Hence $(fg)(A) = \sum_{k=0}^{n+m} c_k A^{n+m-k}$ and

$$f(A)\,g(A) = \left(\sum_{i=0}^{n} a_i A^{n-i}\right)\left(\sum_{j=0}^{m} b_j A^{m-j}\right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j A^{n+m-(i+j)}$$

$$= \sum_{k=0}^{n+m} c_k A^{n+m-k} = fg(A)$$

(iii) By definition

$\alpha f(\lambda) = \alpha a_n \lambda^n + \alpha a_1 \lambda^{n-1} + \ldots + \alpha a_{n-1} \lambda + \alpha a_n$ and so

$\alpha f(A) = \alpha a_0 A^n + \alpha a_1 A^{n-1} + \ldots + \alpha a_{n-1} A + \alpha a_0 I$

$= \alpha (a_0 A^n + a_1 A^{n-1} + \ldots a_{n-1} A + a_n I)$

$= \alpha f(A)$.

9.2 Definition of eigenvalues and eigenvectors

If A is an n×n matrix, then a non-zero vector v in R^n is called an **eigenvector** of A if Av is a scalar multiple of v, that is, $Av = \lambda v$ (1) for some scalar λ. The scalar λ is called an **eigenvalue** of A and v is said to be an **eigenvector** of A corresponding to λ.

"Eigen" is a German word meaning **"Proper"** or **"own"**.

Eigenvectors are also called proper vectors, characteristic vectors or latent vectors and eigenvalues are called proper values, characteristic values or latent roots by some writers.

9.3 Characteristic polynomial and characteristic equation.

To find the eigenvalue of n × n matrix A we write $Av = \lambda v$ as $Av = \lambda I v$, or, equivalently $(\lambda I - A)v = 0$ (2).

The matrix $\lambda I - A$, where I is the n × n identity matrix and λ is an indeterminate, is called the **characte istic matrix** of A.

For λ to be an eigenvalue of the matrix A, there must be a non-zero solution for the vector v of the equation (2) only if the rank of $\lambda I-A$ is less than its order, in which case its determinant is zero i.e $|\lambda I - A| = 0$ (3)

The determinant of the characteristic matrix $\lambda I-A$ is a polynomial in λ and is called the **characteristic polynomial** of A. Also equation no (3) is called the **characteristic equation** of A, the scalars satisfying this equation are the **eigenvalues** of A.

[Eigenvecors and Eigenvalues can be found for linear operators as well as matrices. A scalar λ is called an **eigenvalue** of a linear operator $T : V \to V$ if there is a non-zero vector X in V such that $TX = \lambda X$. The vector X is called an **eigenvector** of T corresponding to λ. Equivalently, the eigenvectors of T corresponding to λ are the non-zero vectors in the kernel of $\lambda I-T$. This kernel is called the **eigenspace** of T corresponding to λ.

It can be shown that if V is a finite dimensional vector space and A is the matrix of T with respect to any basis B, then

(i) The eigenvalues of T are the eigenvalues of the matrix A

(ii) A vector X is an eigenvector of T corresponding to λ if and only if its co-ordinate matrix $[X]_B$ is an eigenvector of A corresponding to λ.]

Theorem 9.2 Any square matrix A and its transpose A′ have the same eigenvalues.

Proof : Let $A = [a_{ij}]$ $i = 1, 2, \ldots, n$

$$j = 1, 2, \ldots, n$$

then $A' = [a_{ji}]$ $i = 1, 2, \ldots, n$

$$j = 1, 2, \ldots, n$$

Thus the eigenvalues of the Hermitian ...

Example 2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{bmatrix}$$

Now the determinant of $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2$$

Therefore, the characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$

or, $\lambda^2 - 2\lambda - \lambda + 2 = 0$ or, $(\lambda - 2)(\lambda - 1) = 0$

$\therefore$ $\lambda = 1, \lambda = 2$ which are the eigenvalues of A.

Example 3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix} = (\lambda-4)\{(\lambda-2)^2-3\}$$

Therefore, the characteristic equation of A is

$$(\lambda-4)\{(\lambda-2)^2-3\}=0$$

or, $(\lambda-4)(\lambda^2-4\lambda+4-3)=0$

or, $(\lambda-4)(\lambda^2-4\lambda+1)=0$

$\therefore\quad \lambda=4$ and $\lambda^2-4\lambda+1=0$

Here $\lambda^2 - 4\lambda + 1 = 0$ is a quadratic equation which can be solved by the quadratic formula

$$\lambda = \frac{4\pm\sqrt{16-4}}{2} = \frac{4\pm\sqrt{12}}{2} = 2\pm\sqrt{3}$$

or, $\lambda = 2\pm\sqrt{3}$

Hence the eigenvalues of A are

$\lambda_1 = 4,\ \lambda_2 = 2+\sqrt{3}$ and $\lambda_3 = 2-\sqrt{3}$.

