Definition-2. The imaginary unit i (iota) is defined as **i = (O,**

Proposition-1. $i^2 = -1$.

Proof : $i^2 = i \cdot i = (0, 1) \cdot (0, 1)$

$= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)$

$= (-1, 0) \equiv -1$

Proposition-2. Every complex number $z = (x, y)$ can b written as $z = x + iy$ or $x + yi$.

Proof : $z = (x, y) = (x, 0) + (0, y)$

$= (x, 0) + (0, 1)(y, 0)$

$= x + iy$

$x + iy = (x, 0) + (0, 1)(y, 0)$

$= (x, 0) + (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y$

$= (x, 0) + (0, y)$

$= (x, y) = z$

Similarly, $z = (x, y) = (x, 0) + (0, y)$

$= (x, 0) + (y, 0)(0, 1)$

$= x + yi$

Thus, definition-1 can be modify as, "any number of the form $x + iy$ is called a complex number where $x, y \in \mathbf{R}$."

With the introduction of i, Definition-1 for addition and multiplication of complex numbers can be translated as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

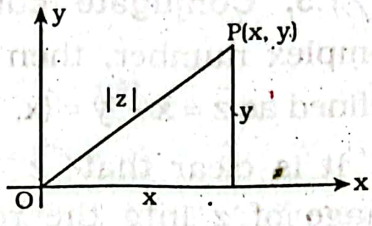
The term imaginary number does not mean that such a number does not exist. The letter i is a symbol which denotes imaginary unit just as 1 denotes the unit of real numbers. Thus the imaginary number iy means y unit of imaginary numbers just as x means x units of real numbers. The expression $x + iy$ is not an imaginary number, it is a complex number. If $z = x + iy$, then x is called the real part of z and y is called the imaginary part of z, written as

$$\text{Re}(z) = x, \text{Im}(z) = y.$$

1.2. Equality of complex numbers : Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal $\Leftrightarrow x_1 = x_2$ and $y_1 = y_2$, that is, the real part of the one is equal to the real part of the other, and the imaginary part of the one is equal to the imaginary part of the other

Note : The phrases "greater than" or "less than" have no meaning in relation between two complex numbers. Inequalities can only occur in relation between the moduli of complex numbers.

1.3. Geometrical representation : Every complex number can be represented geometrically as a point in the xy-plane. We can identify the complex number $z = x + iy$ with the point $P(x, y)$. The set of all real numbers $(x, 0)$ corresponding to the x-axis, called real axis, and the set of all imaginary numbers $(0, y)$ corresponding to the y-axis, called the imaginary axis. The origin identifies complex number $0 = 0 + i0$.



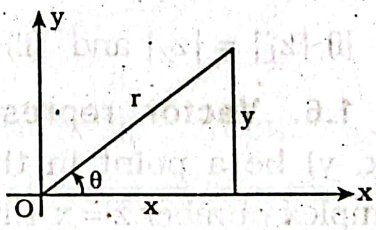
The distance of P from the origin is $\sqrt{x^2 + y^2}$. The nonnegative value of $\sqrt{x^2 + y^2}$ is denoted by $|z|$ and hence $|z| = \sqrt{x^2 + y^2}$. $|z|$ is called the modulas or absolute value of $z = (x, y)$.

We shall use the following inequalities :

$$x \le |x| \le \sqrt{x^2 + y^2} \Rightarrow Re(z) \le |Re(z)| \le |z|$$

$$y \le |y| \le \sqrt{x^2 + y^2} \Rightarrow Im(z) \le |Im(z)| \le |z|$$

1.4. Polar and exponential form of complex numbers: Let r and θ be polar coordinates of a point $z = (x, y)$. For $z \neq 0$, let $x = r\cos\theta$, $y = r\sin\theta$. Then



$$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$\therefore r = +\sqrt{x^2 + y^2}$$

$$\text{and } \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

r is called the modulas or the absolute value and θ is called the amplitude or argument of the complex number z.

$$|z| = r = +\sqrt{x^2 + y^2} = 0 \Leftrightarrow x = 0 \text{ and } y = 0 \Leftrightarrow z = x + iy = 0.$$

The value of θ between $-\pi$ and π is called the principal value of the amplitude. We denote it by Arg z.

Again, $z = x + iy = r\cos\theta + ir\sin\theta$

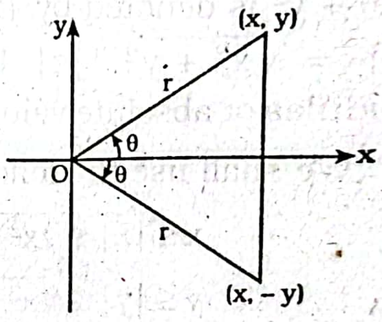
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}$$

which is called the polar or exponential form of the comple number z.

Note : $r = |z|$ is a unique nonnegative number, but $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is a multivalued function.

Argand plane or Argand diagram : When a complex number is represented by a point P(x, y) in the xy-plane, then this plane is called the Argand plane or Argand diagram or simply a complex plane.

1.5. Conjugate complex number : If $z = x + iy$ is any complex number, then its complex conjugate denoted by $\bar{z}$ is defined as $\bar{z} = x - iy = (x, -y)$.



It is clear that $\bar{z}$ is the mirror image of z into the real axis. This indicates that $z = \bar{z} \Leftrightarrow z$ is purely a real number. Also, $\bar{\bar{z}} = z$.

Again, $\bar{z} = x - iy = r\cos\theta - ir\sin\theta$

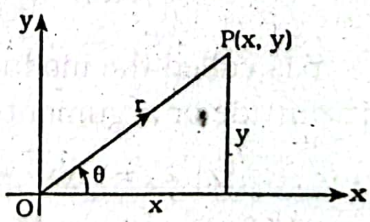
$= r(\cos\theta - i\sin\theta) = r\,e^{-i\theta}$

$\therefore |\bar{z}| = r = |z|$ and $\text{Amp}\,\bar{z} = -\theta = -\text{Amp}\,z$.

Thus condition for two given numbers z_1 and z_2 to be conjugate

(i) $|z_1| = |z_2|$ and (ii) $\text{amp}\,z_1 + \text{amp}\,z_2 = 0$.

