Inverse Kinematics in Robotics

Course Title: Robotics and Automation
Course Code: CSE-4101

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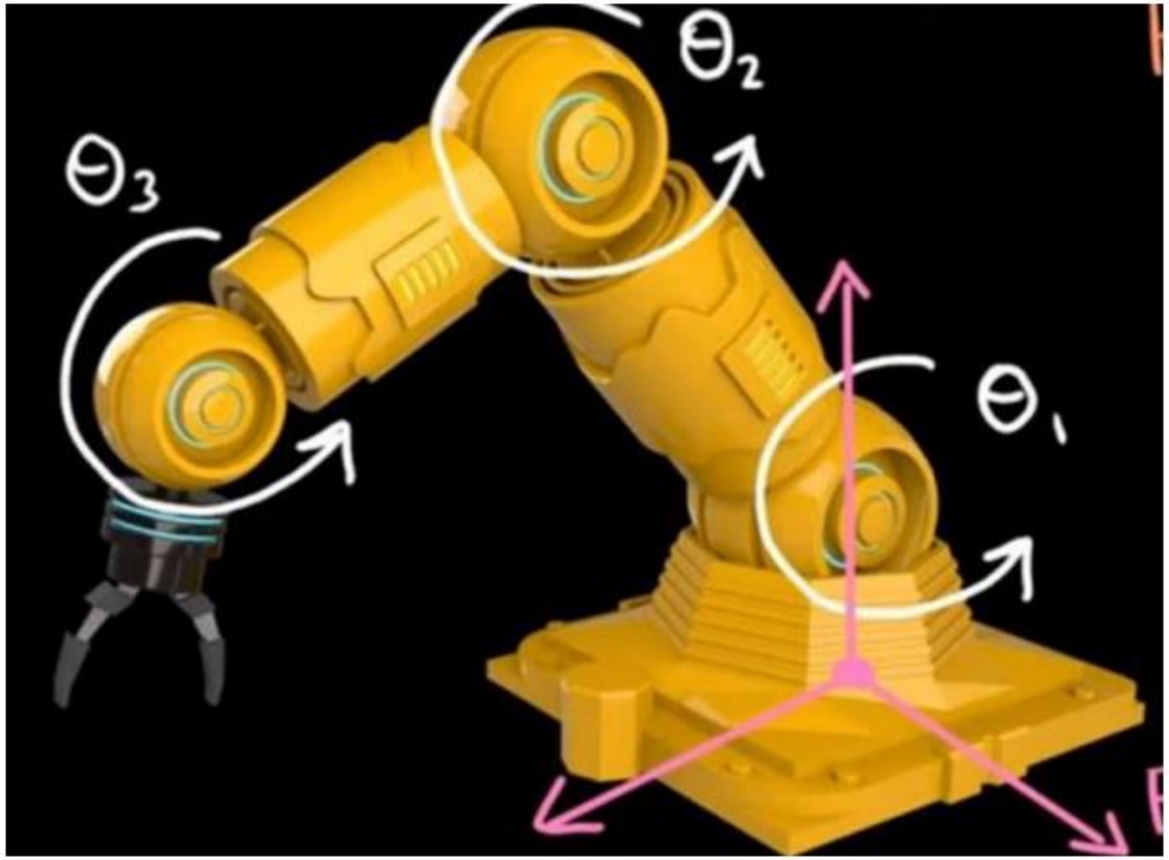
What is Inverse Kinematics?

