Again, $f(z) = |z|^2$

$\Rightarrow f(z) = u + iv = |x + iy|^2$

$\Rightarrow u + iv = x^2 + y^2$

$\Rightarrow u = x^2 + y^2$ and $v = 0$

$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 2y, \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

At $z = 0, \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

Thus the Cauchy-Riemann equations are satisfied at $z = 0$ but not in the neighbourhood $|z - 0| < \delta$.

Thus $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic there.

Example–19. Show that $f(z) = 2x + ixy^2$ is no where analytic. **[RUH–1996]**

Solution : Let $f(z) = u(x, y) + iv(x, y)$

$\Rightarrow 2x + ixy^2 = u(x, y) + iv(x, y)$

$\Rightarrow u(x, y) = 2x$ and $v(x, y) = xy^2$

$\therefore \frac{\partial u}{\partial x} = 2, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = y^2$ and $\frac{\partial v}{\partial y} = 2xy$

Thus we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$.

That is, Cauchy-Riemann equations are not satisfied anywhere. Hence f(z) is not analytic at any point, that is, f(z) is no where analytic.

Example–20. If p and q are functions of x and y satisfying Laplace's equation, then show that (u + iv) is analytic where $u = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}$ and $v = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}$. **[RUH–1999]**

Solution : Given that p and q are functions of x and y satisfying Laplace's equation.

$$\therefore \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \quad \text{...... (1)}$$

$$\text{and} \quad \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0 \quad \text{...... (2)}$$

Similarly, from $v(x, y) = c_2$ we have slope of the second curve

$$m_2 = -\frac{v_x}{v_y} \;\ldots\ldots (3)$$

Product of the slopes, $m_1 m_2 = \frac{-u_x}{u_y} \cdot \frac{-v_x}{v_y}$.

$$\Rightarrow m_1 m_2 = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1 \text{ by (1)}$$

Hence the given system of families of curves are orthogonal.

Example–28. If $f(z) = u + iv$ is analytic in a region R and if u and v have continuous second order partial derivatives in R, then u and v are harmonic in R. **[DUH–1986, 1989, 1991, JUH–1986, 87]**

Solution : Given $f(z) = u + iv$ is analytic in the region R. So, by Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\ldots\ldots (1)$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\ldots\ldots (2)$$

Again given u and v have continuous second order partial derivatives in R. So we have

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \;\ldots\ldots (3)$$

and $$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x} \;\ldots\ldots (4)$$

Now from (3) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right). \quad \text{[by (1) and (2)]}$$

$$\Rightarrow -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Thus, v satisfy Laplace equation and hence it is harmonic.

Again, form (4) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right), \quad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Thus, u satisfy **Laplace equation** and hence it is harmonic.

Example–29. If f(z) = u + iv is a analytic function of z = x + iy and ϕ is any function of x and y with differential coefficient of first order, then show that

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left\{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right\}|f'(z)|^2$$ **[RUH–2001]**

Solution : We have $\phi=\phi(x, y)$

$$\therefore \frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} \quad \ldots\ldots (1)$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} \quad \ldots\ldots (2)$$

From Cauchy-Riemann equations we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \ldots\ldots (3)$$

By (3), (2) becomes

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\left(-\frac{\partial v}{\partial x}\right)+\frac{\partial\phi}{\partial v}\cdot\frac{\partial u}{\partial x} \quad \ldots\ldots (4)$$

Squaring and adding (1) and (4) we get

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

$$+\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2-2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$$

$$=\left(\frac{\partial\phi}{\partial u}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}+\left(\frac{\partial\phi}{\partial v}\right)^2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

N. B. For analyticity Cauchy-Riemann equations must be satisfied and also the first order partial derivatives of u and v should be continuous. Here the second condition is not satisfied and hence f(z) is not analytic at z = 0.