1.6. Vector representation of complex number : Let P(x, y) be a point in the complex plane corresponding to the complex number $z = x + iy$.



Then the modulas $r = \sqrt{x^2 + y^2}$ is represented by the magnitude of the vector $\overrightarrow{OP}$ and its amplitude θ is represented by the direction of the vector $\overrightarrow{OP}$.

Hence the complex number $z = x + iy$ is completely represents by the vector $\overrightarrow{OP}$.

SOLVED EXAMPLES

Example-1. Express $\frac{(1+2i)^2}{(2+i)^2}$ in the form A + iB. Also find modulas and argument. **[DUH-19**

Solution : $\frac{(1+2i)^2}{(2+i)^2} = \frac{1+4i+4i^2}{4+4i+i^2}$

$$= \frac{1+4i-4}{4+4i-1} \quad [\because i^2 = -1]$$

$$= \frac{-3+4i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{-9+12i+12i-16i^2}{3^2-(4i)^2}$$

$$= \frac{-9+24i+16}{9+16} = \frac{7+24i}{25}$$

$$= \frac{7}{25} + i\frac{24}{25} \quad \text{Ans.}$$

Modulas $= \left|\frac{(1+2i)^2}{(2+i)^2}\right| = \left|\frac{7}{25} + i\frac{24}{25}\right|$

$$= \sqrt{\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2} = \sqrt{\frac{49+576}{625}}$$

$$= \sqrt{\frac{625}{625}} = 1$$

The principal argument $= \tan^{-1}\left(\frac{24/25}{7/25}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.

The general argument = $2n\pi$ + principal argument

$$= 2n\pi + \tan^{-1}\left(\frac{24}{7}\right)$$

where $n = 0, \pm 1, \pm 2, \cdots$ etc.

Example-2. Find the real and imaginary parts of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}$ and also find its modulas. **[RUH-1986]**

Solution : $\frac{1+\cos\theta+i\sin\theta}{1+\cos\phi+i\sin\phi} = \frac{2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\phi}{2} + i\,2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}$

$$= \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{2\cos\frac{\phi}{2}\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \frac{e^{i\theta/2}}{e^{i\phi/2}} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} e^{i(\theta-\phi)/2}$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\left[\cos\left(\frac{\theta-\phi}{2}\right) + i\sin\left(\frac{\theta-\phi}{2}\right)\right]$$

Real part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \cdot \cos\frac{\theta-\phi}{2}$

Imaginary part of $\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi} = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \sin\frac{\theta-\phi}{2}$

Modulas $= \left|\frac{1+\cos\theta + i\sin\theta}{1+\cos\phi + i\sin\phi}\right| = \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} e^{i(\theta-\phi)/2}\right|$

$$= \left|\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}\right| \left|e^{i(\theta-\phi)/2}\right|$$

$$= \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \cdot 1 = \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} \quad \text{Ans.}$$

Example–3. Find the modulas and argument of the following complex numbers :

(i) $\frac{-2}{1+i\sqrt{3}}$, (ii) $\frac{1-i}{1+i}$ **[DUH–2005, JUH–1987]**

Solution : (i) $\frac{-2}{1+i\sqrt{3}} = \frac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-2+i2\sqrt{3}}{1-i^2 3}$

$$= \frac{-2+i2\sqrt{3}}{1+3} = \frac{-2}{4} + i\frac{2\sqrt{3}}{4} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\therefore \text{ Modulas} = \left|\frac{-2}{1+i\sqrt{3}}\right| = \left|-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

The principal argument $= \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$

The general argument is $2n\pi + \frac{2\pi}{3}$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.

(ii) $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$

Modulas $= \left|\frac{1-i}{1+i}\right| = |-i| = \sqrt{0+(-1)^2} = \sqrt{1} = 1$

The principal argument $= \tan^{-1}\left(\frac{-1}{0}\right) = -\tan^{-1}(\infty) = -\frac{\pi}{2}$

The general argument $= 2n\pi - \frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2 \cdots$ etc.

Example–4. Find the modulas and argument of the compl[ex] number $\left(\frac{2+i}{3-i}\right)^2$. **[DUHT–199**

Solution : $\left(\frac{2+i}{3-i}\right)^2 = \frac{4+4i+i^2}{9-6i+i^2} = \frac{4+4i-1}{9-6i-1} = \frac{3+4i}{8-6i}$

$$= \frac{3+4i}{8-6i} \times \frac{8+6i}{8+6i} = \frac{24+18i+32i+24i^2}{8^2-6^2i^2}$$

$$= \frac{24+50i-24}{64+36}$$

$$= \frac{i50}{100} = \frac{i}{2}$$

$\therefore$ Modulas $= \left|\left(\frac{2+i}{3-i}\right)^2\right| = \left|\frac{i}{2}\right| = \sqrt{0+\frac{1}{4}} = \frac{1}{2}$

The principal argument $= \tan^{-1}\left(\frac{1/2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

The general argument $= 2n\pi + \frac{\pi}{2}$, where $n = 0, \pm 1, \pm 2, \cdots$ etc.

Example–5. Find the square roots of the complex number $5 - 12i$. **[RUH–2002]**

Solution : $5 - 12i = 9 - 12i - 4$

$$= 3^2 - 2 \cdot 3 \cdot 2i + (2i)^2 \quad [\because i^2 = -1]$$

$$= (3 - 2i)^2$$

$$\Rightarrow \sqrt{5 - 12i} = \sqrt{(3 - 2i)^2} = \pm (3 - 2i)$$

$\therefore$ The square roots of $5 - 12i$ are $3 - 2i$ and $-3 + 2i$.

Example–6. Express $-5 + 5i$ in polar form. **[RUH–1998]**

Solution : Let $-5 = r\cos\theta$ and $5 = r\sin\theta$

Then by squaring and adding we get

$$(-5)^2 + 5^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow 25 + 25 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\Rightarrow 50 = r^2 \Rightarrow r = \sqrt{50} = 5\sqrt{2}$$

Also $\tan\theta = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{5}{-5} = -1 = -\tan\dfrac{\pi}{4} = \tan\left(\pi - \dfrac{\pi}{4}\right) = \tan\dfrac{3\pi}{4}$

$$\Rightarrow \theta = \frac{3\pi}{4}$$

Thus, $-5 + 5i = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

$$\Rightarrow -5 + 5i = 5\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

which is the required polar form.