Example 4. Find the eigenvalues and the corresponding eigevectors of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ (the characteristic polynomial of A) is $|\lambda I - A| = \begin{vmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4)-3.$

Therefore, the characteristic equation of A is

$$(\lambda-2)(\lambda-4)-3=0$$

or, $\lambda^2-6\lambda+8-3=0$

or, $\lambda^2-6\lambda+5=0$

or, $(\lambda-5)(\lambda-1)=0\ \therefore\ \lambda=5,\ \lambda=1$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A) X = 0$, that is, of

$$\begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

If $\lambda = 5$, equation no (1) becomes

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} 3x_1 - 3x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0$$

This system is in echelon form and consistent. Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = a$ (a is an arbitrary real number). Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 5$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ a \end{bmatrix}$

In particular, let $a = 1$, then $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

If $\lambda = 1$, equation no (1) becomes

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or,} \quad \left.\begin{aligned} -x_1 - 3x_2 &= 0 \\ -x_1 - 3x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 + 3x_2 = 0$$

This system is in echelon form and consistent. Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = b$ (b is an arbitrary real number). $x_1 = -3b$, Therefore, the eigen vectors of A corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors of the form

$$X = \begin{bmatrix} -3b \\ b \end{bmatrix}$$

In particular, let $b = 1$, then $X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

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Example 5. For the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 3y, x + 5y)$, find all eigenvalues and a basis of eigenspace.

Solution : First find a matrix representation of T, say relative to the usual basis of $\mathbb{R}^2$.

$$A = [T] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

The characteristic polynomial $\Delta(\lambda)$ of T is then

$$\Delta = |\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda-3 & -3 \\ -1 & \lambda-5 \end{vmatrix} = (\lambda-3)(\lambda-5) - 3$$

Hence the characteristic equation is $(\lambda - 3)(\lambda-5) - 3 = 0$

or, $\lambda^2 - 8\lambda + 12 = 0$

or, $(\lambda - 2)(\lambda - 6) = 0 \quad \therefore \lambda = 2, \lambda = 6$

Thus 2 and 6 are the eigenvalues of T.

Now we find a basis of the eigenspace of the eigenvalue 2.

Putting $\lambda = 2$ into $\lambda I - A$ to obtain

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{matrix} -x_1 - 3x_2 = 0 \\ -x_1 - 3x_2 = 0 \end{matrix}\right\} \Rightarrow x_1 + 3x_2 = 0$

The system has only one independent solution, e. g. $x_1 = 3$, $x_2 = -1$.

Thus $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is an eigenvector which generates the eigenspace of 2, i.e $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ forms a basis of eigenspace of 2.

Again, we find a basis of the eigenspace of the eigenvalue 6. Putting $\lambda = 6$ into $\lambda I - A$ to obtain

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $$\left.\begin{aligned} 3x_1 - 3x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 - x_2 = 0.$$

The system has only one independent solution, e. g. $x_1 = 1$, $x_2 = 1$. Thus $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector which generates the eigenspace of 6, i. e $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ forms basis of the eigenspace of 6.

Example 6. Find all eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$ **[D. U. P. 1991]**

Solution : The characteristic matrix of A is

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix}$$

The characteristic polynomial of A is

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-1)(\lambda+2)(\lambda-2).$$

Therefore, the characteristic equation of A is

$$(\lambda-1)(\lambda+2)(\lambda-2) = 0$$

$$\therefore \lambda = 1, \lambda = -2, \lambda = 2$$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A

corresponding to the eigenvalue λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$.

i.e $= \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda = 1$, $\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we have

$$\left.\begin{aligned} -2x_2 + x_3 &= 0 \\ 3x_2 \quad &= 0 \\ 5x_2 - x_3 &= 0 \end{aligned}\right\} \text{Solving we get } x_2 = x_3 = 0$$

Hence x_1 is a free variable. Let $x_1 = a$ where a is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 1$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$.

In particular, let $a = 1$, then $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

When $\lambda = -2$, $\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we get

$$\left.\begin{aligned} -3x_1 - 2x_2 + x_3 &= 0 \\ 5x_2 - 4x_3 &= 0 \end{aligned}\right\}$$

This system is in echelon form and has one free variable which is x_3. Let $x_3 = b$ where b is any real number. Then from second equation we get $x_2 = \frac{4b}{5}$. Putting $x_2 = \frac{4b}{5}$ and $x_3 = b$ in the first equation. we get $x_1 = -\frac{b}{5}$.

Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = -2$ are the non-zero vectors of the form

$$X = \begin{bmatrix} \frac{-b}{5} \\ \frac{4b}{5} \\ b \end{bmatrix}$$

In particular, let b = 5, then $X = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$.

When $\lambda = 2$, $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming linear system, we get

$$\left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 4x_2 \quad\quad &= 0 \\ 5x_2 \quad\quad &= 0 \end{aligned}\right\} \text{i. e} \quad \left.\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\}$$

Here x_3 is a free variable. Let $x_3 = c$ where c is any real number, Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 2$ are non-zero vectors of the form.

$X = \begin{bmatrix} -c \\ o \\ c \end{bmatrix}$ In particular, let c = 1, then

$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

9.4 Diagonalization

A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal, the matrix P is said to diangonalize A.