Example–35(a). Prove that the function $f(z) = z^2 + 5iz + 3 - i$ satisfy Cauchy-Riemann equations. **[NUH(Phy)–2005]**

Solution : Given that $f(z) = z^2 + 5iz + 3 - i$

$$\Rightarrow u + iv = (x + iy)^2 + 5i(x + iy) + 3 - i$$
$$= x^2 + 2ixy - y^2 + 5ix - 5y + 3 - i$$
$$= (x^2 - y^2 - 5y + 3) + i(2xy + 5x - 1)$$

Equating real and imaginary parts we get

$$\Rightarrow u = x^2 - y^2 - 5y + 3 \text{ and } v = 2xy + 5x - 1$$

$$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y - 5, \frac{\partial v}{\partial x} = 2y + 5 \text{ and } \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \text{ ...... (1)}$$

$$\text{and, } \frac{\partial u}{\partial y} = -(2y + 5) = -\frac{\partial v}{\partial x} \text{ ...... (2)}$$

From (1) and (2) we see that the given equation satisfy the Cauchy-Riemann equations. **(Proved)**

Example–35(b). Test for analyticity of $W_1 = f_1(z) = |z|^2$ and $W_2 = f_2(z) = \frac{1}{2}$. **[NU(Pre)–2006]**

Solution : (i) Given that $W_1 = f_1(z) = |z|^2$

$$\Rightarrow u_1 + iv_1 = |x + iy|^2$$
$$\Rightarrow u_1 + iv_1 = x^2 + y^2$$
$$\Rightarrow u_1 = x^2 + y^2 \text{ and } v_1 = 0$$

$$\therefore \frac{\partial u_1}{\partial x} = 2x, \frac{\partial u_1}{\partial y} = 2y, \frac{\partial v_1}{\partial x} = 0, \frac{\partial v_1}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial x} = 2x \neq \frac{\partial v_1}{\partial y} \text{ and } \frac{\partial u_1}{\partial y} = 2y \neq -\frac{\partial u_1}{\partial x}$$

Therefore, $W_1 = f_1(z) = |z|^2$ does not satisfy Cauchy-Riemann equations and hence not analytic.

(ii) Given that $w_2 = f_2(z) = \frac{1}{2}$

$$\Rightarrow u_2 + iv_2 = \frac{1}{2}$$

$$\Rightarrow u_2 = \frac{1}{2} \text{ and } v_2 = 0$$

$\therefore f'(z) = u + iv$

$= 6x^2 - 6y^2 + 6y + c + i(12xy - 6x)$

$= 6(x^2 - y^2 + 2ixy) - 6i(x + iy) + c$

$= 6(x^2 + i^2 y^2 + 2ixy) - 6i(x + iy) + c$

$= 6(x + iy)^2 - 6i(x + iy) + c$

$\Rightarrow \dfrac{df}{dz} = 6z^2 - 6iz + c$

$\Rightarrow \int df = \int (6z^2 - 6iz + c)\, dz$

$\Rightarrow f = 6 \cdot \dfrac{z^3}{3} - 6i\dfrac{z^2}{2} + cz + D,$ [D = integrating constant]

$\Rightarrow f(z) = 2z^3 - 3iz^2 + cz + D \cdots\cdots (5)$

Given $f(0) = 3 - 2i$ and $f(1) = 6 - 5i \cdots\cdots (6)$

Putting $z = 0$ in (5) we get

$f(0) = 0 - 0 + 0 + D$

$\Rightarrow 3 - 2i = D \cdots\cdots (7)$ [by (6)]

Putting $z = 1$ in (5) we get

$f(1) = 2 - 3i + c + D$

$\Rightarrow 6 - 5i = 2 - 3i + c + 3 - 2i;$ [by (6) and (7)]

$\Rightarrow 6 - 5i = 5 - 5i + c$

$\Rightarrow 6 - 5 = c \Rightarrow c = 1$

Putting the values of C and D in (5) we get

$f(z) = 2z^3 - 3iz^2 + z + 3 - 2i$

$\therefore f(1 + i) = 2(1 + i)^3 - 3i(1 + i)^2 + (1 + i) + 3 - 2i;$ by putting $z = 1 + i$

$= 2(1 + 3i - 3 - i) - 3i(1 + 2i - 1) + 1 + i + 3 - 2i$

$= 2 + 6i - 6 - 2i - 3i + 6 + 3i + 4 - i = 6 + 3i.$ (Ans)

Example–39. Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find its harmonic conjugate v and express u + iv as an analytic function of z. **[RUH–1997, 2002, 2004, CUH–1989]**

Solution : Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$\therefore \dfrac{\partial u}{\partial x} = 6xy + 4x \cdots\cdots (1)$

$\dfrac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y \cdots\cdots (2)$

$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \cdots\cdots (3)$

$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \cdots\cdots (4)$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$

$\Rightarrow$ u satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 4y$ [by (2)]

By integrating this w. r. to x keeping y as constant

$v = \int(-3x^2 + 3y^2 + 4y)\,dx$

$\Rightarrow v = -x^3 + 3xy^2 + 4xy + F(y) \cdots\cdots (5)$

$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x + F'(y)$

$\Rightarrow \frac{\partial u}{\partial x} = 6xy + 4x + F'(y)$; by C-R equation.