Example–7. Show that the sum of the products of all the nth roots of unity taken 2, 3, 4, ⋯ , (n – 1) at a time is zero. **[RUH–1986]**

Solution : Let $z = (1)^{1/n}$. Then $z^n = 1 \Rightarrow z^n - 1 = 0$

$$\Rightarrow z^n + 0 \cdot z^{n-1} + 0 \cdot z^{n-2} + \cdots + 0 \cdot z - 1 = 0$$

Let $z_1, z_2, z_3, \cdots, z_n$ be the roots of this equation.

Sum of the products of the roots

taken two at a time is $\sum z_1 z_2 = 0$,

taken three at a time is $\sum z_1 z_2 z_2 = 0$

… … … …

taken $(n - 1)$ at a time is $\sum z_1 z_2 \cdots z_{n-1} = 0$ (Proved)

Now $|z_1z_2| = |(x_1 + iy_1)(x_2 + iy_2)|$

$$= |x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2|$$

$$= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)|$$

$$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}$$

$$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2 + x_2^2y_1^2 + 2x_1x_2}$$

$$= \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2}$$

$$= |z_1|\,|z_2|$$

$\therefore |z_1z_2| = |z_1|\,|z_2|$

Other way : We know that $|z|^2 = z\bar{z}$ and $\overline{z_1z_2} = \bar{z}_1\,\bar{z}_2$.

$\therefore |z_1z_2|^2 = (z_1z_2)\,(\overline{z_1z_2})$

$$= z_1z_2\,\bar{z}_1\,\bar{z}_2 = (z_1\bar{z}_1)\,(z_2\bar{z}_2) = |z_1|^2\,|z_2|^2$$

$$= (|z_1|\,|z_2|)^2$$

$\Rightarrow |z_1z_2| = |z_1|\,|z_2|$

(ii) $|z_1z_2 \cdots z_n|^2 = (z_1z_2 \cdots z_n)\,(\overline{z_1z_2 \cdots z_n})$

$$= (z_1z_2 \cdots z_n)\,(\bar{z}_1\,\bar{z}_2 \cdots \bar{z}_n)$$

$$= (z_1\,\bar{z}_1)\,(z_2\bar{z}_2) \cdots (z_n)\,(\bar{z}_n)$$

$$= |z_1|^2\,|z_2|^2 \cdots |z_n|^2$$

$$= (|z_1|\,|z_2| \cdots |z_n|)^2$$

$\Rightarrow |z_1z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$ (proved)

Example–22. For any complex number $z_1, z_2, \cdots, z_n$ prove that

(i) $|z_1 + z_2| \le |z_1| + |z_2|$.

[NUH–02 (Old), 03, 04, 05, 07, NU(Pre)–08, DUH–98, 05]

(ii) $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$

[RUH–1998, CUH–2004]

(iii) $|z_1 - z_2| \le |z_1| + |z_2|$ **[NUH–2006]**

Solution : We know that $|z|^2 = z\bar{z}$ and $\bar{\bar{z}} = z$.

$$\therefore |z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$
$$= (z_1+z_2)(\overline{z_1}+\overline{z_2})$$
$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$
$$= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2$$
$$= |z_1|^2 + 2\text{Re}(z_1\overline{z_2}) + |z_2|^2 \qquad [\because z+\bar{z} = 2\text{Re}(z)]$$
$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \qquad [\because \text{Re}(z) \le |z|]$$
$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + 2|z_1|\,|\overline{z_2}| + |z_2|^2 \qquad [\because |z_1z_2| = |z_1|\,|z_2|]$$
$$\Rightarrow |z_1+z_2|^2 \le |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 \qquad [\because |z| = |\bar{z}|]$$
$$\Rightarrow |z_1+z_2|^2 \le (|z_1| + |z_2|)^2$$
$$\Rightarrow |z_1+z_2| \le |z_1| + |z_2|.$$

Other way : Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then $|z_1| = |r_1 e^{i\theta_1}|$
$$= |r_1|\,|e^{i\theta_1}| = r_1 \cdot 1 = r_1 \text{ and } |z_2| = r_2$$
$$\therefore z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$$
$$\Rightarrow |z_1+z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$$
$$= r_1^2\cos^2\theta_1 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos\theta_2^2$$
$$+ r_1^2\sin^2\theta_1 + 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin\theta_2^2$$
$$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + 2r_1r_2(\cos\theta_1\cos\theta_2$$
$$+ \sin\theta_1\sin\theta_2) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$$
$$= r_1^2 \cdot 1 + 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2 \cdot 1$$
$$\Rightarrow |z_1+z_2|^2 \le r_1^2 + 2r_1r_2 + r_2^2 \qquad [\because \cos(\theta_1 - \theta_2) \le 1]$$
$$\Rightarrow |z_1+z_2|^2 \le (r_1+r_2)^2$$
$$\Rightarrow |z_1+z_2| \le r_1 + r_2$$
$$\Rightarrow |z_1+z_2| \le |z_1| + |z_2|$$

(ii) $|z_1 + z_2 + \cdots + z_n| = |z_1 + (z_2 + \cdots + z_n)|$
$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2 + \cdots + z_n| \qquad [\text{by (i)}]$$
$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2 + (z_3 + \cdots + z_n)|$$
$$\Rightarrow |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + |z_3 + \cdots + z_n|$$

Proceding in the same way we get
$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

(iii) $|z_1 - z_2|^2 = |z_1 - z_2)\ \overline{(z_1 - z_2)}$ $[\because |z|^2 = z\bar{z}]$

$= (z_1 - z_2)\ (\overline{z_1} - \overline{z_2})$ $[\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}]$

$= z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$

$= |z_1|^2 - (z_1\overline{z_2} + z_2\overline{z_1}) + |z_2)^2$ $[\because |z|^2 = z\bar{z}]$

