $dex(q)=0$.

Boundary singularity occurs when the tool tip is on the surface of the work envelope.

Boundary Singularities of SCARA

$$J = \begin{bmatrix} -l_1S_1 - l_2S_{1-2} & -l_2S_{1-2} & 0 & 0 \\ l_1C_1 + l_2C_{1-2} & l_2C_{1-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{aligned} dex = \det(J^TJ) &= (-l_1S_1 - l_2S_{1-2}).(-l_2C_{1-2}) - l_2S_{1-2}(l_1C_1 + l_2C_{1-2}) \\ &= l_1l_2(S_1C_{1-2} - C_1S_{1-2}) \\ &= l_1l_2S_2 \end{aligned}$$

dex() =0 iff $S_2=0$; $\longrightarrow \theta_2=0,\pi$

When θ2=0, the arm is fully stretched and the tip is on the surface of the work envelope.

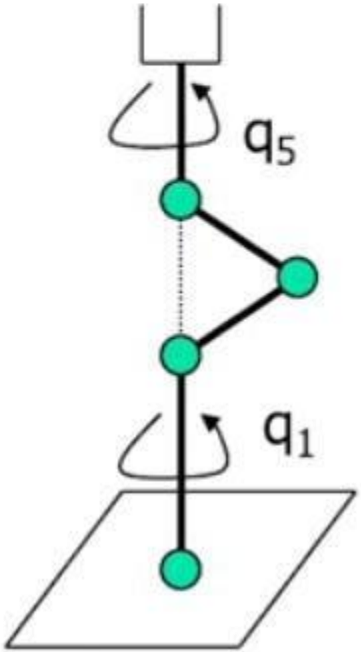
Interior Singularity: Potentially troublesome; formed when two or more axes form a straight line. The effects of rotation about one axis may be cancelled due to a counteracting rotation about the other axis. Tool configuration may remain the same even though the robot moves in joint space.

Example: Microbot-Alpha2

Consider the following locus of points in Joint space

$q(\beta)=[q_1, -\beta, 2\beta-\pi, -\beta, q_5]$ $\quad 0<\beta<\pi/2$

If $a_3=a_2$ and $a_4=0$, then $J(q)$ loses full rank along the line $q=q(\beta)$ and $q(\beta)$ represents interior singularities for the articulated robot.



Generalised Inverse

$$\dot{\theta} = [J(\theta)]^{-1}\dot{x}$$

Generalised Inverse: If A is an mxn matrix, then an nxm matrix X is a generalised inverse of A if and only if it satisfies at least property 1 or 2 of the following list of properties:

1. AXA=A
2. XAX=X
3. $(AX)^T$=AX
4. $(XA)^T$=XA

Most well known generalised Inverse is Moore-Penrose Inverse or **Pseudo Inverse (A^+) which satisfies all 4 properties. If A is of full rank then,**

$$A^+ = A^T(AA^T)^{-1} \quad m \le n$$
$$= A^{-1} \quad m = n$$
$$= (A^TA)^{-1}A^T \quad m \ge n$$

Example:

Find the pseudo Inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

The rank of A is 2.

$$A^+ = A^T (AA^T)^{-1} \quad \Longrightarrow \quad A^+ = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

Thank you!