$\Rightarrow 6xy + 4x = 6xy + 4x + F'(y)$; [by (1)]

$\Rightarrow 0 = F'(y)$

$\therefore F(y) = c_1$ by integrating

Putting this value in (5) we get

$$v = -x^3 + 3xy^2 + 4xy + c_1$$

Let $f(z) = u + iv$

$= 3x^2y + 2x^2 - y^3 - 2y^2 + i(-x^3 + 3xy^2 + 4xy + c_1)$

$= (-ix^3 + 3ixy^2 + 3x^2y - y^3) + 2(x^2 - y^2 + 2ixy) + ic_1$

$= -i(x^3 + 3ix^2y + 3i^2xy^2 + i^3y) + 2(x^2 + 2ixy + i^2y^2) + c$

$= -i(x + iy)^3 + 2(x + iy)^2 + c$, where $c = ic_1$

$\Rightarrow f(z) = u + iv = -iz^3 + 2z^2 + c.$ (Ans)

By Milne's methd :

Given that $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$\therefore \frac{\partial u}{\partial x} = 6xy + 4x = \phi_1(x, y)$, say $\cdots\cdots (1)$

$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \cdots\cdots (2)$

$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y = \phi_2(x, y)$, say $\cdots\cdots (3)$

$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \cdots\cdots (4)$

(2) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$

$\Rightarrow$ u satisfy Laplace equation

$\Rightarrow$ u is harmonic.

Putting x = z and y = 0 in (1) and (3) we get

$$\phi_1(z, 0) = 0 + 4z = 4z$$

and $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

By Milne's theorem we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 4z - i3z^2$$

$\Rightarrow f(z) = 2z^2 - iz^3 + c$ by integrating

$$\Rightarrow u + iv = 2(x + iy)^2 - i(x + iy)^3 + c$$
$$= 2(x^2 + 2ixy + i^2y^2) - i(x^3 + 3ix^2y + 3i^2xy^2 + i^3 y^3) + c$$
$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y - 3ixy^2 - y^3 + c_1 + ic_2$$

where $c = c_1 + ic_2$

Equating imaginary parts we get,

$$v = 4xy - x^3 - 3xy^2 + c_2$$

Thus, $v = -x^3 - 3xy^2 + 4xy + c_2$
and $u + iv = -iz^3 + 2z^2 + c$ (Ans)

Example–40. Show that the function $u = 2x - x^3 + 3xy^2$ is harmonic and also find the harmonic conjugate if f(z) = u + iv is analytic. **[DUH–1991, 2003, CUH–1985, JUH–1987, 1989]**

Solution : Given that $u = 2x - x^3 + 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 6xy = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6x \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

This shows that u satisfy Laplace equation and hence u is harmonic.

To find the harmonic conjugate v, we have by putting x = z and y = 0 in (1) and (2) we get

$$\phi_1(z, 0) = 2 - 3z^2$$

and $\phi_2(z, 0) = 0$

By Milne's theorem we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 2 - 3z^2 - 0$$
$$\therefore f(z) = \int (2 - 3z^2)\, dz$$
$$= 2z - z^3 + c$$
$$\Rightarrow u + iv = 2(x + iy) - (x + iy)^3 + c$$
$$= 2(x + iy) - (x^3 + 3ix^2y + 3i^2 xy^2 + i^3 y^3) + c$$
$$= 2(x + iy) - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$
$$= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3 + c_1); \text{ where } ic_1 =$$

Equating imaginary parts we have

$$v = 2y - 3x^2y + y^3 + c_1 \quad \text{(Ans)}$$

Example–41. Find the harmonic conjugate of the func
$= x^3 + 6x^2y - 3xy^2 - 2y^3$. **[DUH–1987, 1991, JUH-1**

Solution : Given $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 + 6x^2y - 3xy^2 - 2y^3)$$
$$= 3x^2 + 12xy - 3y^2 = \phi_1(x, y), \text{ say} \cdots\cdots (1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 + 6x^2y - 3xy^2 - 2y^3)$$
$$= 6x^2 - 6xy - 6y^2 = \phi_2(x, y), \text{ say} \cdots\cdots (2)$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 12xy - 3y^2)$$
$$= 6x + 12y \cdots\cdots (3)$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(6x^2 - 6xy - 6y^2)$$
$$= -6x - 12y \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y$

Since u satisfies Laplace equation, so u is harmonic.