$= |z_1|^2 - (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + |z_2|^2$ $[\because \bar{\bar{z}} = z]$

$= |z_1|^2 - 2\text{Re}(z_1\overline{z_2}) + |z_2|^2$

$\Rightarrow |z_1 - z_2|^2 \le |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2$ $\left[\begin{array}{l}\because -x \le \sqrt{x^2 + y^2} \\ \Rightarrow -\text{Re}(z) \le |z|\end{array}\right.$

$\Rightarrow |z_1 - z_2|^2 \le (|z_1| + |z_2|)^2$

$\Rightarrow |z_1 - z_2| \le |z_1| + |z_2|$ (Proved)

Example–23. Let **C** be the set of all complex numbers. Consider $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in$ **C** with $x_1 < x_2$, $y_1 < y_2$. Do you agree that $z_1 < z_2$? What about $|z_1| < |z_2|$? Prove that $\left|\sum_{j=1}^{n} z_j\right| \le \sum_{j=1}^{n} |z_j|$ and $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$, where $z_1, z_2, \cdots, z_n$ are complex numbers. **[NUH–1997]**

Solution : Given that $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$. That is, z_1 and z_2 are two points in the Argand (complex) plane. We know that greater than or less than have no meaning in relation between two complex numbers. So $z_1 < z_2$ has no meaning. Thus, I do not agree that $z_1 < z_2$.

2nd Part : Given that $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$,

where $x_1 < x_2$, $y_1 < y_2$.

$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$

Now $x_1 < x_2 \Rightarrow x_1^2 < x_2^2$

and $y_1 < y_2 \Rightarrow y_1^2 < y_2^2$

$\Rightarrow x_1^2 + y_1^2 < x_2^2 + y_2^2$ [by adding]

$\Rightarrow \sqrt{x_1^2 + y_1^2} < \sqrt{x_2^2 + y_2^2}$ [by taking squre root]

$\Rightarrow |z_1| < |z_2|$

But this inequality is not always true. We counter this by the following example.

Let $z_1 = x_1 + iy_1 = 1 + i(-7)$

and $z_2 = x_2 + iy_2 = 3 + i2$

Here $x_1 = 1, x_2 = 3, y_1 = -7, y_2 = 2$

$$1 < 3 \Rightarrow x_1 < x_2 \text{ and } -7 < 2 \Rightarrow y_1 < y_2$$

$$|z_1| = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2 + (-7)^2} = \sqrt{50}$$

$$|z_2| = \sqrt{x_2^2 + y_2^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

It is true that $\sqrt{50} > \sqrt{13} \Rightarrow |z_1| > |z_2|$

Thus, $|z_1| < |z_2|$ is not always true under the given conditions. $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

3rd Part : Given that $\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|$

$\Rightarrow |z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$

Now do as example–22.

4th Part : Given that $\left|\prod_{j=1}^{n} z_j\right| = \prod_{j=1}^{n} |z_j|$

that is, $|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$

Now do as example–22.

Example–24. Prove that $|z_1 - z_2| \geq \big||z_1| - |z_2|\big| \geq |z_1| - |z_2|$

[NUH–2002 (Old), DUH–1998, 2005]

Solution : We know that $|z|^2 = z\bar{z}$ and $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$$\therefore |z_1 - z_2|^2 = (z_1 - z_2)\,\overline{(z_1 - z_2)}$$

$$= (z_1 - z_2)\,(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$$

$$= |z_1|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2) + |z_2|^2$$

$$= |z_1|^2 - (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + |z_2|^2 \qquad [\because \bar{\bar{z}} = z]$$

$$= |z_1|^2 - 2\mathrm{Re}(z_1\bar{z}_2) + |z_2|^2 \qquad [\because z + \bar{z} = 2\mathrm{Re}(z)]$$

$$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1\bar{z}_2| + |z_2|^2 \qquad \begin{bmatrix}\because x = \mathrm{Re}(z) \leq |z| \\ \Rightarrow -\mathrm{Re}(z) \geq -|z|\end{bmatrix}$$

$$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1|\,|\bar{z}_2| + |z_2|^2$$

$$\Rightarrow |z_1 - z_2|^2 \geq |z_1|^2 - 2|z_1|\,|z_2| + |z_2|^2 \qquad [\because |z| = |\bar{z}|]$$

$\Rightarrow |z_1 - z_2|^2 \geq (|z_1| - |z|)^2 = (||z_1| - |z_2||)^2$

$\Rightarrow |z_1 - z_2| \geq ||z_1| - |z_2||$ ...... (1)

Again, we have $||z_1| - |z_2|| \geq |z_1| - |z_2|$ ...... (2)

From (1) and (2) we have

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \geq |z_1| - |z_2| \qquad \text{(Proved)}$$

Example–25. If z_1, z_2, z_3, z_4 are complex numbers then show that

(i) $\left|\dfrac{z_1}{z_2 + z_3}\right| \leq \dfrac{|z_1|}{||z_2| - |z_3||}$, where $|z_2| \neq |z_3|$

[NUH–2001, DUH–98]

(ii) $\left|\dfrac{z_1 + z_2}{z_3 + z_4}\right| \leq \dfrac{|z_1| + |z_2|}{||z_3| - |z_4||}$, where $|z_3| \neq |z_4|$.

[CUH–2000, DUH–2001, 2003, 2006]

Solution : (i) We know that $|z_1 - z_2| \geq ||z_1| - |z_2||$

Replacing z_2 by $-z_2$ we get $|z_1 + z_2| \geq ||z_1| - |-z_2||$

$$\Rightarrow |z_1 + z_2| \geq ||z_1| - |z_2|| \qquad [\because |z| = |-z| \text{ ...... (1)}]$$

$$\Rightarrow \frac{1}{|z_1 + z_2|} \leq \frac{1}{||z_1| - |z_2||}$$

Multiplying both sides by $|z_1|$ we get

$$\frac{|z_1|}{|z_2 + z_3|} \leq \frac{|z_1|}{||z_2| - |z_3||}$$

(ii) We know that $|z_1 + z_2| \leq |z_1| + |z_2|$ ...... (2)

and by (1) we have $|z_3 + z_4| \geq ||z_3| - |z_4||$

$$\Rightarrow \frac{1}{|z_3 + z_4|} \leq \frac{1}{||z_3| - |z_4||} \text{ ...... (3)}$$

Combining (2) and (3) we get,

$$\frac{|z_1 + z_2|}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||} \qquad \text{(Proved)}$$

Example–26. Find two complex numbers whose sum is 4 and whose product is 8. **[NUH–2000, 2006 (Old)]**

Solution : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. According to the question,

$z_1 + z_2 = 4$ ...... (1)

and $z_1 z_2 = 8$ ...... (2)

$\Rightarrow p^2 + q^2$

b is real, since p and q are real.