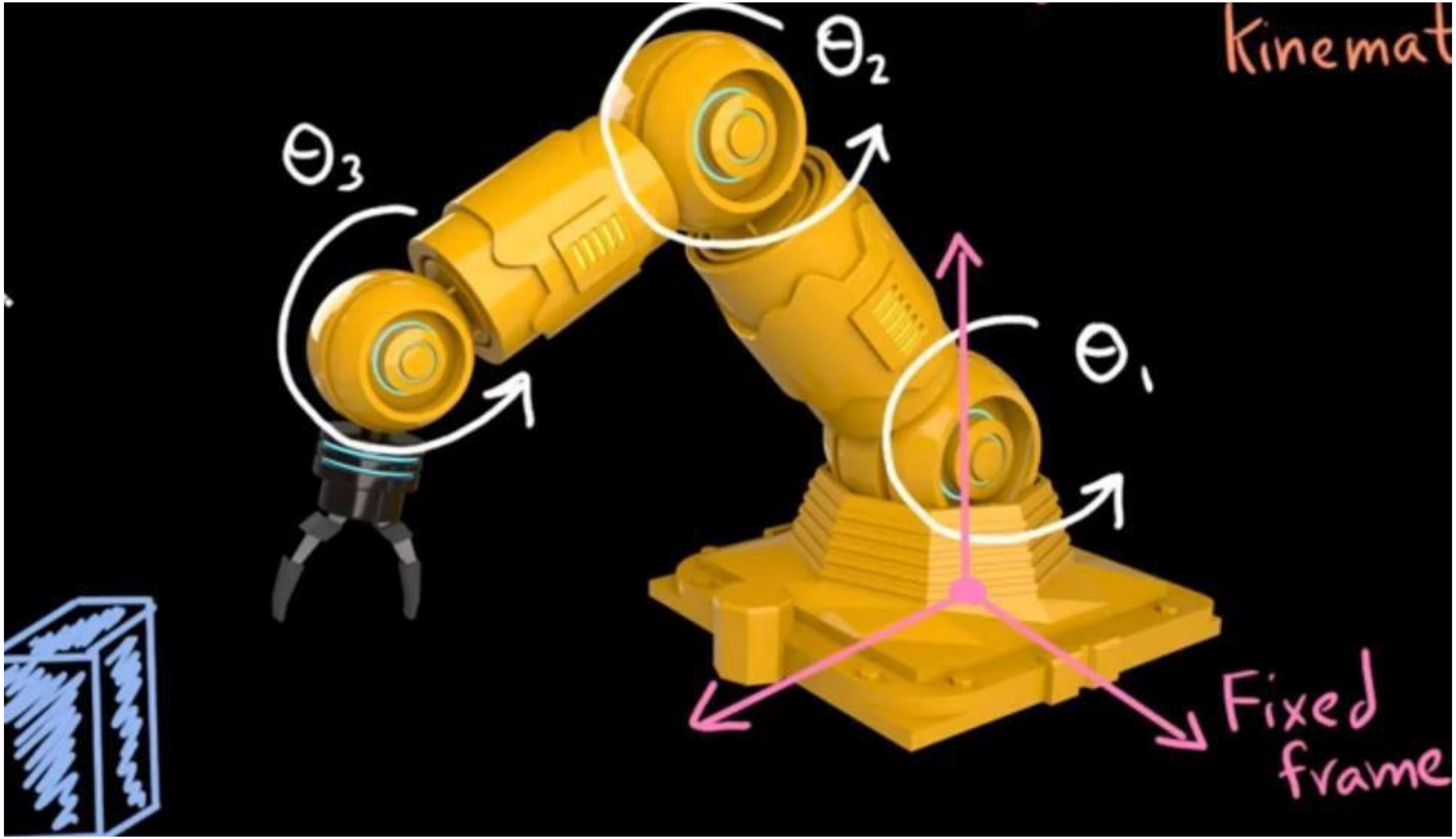
A Branch of robotics that deals with determining the joint configurations of a robotic system or a mechanism to achieve a desired end-effector position and orientation.



Importance & Applications of Inverse Kinematics

- Precision Control
- Task Planning and Execution
- Human-Robot Interaction
- Animation and Virtual Reality
- Efficient Resource Utilization

- Robotic Manipulation
- Humanoid Robots
- Medical Robotics
- Animation and Film Production
- Space Exploration
- Educational Robots