Let v be the harmonic conjugate of u, so that

f (z) = u + iv is analytic.

Now from (1), $\phi_1(z, 0) = 3z^2 + 0 - 0 = 3z^2$

and from (2), $\phi_2(z, 0) = 6z^2 - 0 - 0 = 6z^2$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 3z^2 - 6iz^2$$

$$\Rightarrow f(z) = \int (3z^2 - 6iz^2)\, dz$$

$$\Rightarrow u + iv = \int (3 - 6i)\, z^2\, dz$$

$$= (3 - 6i)\frac{z^3}{3} + c_1 + ic_2, \text{ where } c_1 + ic_2 = \text{complex constant}$$

$$= (1 - 2i)\, z^3 + c_1 + ic_2$$

$$= (1 - 2i)\,(x + iy)^3 + c_1 + ic_2$$

$$= (1 - 2i)\,(x^3 + 3ix^2y - 3xy^2 - iy^3) + c_1 + ic_2$$

$$= x^3 + 3ix^2y - 3xy^2 - iy^3 - i2x^3 + 6x^2y + 6ixy^2 - 2y^3 + c_1 + ic_2$$

$$= (x^3 - 3x\,y^2 + 6x^2y - 2y^3 + c_1) + i(3x^2y - y^3 - 2x^3 + 6xy^2 + c_2)$$

Equating imaginary parts we get,

$$v = 3x^2y - y^3 - 2x^3 + 6xy^2 + c_2. \quad \text{(Ans)}$$

Example-42. Show that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic fuunction. Find v such that u + iv is analytic.

[NUH–1998, 2006, RUH–1986, CUH–1986, JUH (Phy)–1996]

Solution : Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$

$$= 3x^2 - 3y^2 + 6x = \phi_1(x, y), \text{ say} \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2 + 3x^2 - 3y^2 + 1)$$

$$= -6xy - 6y = \phi_2(x, y), \text{ say} \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 6x)$$

$$= 6x + 6 \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-6xy - 6y)$$

$$= -6x - 6 \cdots\cdots (4)$$

(3) + (4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$

$$\Rightarrow \nabla^2 u = 0$$

$\therefore$ u satisfies Laplace equation, so u is a harmonic function. Let v is the harmonic conjugate of u, so that f(z) = u + iv is analytic.

2nd Part : Putting $x = z$, $y = 0$ in (1) and (2) we get

$$\phi_1(z, 0) = 3z^2 + 6z$$

and $\phi_2(z, 0) = -0 - 0 = 0$

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= 3z^2 + 6z - 0i$$

$$\Rightarrow f(z) = \int (3z^2 + 6z)\, dz$$

$\Rightarrow u + iv = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c_1 + ic_2$, where $c_1 + ic_2$ is complex constant.

$$= z^3 + 3z^2 + c_1 + ic_2$$
$$= (x + iy)^3 + 3(x + iy)^2 + c_1 + ic_2$$
$$= x^3 + 3ix^2y - 3xy^2 - iy^3 + 3x^2 + 6ixy - 3y^2 + c_1 + ic_2$$
$$= x^3 - 3xy^2 + 3x^2 - 3y^2 + c_1 + i(3x^2y - y^3 + 6xy + c_2)$$

Equating imaginary parts we get

$$v = 3x^2y - y^3 + 6xy + c_2. \quad \text{(Ans)}$$

Example–43. Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic and hence find its harmonic conjugate v if $f(z) = u + iv$ is analytic. **[NUH–2002]**

Solution : Given $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 6xy + 4x = \phi_1(x, y), \text{ say } \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^2y + 2x^2 - y^3 - 2y^2)$$
$$= 3x^2 - 3y^2 - 4y = \phi_2(x, y), \text{ say } \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(6xy + 4x)$$
$$= 6y + 4 \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(3x^2 - 3y^2 - 4y)$$
$$= -6y - 4 \cdots\cdots (4)$$

(3) + (4) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4$

$$\Rightarrow \nabla^2 u = 0$$

Since u satisfies Laplace equation, so u is harmonic. Let v is the harmonic conjugate of u, so that $f(z) = u + iv$ is analytic.