Also, $p^2 + q^2 = b$ gives, $q^2 = b - p^2 > 0$

$$\Rightarrow b - p^2 > 0 \Rightarrow b - \left(\frac{-a}{2}\right)^2 > 0$$

$$\Rightarrow b - \frac{a^2}{4} > 0 \Rightarrow 4b - a^2 > 0$$

$$\Rightarrow 4b > a^2 \Rightarrow a^2 < 4b$$

Thus, we have a, b are both real and $a^2 < 4b$.

Example–32. Solve the equation $|z| - z = 2 + i$.

Solution : Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$

Given that $|z| - z = 2 + i$

$$\Rightarrow \sqrt{x^2 + y^2} - (x + iy) = 2 + i$$

Equating real and imaginary parts we get,

$$\sqrt{x^2 + y^2} - x = 2 \;\cdots\cdots (1)$$

$$-y = 1 \Rightarrow y = -1 \;\cdots\cdots (2)$$

Using (2) in (1) we get,

$$\sqrt{x^2 + 1} - x = 2$$

$$\Rightarrow \sqrt{x^2 + 1} = x + 2$$

$$\Rightarrow x^2 + 1 = x^2 + 4x + 4 \quad \text{by squaring}$$

$$\Rightarrow 4x = -3 \Rightarrow x = \frac{-3}{4}.$$

$$\therefore z = x + iy = \frac{-3}{4} - i \qquad \text{(Ans)}$$

Example–33. Describe geometrically the region of the following :

(i) $|z - 4| > |z|$ **[NUH–1998, DUH–1988, 1998]**

(ii) $|z - i| = |z + i|$ **[NUH–2003, 2006, DUH–1987, RUH–2004]**

(iii) $\text{Im}(z) > 1$ **[DUH–1988]**

(iv) $\text{Re}(\bar{z} - 1) = 2$

(v) $|z + 3i| > 4$ **[DUH–1989]**

(vi) $|z| > 4$ **[DUH–1986]**

(vii) $|z - 2 + i| \le 1$ **[DUH–1988]**

(viii) $|2z + 3| > 4$ **[DUH–1988]**

(ix) $\left|\frac{z-3}{z+3}\right| = 3$

(x) $\left|\frac{z-3}{z+3}\right| > 3$

(xi) $\left|\frac{z-3}{z+3}\right| < 3$ **[DUH–1989]**

(xii) $\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$ [NU(Pre)–08, DUH–1988]

(xiii) $\text{Re}\left(\frac{1}{z}\right) < \frac{1}{2}$ [NUH–00, 06 (Old), DUH– 86, 88, 90,98, 03]

(xiv) $\text{Im}\left(\frac{1}{z}\right) < \frac{1}{2}$ [DUH–1987]

(xv) $1 < |z+i| \le 2$ [NUH–2007, DUH–1989]

(xvi) $1 < |z+i| < 2$ [DUH–1990]

(xvii) $1 < |z-2i| \le 2$ [DUH–1988]

(xviii) $0 < \text{Re}(iz) < 1$ [DUH–1990]

(xix) $\frac{\pi}{3} \le \arg z \le \frac{\pi}{2}$

(xx) $-\pi < \arg z < \pi$ [DUH–1986]

(xxi) $-\pi < \arg z < \pi,\ z \ne 0$ [DUH–1988]

(xxii) $0 < \arg z < 2\pi,\ |z| > 0$ [DUH–1989]

(xxiii) $-\pi < \arg z < \pi,\ |z| > 2$ [DUH–1988]

(xxiv) $|z-1| + |z+1| \le 3$ [NUH–2005, CUH–2004]

(xxv) $|z+2-3i| + |z-2+3i| < 10$ [DUH–1989]

(xxvi) $|z-2| - |z+2| > 3$ [DUH–1990]

(xxvii) $\text{Re}(z^2) > 1$ [DUH–1989]

(xxviii) $\text{Im}(z^2) > 0$ [DUH–1989]

(xxix) $|z+i| + |z-i| \le 3$ [NUH–1995, 2008]

(xxx) $|z+2i| + |z-2i| = 6$ [NUH–2004 (Old)]

(xxxi) $1 \le |z+1+i| < 2$ [NUH–1994]

(xxxii) $\text{Re}\left(\frac{1}{z}\right) > \frac{1}{2}$ NUH–1994]

(xxxiii) $\text{Im}(z^2) > 2$ [NUH–1994]

(xxxiv) $|z+1+i| = |z-1+i|$ [NUH–1994

Solution : (i) Let $z = x + iy$.