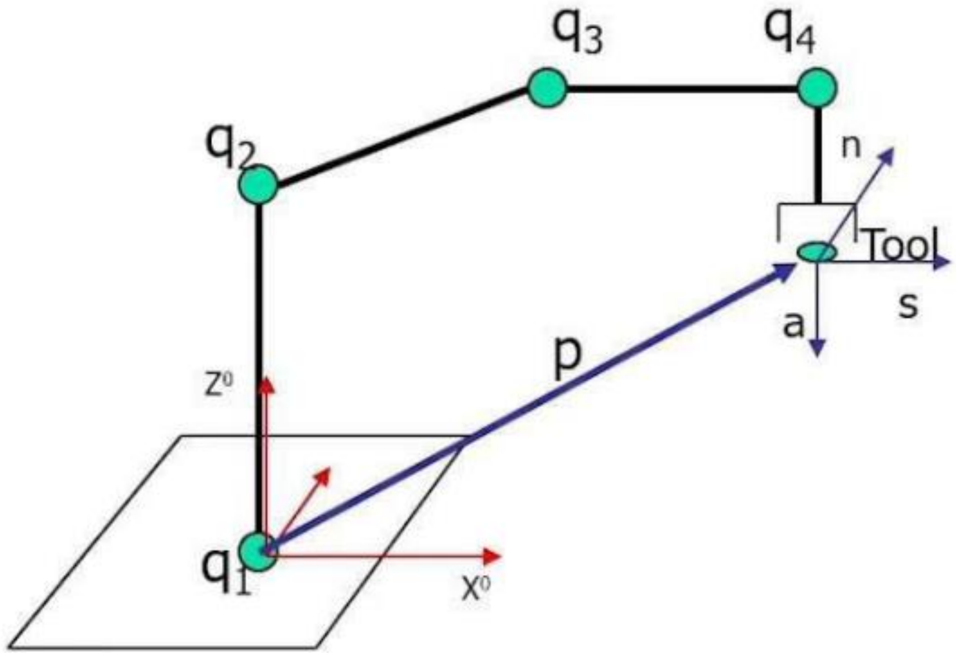


Kinemat
θ2
θ3
θ1
Fixed
frame

Content

- Inverse Kinematics
  - Problem definition
  - Solvability
    - Existence of Solutions
    - Multiple Solutions
  - Solution Strategies
    - Closed Form: Algebraic, Geometric
    - Numeric
- Velocity Relationships
  - Jacobian
  - Singularly
- Statics
  - Examples

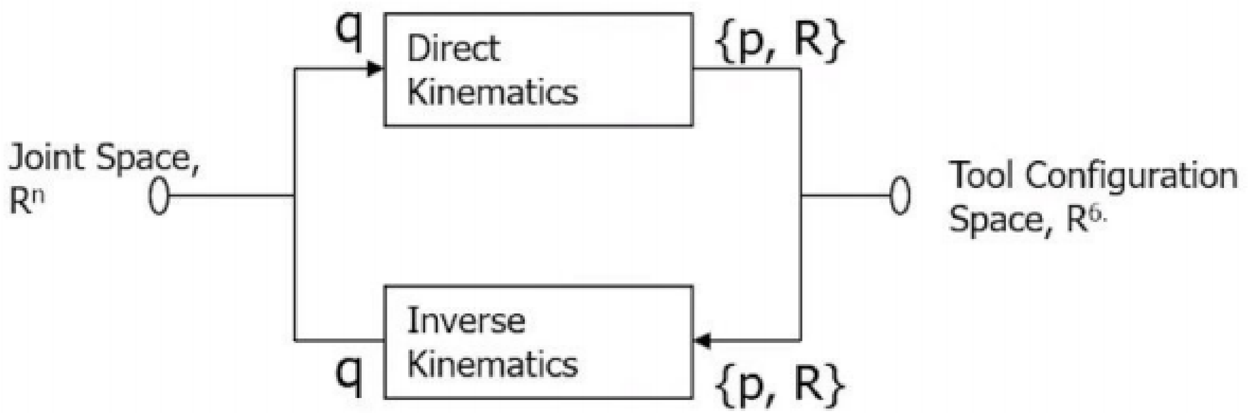
- Manipulator tasks are normally formulated in terms of the desired position and orientation.
- A systematic closed form solution applicable to robots in general is not available.
- Unique solutions are rare; multiple solutions exist.
- Inverse problem is more difficult than forward problem.