Putting $x = z$, $y = 0$ in (1) and (2) we get

$$\phi_1(z, 0) = 0 + 4z = 4z$$

and $\phi_2(z, 0) = 3z^2 - 0 - 0 = 3z^2$

$\therefore$ By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$\Rightarrow f'(z) = 4z - i3z^2$$

$$\Rightarrow f(z) = \int (4z - i3z^2)\, dz$$

$\Rightarrow u + iv = 4 \cdot \dfrac{z^2}{2} - i3\dfrac{z^3}{3} + c_1 + ic_2$, where $c_1 + ic_2$ is complex constant.

$$= 2z^2 - iz^3 + c_1 + ic_2$$
$$= 2(x + iy)^2 - i(x + iy)^3 + c_1 + ic_2$$
$$= 2x^2 + 4ixy - 2y^2 - ix^3 + 3x^2y + i3xy^2 - y^3 + c_1 + ic_2$$
$$= (2x^2 - 2y^2 + 3x^2y - y^3 + c_1) + i(4xy - x^3 + 3xy^2 + c_2)$$

Equating imaginary parts we get

$$v = 4xy - x^3 + 3xy^2 + c_2. \quad \text{(Ans)}$$

Example–44. Show that $\psi(x, y) = \dfrac{1}{2}\log(x^2 + y^2)$ is a harmonic in the region $C - \{(0, 0)\}$. Find the harmonic conjugate of this function such that $f(z) = \psi + i\phi$ is analytic and also find $f(z)$ interms of z. **[NUH–2004, DUH–1986]**

Solution : Given $\psi = \psi(x, y) = \dfrac{1}{2}\log(x^2 + y^2)$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2} = \phi_1(x, y), \text{ say} \;\cdots\cdots (1)$$

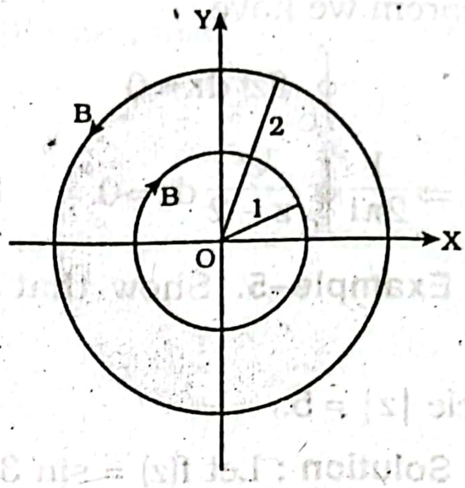
$$\frac{\partial \psi}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2} = \phi_2(x, y), \text{ say} \;\cdots\cdots (2)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \;\cdots\cdots (3)$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \;\cdots\cdots (4)$$

Example–3. Evaluate $\int_B \frac{dz}{z^2(z^2+9)}$ where B is the circle $|z| = 2$, described in the positive direction, together with the circle $|z| = 1$ described in the negative direction. **[RUH–1999]**

Solution :



Let $\int_B \frac{dz}{z^2(z^2+9)} = \int_B f(z)\,dz$

where $f(z) = \frac{1}{z^2(z^2+9)}$

f(z) is fails to be analytic at $z^2 = 0$ and $z^2 + 9 = 0$

That is, at $z = 0$ and $z = \pm 3i$

all of which lie outside the annular region with boundary B. Hence by Cauchy's integral theorem

$$\int_B f(z)\,dz = 0$$

$$\Rightarrow \int_B \frac{dz}{z^2\,(z^2+9)} = 0$$

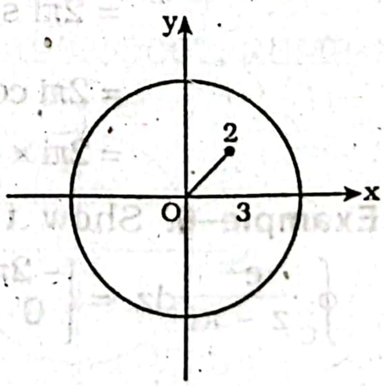
Example–4. Show that

$$\frac{1}{2\pi i}\oint \frac{e^z}{z-2}\,dz = \begin{cases} e^2, & \text{if C is the circle } |z| = 3 \\ 0, & \text{if C is the circle } |z| = 1 \end{cases}$$ **[DUH–1984]**

Solution : From Cauchy's integral formula we have

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \text{ ...... (1)}$$

Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$.