Then $|z-4| > |z|$ gives

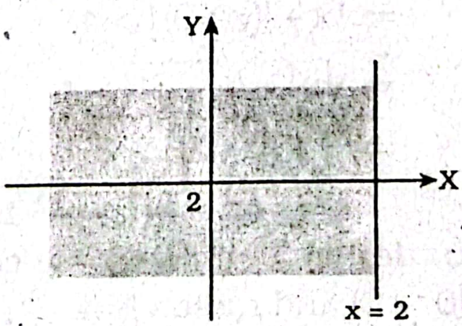
$$|x+iy-4| > |x+iy|$$

$$\Rightarrow \sqrt{(x-4)^2+y^2} > \sqrt{x^2+y^2}$$

$$\Rightarrow x^2 - 8x + 16 + y^2 > x^2 + y^2$$

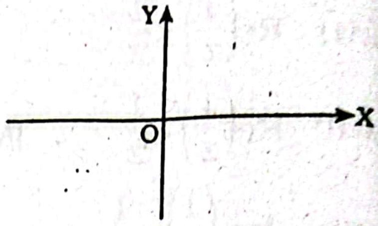
$$\Rightarrow -8x + 16 > 0$$

$$\Rightarrow 8x < 16 \Rightarrow x < 2$$



∴ The region is the set of all points (x, y) such that x < 2. That

(ii) $|z - i| = |z + i|$



$\Rightarrow |x + iy - i| = |x + iy + i|$

$\Rightarrow |x + i(y - 1)| = |x + i(y + 1)|$

$\Rightarrow \sqrt{x^2 + (y - 1)^2} = \sqrt{x^2 + (y + 1)^2}$

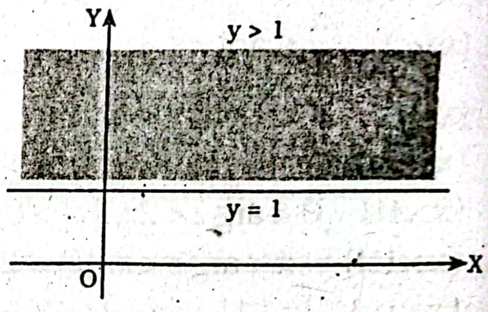
$\Rightarrow x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2y + 1$

$\Rightarrow -2y = 2y$

$\Rightarrow 4y = 0 \Rightarrow y = 0$, which is the equation of real axis **(x – axi**

Thus the region is the set of all points lie on the real axis.

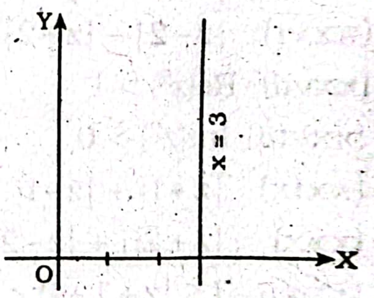
(iii) $\text{Im}(z) > 1$



$\Rightarrow \text{Im}(x + iy) > 1 \Rightarrow y > 1$

The region is the set of all points (x, y) such that $y > 1$, that is, the set of all points (x, y) lie in the upper of the line $y = 1$.

(iv) $\text{Re}(\overline{z} - 1) = 2$



$\Rightarrow \text{Re}(\overline{x + iy} - 1) = 2$, where $z = x + iy$

$\Rightarrow \text{Re}(x - iy - 1) = 2$

$\Rightarrow x - 1 = 2$

$\Rightarrow x = 3$

Which is the equation of a straight line parallel to y-axis. Thus the region is the set of all points $(3, y)$, where $y \in \mathbf{R}$.

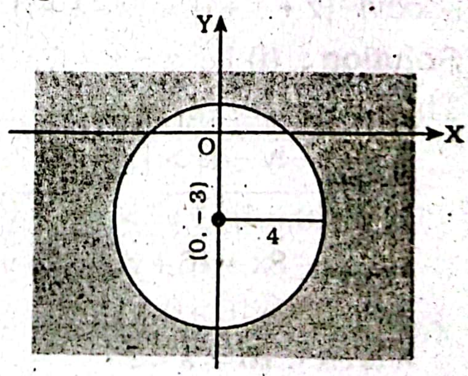
(v) Let $z = x + iy$. Then $|z + 3i| > 4$ gives

$|x + iy + 3i| > 4$

$\Rightarrow |x + i(y + 3)| > 4.$

$\Rightarrow \sqrt{x^2 + (y + 3)^2} > 4$

$\Rightarrow x^2 + (y + 3)^2 > 4^2$



$x^2 + (y + 3)^2 = 4^2$ is the equation of a circle whose centre is $(0, -3)$ and radius is 4.

Thus, the region is the set of all external points of the circle whose centre is $(0, -3)$ and radius is 4.

$$\Rightarrow \operatorname{Im}\left(\frac{x - iy}{(x + iy)(x - iy)}\right) < \frac{1}{2}$$

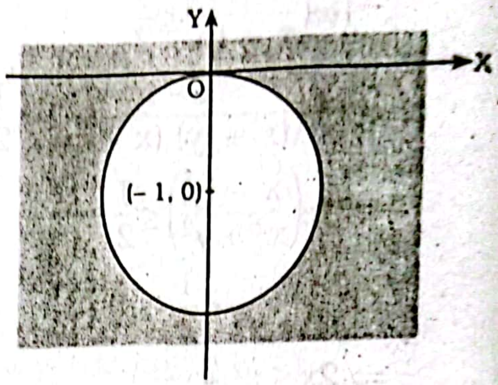
$$\Rightarrow \operatorname{Im}\left(\frac{x - iy}{x^2 + y^2}\right) < \frac{1}{2}$$

$$\Rightarrow \frac{-y}{x^2 + y^2} < \frac{1}{2}$$

$$\Rightarrow -2y < x^2 + y^2$$

$$\Rightarrow x^2 + y^2 + 2y > 0$$

$$\Rightarrow x^2 + (y + 1)^2 > 1$$



This represents the set of all external points of the cir whose centre is (– 1, 0) and radius is 1.

(xv) Let $z = x + iy$. Given that $1 < |z + i| \le 2$

$$\Rightarrow 1 < |x + iy + i| \le 2$$

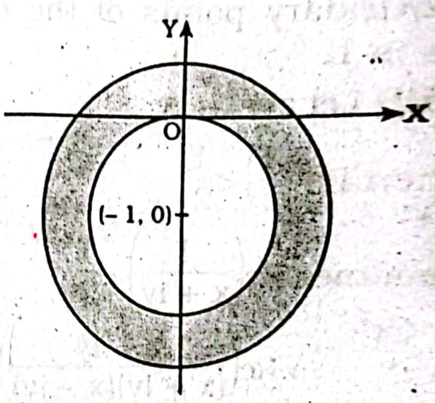
$$\Rightarrow 1 < |x + i(y + 1)| \le 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y + 1)^2} \le 2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2 \le 2^2$$

$$\Rightarrow 1 < x^2 + (y + 1)^2$$

and $x^2 + (y + 1)^2 \le 2^2$



$x^2 + (y + 1)^2 > 1$ represents all points external to the circl whose centre is (0, – 1) and radius is 1. Also, $x^2 + (y + 1)^2 \le 2$ represents all points internal and boundary of the circle whos centre is (0, – 1) and radius is 2. Thus the region is the set of al common points of external points of the circle $x^2 + (y + 1)^2 = 1$ and internal points of the circle $x^2 + (y + 1)^2 = 2^2$ including its boundary points.