The Arm matrix represents the position p and orientation R of the tool in base coordinate frame as a function of joint variable q.

$$T_{base}^{tool}(q) = \left[\begin{array}{ccc|c} & R(q) & & p(q) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Inverse Kinematics: Given a desired position p and orientation R for the tool, find values for the joint variables which satisfy the arm equation



Solvability

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1C_{234}C_5 + S_1S_5 & -C_1C_{234}S_5 + S_1C_5 & -C_1S_{234} & C_1(177.8C_2 + 177.8C_{23} - 129.5S_{234}) \\ S_1C_{234}C_5 - C_1S_5 & -S_1C_{234}S_5 - C_1C_5 & -S_1S_{234} & S_1(177.8C_2 + 177.8C_{23} - 129.5S_{234}) \\ -S_{234}C_5 & S_{234}S_5 & -C_{234} & 215 - 177.8S_2 - 177.8S_{23} - 129.5C_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 12 equations and ‘n’ unknowns. (n=6 for 6 axis robot)
- Out of 9 equations from the rotation part, only 3 are independent
- From the position vector part, there are 3 independent equations
- 6 independent equations and ‘n’ unknowns
- Nonlinear equations, difficult to solve

Existence of Solutions

- A manipulator is solvable if all the sets of joint variables can be found corresponding to the given end-effector location.

Necessary conditions

- Tool point within the workspace
- N>=6, to have any arbitrary orientation of tool.
- Tool orientation is such that none of the joint limitations are violated

Two kinds of solutions:

- **Closed form solutions:** analytical expressions
- **Numerical solutions:** iterative search – time consuming
  - For closed-form solutions: sufficiency condition
    - Three adjacent joint axes intersecting or
    - Three adjacent joint axes parallel to one another
- **Uniqueness of solution:-Multiple solutions**
- Kinematically redundant robots
- Elbow-up, Elbow-down solutions

There are two approaches for deriving closed form solutions:

Algebraic vs geometric.

Algebraic Approach:

Obtain scalar equations from the matrix form (at most 12 of them)

There are some tricks to solve the equations.

Trick 1: use trigonometric identities to combine two equations(such as first square them and then add them) and eliminate certain variable.

Trick 2: use the following variable substitution

$$u = \tan\frac{\theta}{2}$$
$$\cos\theta = (1-u^2)(1+u^2)$$
$$\sin\theta = \frac{2u}{1+u^2}$$

To convert these equations to polynomial ones, and solve the polynomial ones instead.

Trick 3: find out expression for both sin and cos of a joint angle θ, and express the angle using the function Atan $2(\sin\theta, \cos\theta)$. With Atan 2 rather than the conventional arctan, the angle can be uniquely expressed

2 DoF Planner manipulator

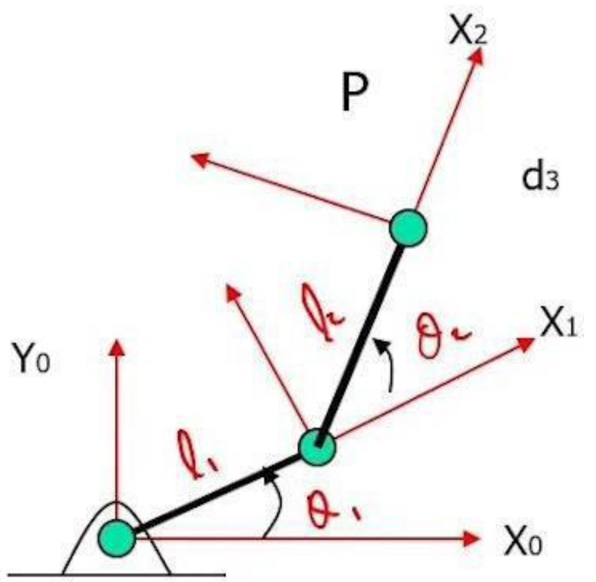
$$p_x = l_1 C_1 + l_2 C_{12}$$
$$p_y = l_1 S_1 + l_2 S_{12}$$

$$P_x^2 + P_y^2 = l^2_1 + l^2_2 + 2 l_1 l_2 C_2$$

$$C_2 = \frac{P_x^2 + P_y^2 - (l^2_1 + l^2_2)}{2 l_1 l_2}$$

$$S_2 = \pm\sqrt{1 - C_2^2}$$

$$\theta_2 = a\tan 2(S_2, C_2)$$



The function atan2 denotes a four quadrant version of arctan function. It allows us to recover angles over the entire range of [$-\pi$ to π]

if $x = y = 0$, then the result is indefinite,
if $x > 0$ and $y = 0$, then atan2 = 0,
if $x < 0$ and $y = 0$, then atan2 = π, else
if $y < 0$, then $-\pi <$ atan2 < 0,
if $y > 0$, then $0 <$ atan2 $< \pi$.