Again $z = a = 2$

$\Rightarrow |z| = |2| = 2 < 3$

$\therefore z = 2$ lies inside the circle $|z| = 3$

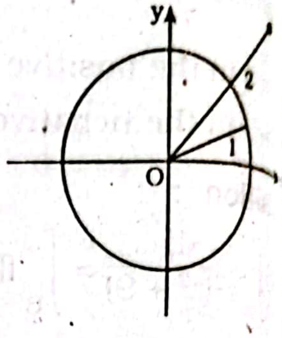
Hence by (1), $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = f(2)$

$= e^2$.

(Showed)

Again, $|z| = |2| = 2 > 1$

$\therefore z = 2$ lies outside the circle $|z| = 1$.



Hence by Cauchy's integral theorem we have

$$\oint_C f(z)\, dz = 0$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\, dz = 0. \qquad \text{(Showed)}$$

Example–5. Show that $\oint_C \dfrac{\sin 3z}{z + \frac{\pi}{2}}\, dz = 2\pi i$, where C is circle $|z| = 5$. [RUH-1...]

Solution : Let $f(z) = \sin 3z$

Then $f(z) = \sin 3z$ is analytic inside and on the circle $|z| = 5$

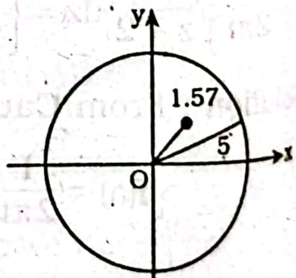
Also, here $z = -\dfrac{\pi}{2}$

$$\Rightarrow |z| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2} = \frac{3{\cdot}14}{2} = 1{\cdot}57 < 5$$

$\therefore z = -\dfrac{\pi}{2}$ lies in the circle $|z| = 5$.

Hence by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz = f(a)$$



$$\Rightarrow \oint_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

$$\Rightarrow \oint_C \frac{\sin 3z}{z + \frac{\pi}{2}}\, dz = 2\pi i\, f\left(\frac{-\pi}{2}\right)$$

$$= 2\pi i \sin\left(\frac{-3\pi}{2}\right)$$

$$= 2\pi i \cos 0^0$$

$$= 2\pi i \times 1 = 2\pi i. \qquad \text{(Showed)}$$

Example–6. Show that

$$\oint_C \frac{e^{3z}}{z - \pi i}\, dz = \begin{cases} -2\pi i, & \text{if C is the circle } |z-1| = 4 \\ 0, & \text{if C is the ellipse } |z-2| + |z+2| = \end{cases}$$

[NUH-1999, RUH-1981, 1...]

Solution : $f(z) = e^{3z}$ is analytic inside and on the circle $|z - 1|$

⭘ The Theorem is valid for both simply and mu connected regions.

Cauchy's Theorem or Cauchy's Fundamental Theore Cauchy's Integral Theorem.

Theorem–5. If f(z) is analytic in a region R and on its cl boundary C with derivative f′(z) which is continuous at all p inside R and on C, then

$$\oint_C f(z)\,dz = 0$$

[NUH–95, 97, 04 (Old), 06, NU(Phy)–04, RUH

Proof : Let z = x + iy. Then $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = i$.

Given, f(z) is analytic and f′(z) is continuous.

$\frac{\partial}{\partial z}[f(z)] = \frac{\partial}{\partial z}(u + iv)$, where $f(z) = u + iv = u(x, y) + iv(x, y)$

$$\Rightarrow \frac{\partial}{\partial x}(u + iv)\cdot\frac{\partial x}{\partial z} = \frac{\partial}{\partial y}(u + iv)\cdot\frac{\partial y}{\partial z};$$

Since f′(z) exist so f will be independent manner.

$$\Rightarrow \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot 1 = \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\cdot\frac{1}{i}$$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\ldots\ldots (1) \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\ldots\ldots (2)$$

which are continuous inside R and on C.

∴ By applying Green's theorem we get,

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dx\,dy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy, \quad \text{[by (1) and (2)]}$$

$$= 0 + i0 = 0$$

$$\therefore \oint_C f(z)\,dz = 0 \qquad \text{(Proved).}$$