(xvi) As example 33(xv), $1 < |z + i| < 2$ represents the common points of the external of the circle $x^2 + (y + 1)^2 = 1$ and internal of the circle $x^2 + (y + 1)^2 = 2^2$. Thus the region is the set of all common points of external points of the circle $x^2 + (y - 2)^2 = 1$ and internal points of the circle $x^2 + (y - 2)^2 = 2^2$ including its boundary points.

(xvii) Let $z = x + iy$. Given that $1 < |z - 2i| \leq 2$

$\Rightarrow 1 < x + iy - 2i| \leq 2$

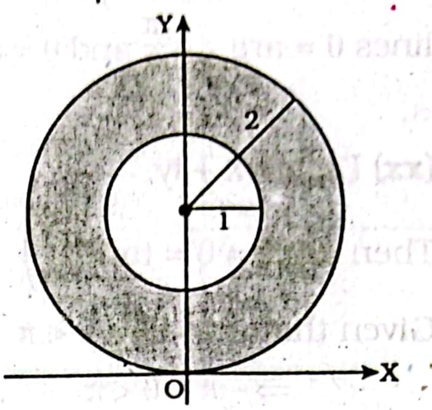
$\Rightarrow 1 < |x + i(y - 2)| \leq 2$

$\Rightarrow 1 < \sqrt{x^2 + (y-2)^2} \leq 2$

$\Rightarrow 1 < x^2 + (y-2)^2 \leq 2^2$

$\Rightarrow 1 < x^2 + (y-2)^2$

and $x^2 + (y-2)^2 \leq 2^2$.



$1 < x^2 + (y-2)^2$ represents all points external to the circle whose centre is $(0, 2)$ and radius is 1. Also, $x^2 + (y-2)^2 \leq 2^2$ represents all points internal and boundary of the circle whose centre is $(0, 2)$ and radius is 2. Thus the region is the set of all common points of external points of the circle $x^2 + (y-2)^2 = 1$ and internal points of the circle $x^2 + (y-2)^2 = 2^2$ including its boundary points.

(xviii) Let $z = x + iy$. Then $0 < \text{Re}(iz) < 1$ becomes

$0 < \text{Re}(ix + i^2 y) < 1$

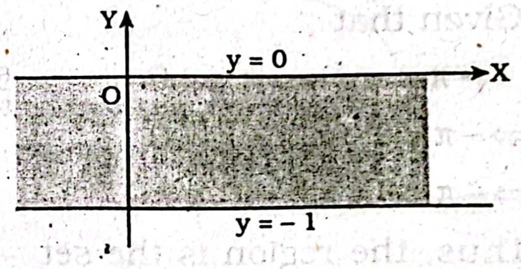
$\Rightarrow 0 < \text{Re}(ix - y) < 1$

$\Rightarrow 0 < -y < 1$

$\Rightarrow 0 < -y$ and $-y < 1$

$\Rightarrow 0 > y$ and $y > -1$

$\Rightarrow y < 0$ and $y > -1$



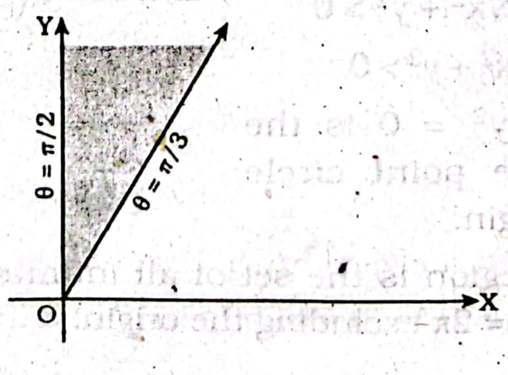
Thus the region lies between the lines $y = 0$ and $y = -1$.

(xix) Let $z = x + iy$.

Then $\arg z = \theta = \tan^{-1}\dfrac{y}{x}$.

Given that $\dfrac{\pi}{3} \leq \arg z \leq \dfrac{\pi}{2}$

$\Rightarrow \dfrac{\pi}{3} \leq \theta \leq \dfrac{\pi}{2}$



which is the equation of a circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius is $\frac{4}{3}$.

Thus the region is the set of boundary points of the circle whose centre is $\left(\frac{-5}{3}, 0\right)$ and radius $= \frac{4}{3}$.

Example–38. Draw the sketch of the following region :

$1 < |z - 2i| < 2$ **[NUH–1998]**

Solution : Let $z = x + iy$.

Given that $1 < |z - 2i| < 2$

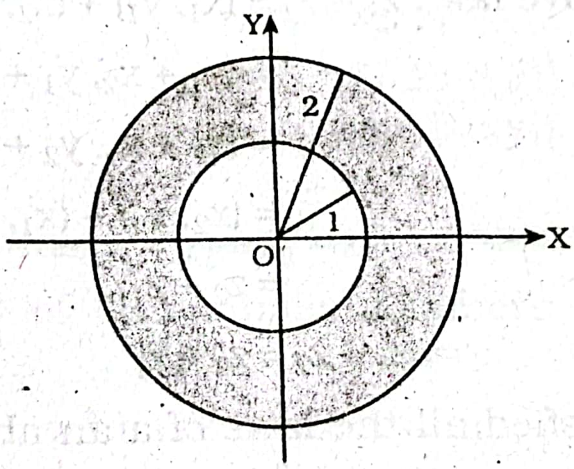
$\Rightarrow 1 < |x + iy - 2i| < 2$

$\Rightarrow 1 < \sqrt{x^2 + (y-2)^2} < 2$

$\Rightarrow 1 < x^2 + (y-2)^2 < 2^2$

$\Rightarrow 1 < x^2 + (y-2)^2$

and $x^2 + (y-2)^2 < 2^2$



Thus the region is the set of all common points of external points of the circle $x^2 + (y-2)^2 = 1$ and internal points of the circle $x^2 + (y-2)^2 = 2^2$.