Example- 3 DOF

$$T_{base}^{tool} = \begin{bmatrix} n_x & s_x & 0 & p_x \\ n_y & s_y & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{base}^{tool} = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1C_1 + l_2C_{12} \\ S_{123} & C_{123} & 0 & l_1S_1 + l_2S_{12} \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x = C_{123}$.......(1), $n_y = S_{123}$........(2)

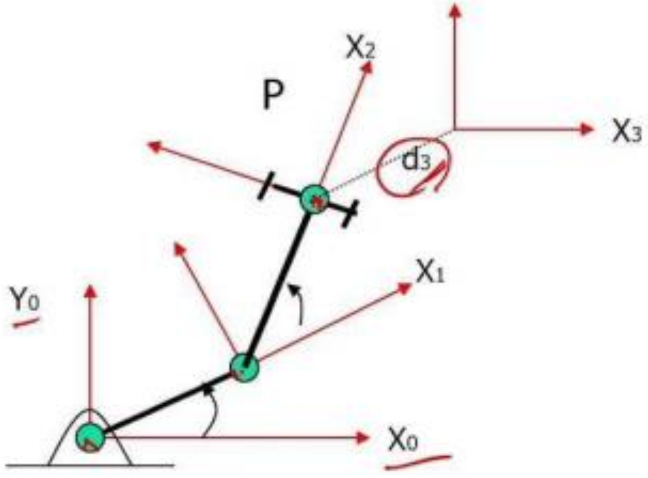
$s_x = -S_{123}$.......(3), $s_y = C_{123}$.......(4)

$p_x = l_1C_1 + l_2C_{12}$.........(5), $p_y = l_1S_1 + l_2S_{12}$.....(6)

On squaring and adding 5 and 6 we get

$$P_x^{\,2} + P_y^{\,2} = l^2{}_1 + l^2{}_2 + 2l_1l_2C_2$$

$$C_2 = \frac{P_x^{\,2} + P_y^{\,2} - (l^2{}_1 + l^2{}_2)}{2l_1l_2}$$



a	d	a	θ
l_1	0	0	
l_2	0	0	
0	d_3	0	

$$S_2 = \pm\sqrt{1 - {C_2}^2}$$

$$\theta_2 = a\tan 2(S_2, C_2)$$

solving eqn. (5), (6) for θ_1,

$$P_x = K_1C_1 - K_2S_1$$

$$P_y = K_1S_1 + K_2C_1;\ \text{where } K_1 = l_1 - l_2C_2;\ K_2 = l_2S_2$$

Substituting

$$K_1 = r\cos\gamma; K_2 = r\sin\gamma;\quad \text{where } r = \sqrt{{K_1}^2 + {K_2}^2};\quad \gamma = a\tan 2(K_2, K_1)$$

We get,

$$\frac{P_x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

$$\frac{P_x}{r} = \cos\gamma\sin\theta_1 - \sin\gamma\cos\theta_1$$

$$\cos(\gamma + \theta_1) = \frac{P_x}{r}$$

$$\sin(\gamma + \theta_1) = \frac{P_y}{r}$$

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$$\gamma + \theta_1 = a\tan 2\left(\frac{P_y}{r}, \frac{P_x}{r}\right) = a\tan 2\left(P_y, P_x\right)$$

$$\theta_1 = a\tan 2\left(P_y, P_x\right) - a\tan 2(K_2, K_1)$$

from equation (1) and (2), we have

$$a\tan 2\left(n_y, n_x\right) = \theta_{123}$$

$$\theta_3 = \theta_{123} - \theta_1 - \theta_2$$

Thank You