Example–39. Prove that the set of complex numbers form an abelian group. **[RUH–2002]**

Solution : Let **C** be the set of complex numbers and

$\mathbf{C} = \{z : z \text{ is a complex number } (x, y)\}$.

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3) \in \mathbf{C}$.

(i) Closure law : $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2) \in \mathbf{C}$

Proof : Let the function f(z) is differentiable at z_0.

$$\text{Now, } \lim_{z\to z_0} [f(z) - f(z_0)] = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

$$\Rightarrow \lim_{z\to z_0} f(z) - f(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z\to z_0} (z - z_0)$$

$$= f'(z_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{z\to z_0} f(z) = f(z_0)$$

Hence f(z) is continuous at z_0. Thus every differentiable function is continuous.

2.5. Analytic (or regular or holomorphic) functions :

[NUH–2000, 03, 06 (Phy), 05, 05 (Old), 06 (Old), 06, DUH–05]

The concept of analytic function is the core heart of complex analysis. First we give the definition of an analytic function at a point and in a domain (region).

Definition : A complex function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

The above definition follows that if f(z) is analytic at z_0, it is actually analytic at each point in a neighbourhood of z_0.

Definition : A function f(z) is said to be analytic in a domain D (or region R) if it is analytic at each point of D (or R).

The terms holomorphic and regular are used with identical meaning in the literature.

Entire function : A complex function f(z) is said to be entire if it is analytic in the whole complex plane.

Singular point or Singularity : **[JUH (Phy)–2000, 2003, 3005]**

If a function f(z) fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a singular point or singularity of f(z).

Cauchy-Riemann Partial Differential Equations :

[JUH (Phy)–2000, 200

Theorem–4. Necessary conditions for f(z) to be analytic. Th necessary condition for $w = f(z) = u(x, y) + iv(x, y)$ to be analytic at point $z = x + iy$ of its domain is that the four partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ should exist and satisfy the Cauchy-Rieman partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[NUH–98, 2000, 02, 05 (Old), 06 (Old), 07, DUH–03, 05

Proof : Given that $f(z) = u(x, y) + iv(x, y)$ is analytic (differentiable) at any point $z = x + iy$ of its domain.

$$\therefore f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \quad \ldots (1)$$

Since f(z) is analytic, so $f'(z)$ exists, finite and unique independent of the path along which $\Delta z \to 0$.

Now there are two cases :

Case–1. Along real axis (x-axis) we have $\Delta y = 0$ and $\Delta x \to 0$, so from (1) we have

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots\ldots (2)$$

provided the partial derivatives exist.

Case–2. Along imaginary axis (y-axis) we have $\Delta x = 0$ and $\Delta y \to 0$, so from (1) we have

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \ldots\ldots (2)$$

$$\Rightarrow -\frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial u}{\partial \theta} = (\sin^2\theta + \cos^2\theta)\frac{\partial v}{\partial r}$$

$$\Rightarrow -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \ldots\ldots (8)$$

Thus, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ are the required polar form of C–R equations.

2.7. Harmonic Functions : [NUH-93, 94, 01, 04, NU(Pre)-08]

Any real valued function of x and y is said to be harmonic in a domain of the xy plane if throughout the domain it has continuous partial derivatives of the first and second order and satisfy the Laplace equation. **[DUH-2005]**

Harmonic conjugate : **[NUH-1993, 1994, 2004, DUH-2005]**

The function v is said to be a harmonic conjugate of u if u and v are harmonic and u, v satisfy the C–R equations.

If $f(z) = u(x, y) + iv(x, y)$, then u, v are the component functions of f. For component functions we shall prove the following results.

Theorem–6. If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D. **[NUH-2001]**

Or, The real and imaginary parts of an analytic function ore harmonic function. **[NUH-2001]**

Proof : Since f is analytic in D, so its component functions u and v satisfy the Cauchy-Riemann equations throughout D.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots (1)$$

Differentiating partially w. r. to x both sides of (1) we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} \text{ and } \frac{\partial^2 u}{\partial x\, \partial y} = -\frac{\partial^2 v}{\partial x^2} \quad \ldots\ldots (2)$$

Again, differentiating partially w. r. to y both sides of (1) we get

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 v}{\partial y^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\, \partial x} \quad \ldots\ldots (3)$$

By calculas, the continuity of the partial derivatives ensures that

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x} \text{ and } \frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x}$$

After making one complete circuit about the origin in the positive or counterclockwise direction, on returning to z_1 we find $r = r_1$, $\theta = \theta_1 + 2\pi$ so that

$$\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi).$$

Thus we have another branch of f(z) and so z = 0 is a branch point.

Example–3. For the function $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$, locate and name all the singularities in the finite z-plane and also determine where f(z) is analytic.

Solution : Given that $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3 (3z+2)^2}$ ...... (1)

In the finite z-plane the singularities will be obtained by solving the equation

$(z-1)^3 (3z+2)^2 = 0$

$\Rightarrow (z-1)^3 = 0 \quad$ or $(3z+2)^2 = 0$

$\Rightarrow z = 1, 1, 1 \quad$ or $z = \dfrac{-2}{3}, \dfrac{-2}{3}$

$\therefore$ The singularities in the finite z-plane are z = 1 of order 3 and $z = \dfrac{-2}{3}$ of order 2.

In the finite z-plane f(z) is analytic everywhere excepts the points z = 1 and $z = \dfrac{-2}{3}$.

Example–4. Determine the singular points of the function

$$f(z) = \frac{z^3 + 7}{(z^2 - 2z + 2)(z-3)}.$$

Solution : The singular points are obtained by solving the equation

$(z^2 - 2z + 2)(z-3) = 0$

$\Rightarrow z - 3 = 0$ or $z^2 - 2z + 2 = 0$

$\Rightarrow z = 3$ or $z = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 + i, 1 - i$

Thus the singular points are z = 3, z = 1 + i, z = 1 − i.