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dependent. Hence we can choose scalars $\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}$ not all zero such that $\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_r X_r + \alpha_{r+1} X_{r+1} = 0$ (1)

Multiplying both sides of (1) on the left by A, we get

$$A\alpha_1 X_1 + A\alpha_2 X_2 + \ldots + A\alpha_{r+1} X_{r+1} = 0$$

or. $\alpha_1 AX_1 + \alpha_2 AX_2 + \ldots + \alpha_{r+1} AX_{r+1} = 0$

or. $\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \ldots + \alpha_{r+1} \lambda_{r+1} X_{r+1} = 0$ (2)

since $AX_i = \lambda_i X_i$

Now multiplying (1) by the scalar λ_{r+1} on the left and then subtracting from (2) we get

$\alpha_1 (\lambda_1 - \lambda_{r+1}) X_1 + \alpha_2 (\lambda_2 - \lambda_{r+1}) X_2 + \ldots + \alpha_r (\lambda_r - \lambda_{r+1}) X_r = 0$ (3)

But by our assumption $X_1, X_2, \ldots, X_r$ are linearly independent and $\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}$ are all distinct, therefore, from (3), we get $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$

Putting $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$ in (1), we get

$\alpha_{r+1} X_{r+1} = 0 \Rightarrow \alpha_{r+1} = 0$ since $X_{r+1} \neq 0$.

Thus the relation (1) implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_r = \alpha_{r+1} = 0$ which contradicts our assumption that the scalars

$\alpha_1, \alpha_2, \ldots, \alpha_r, \alpha_{r+1}$ are not all zero.

Hence our initial assumption is wrong and the vectors $X_1, X_2, \ldots, X_n$ are linearly independent. Since the matrix A has n linearly independent eigenvectors, therefore, it is diagonalizable.

Example 7. Find the matrix P that diagonalizes the matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ and also determine $P^{-1}AP$.

Solution : The charateteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix}$$

Now the characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -4 \\ -9 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 36$$

Therefore, the characteristic equation of A is $(\lambda-1)^2 - 36 = 0$

or, $\lambda^2 - 2\lambda + 1 - 36 = 0$

or, $\lambda^2 - 2\lambda - 35 = 0,$

or, $(\lambda+5)(\lambda-7) = 0$

$\therefore \lambda = -5, \lambda = 7$

which are the eigenvalues of the matrix A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$, that is, of $\begin{bmatrix} \lambda-1 & -4 \\ -9 & \lambda-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (1)

When $\lambda = -5$ equation (1) becomes

$$\begin{bmatrix} -6 & -4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{aligned} -6x_1 - 4x_2 &= 0 \\ -9x_1 - 6x_2 &= 0 \end{aligned}\right\}$ or, $\left.\begin{aligned} 3x_1 + 2x_2 &= 0 \\ 3x_1 + 2x_2 &= 0 \end{aligned}\right\}$

or, $3x_1 + 2x_2 = 0$. (2)

Now it is clear that $x_1 = 2$ and $x_2 = -3$ is a solution of equation (2).

Therefore, $X_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -5$.

When $\lambda = 7$. equation (1) becomes

$$\begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ or, $\left.\begin{aligned} 6x_1 - 4x_2 &= 0 \\ -9x_1 + 6x_2 &= 0 \end{aligned}\right\}$

or, $\left.\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}\right\}$

or, $3x_1 - 2x_2 = 0$ (3)

Now it is clear that $x_1 = 2$, $x_2 = 3$ is a solution of the equation given by (3). Therefore, $X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$.

Suppose that P is the matrix which has the above two eigenvectors as columns.

Then $p = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

One can casily find that the inverse of P is $P^{-1} = \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$

$$\text{Now } P^{-1}AP = \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -10 & 14 \\ 15 & 21 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -60 & 0 \\ 0 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix} = D$$

which is the diagonal matrix of the eigenvalues of the matrix A.

Hecnce $P = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$ is the requred matrix that diagonalizes the given matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$

Example 8. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Also find the matrix P that diagonalizes A and determine $P^{-1}AP$.

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix}$$

Let $x_3 = -1$, then $x_2 = 1$.

Therefore, $X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

If $\lambda = -1$, equation (1) becomes

$$\begin{bmatrix} -5 & -6 & -6 \\ -1 & -4 & -2 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or,}\quad \left.\begin{aligned} -5x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 - 4x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by elementary transformations. we add 3rd eqn with 2nd eqn. Also we multiply 1st equation by – 1.

Then we have the equivalent system $\left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}\right\}$

$$\text{or,}\quad \left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ -14x_2 - 4x_3 &= 0 \end{aligned}\right\} \quad \text{or,}\quad \left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ 7x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

Since in echelon form there are two equations in three unknowns, the system has a non-zero solution. Here x_3 is a free variable,

Let $x_3 = -7$, then $x_2 = 2$ and $x_1 = 6$.

Therefore, $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$.

Let us take $P = \begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$

Now one can easily find that $P^{-1} = -\dfrac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

$$P^{-1}AP = -\frac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$= -\frac{1}{15}\begin{bmatrix} -20 & -24 & -24 \\ 5 & -15 & 0 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$= -\frac{1}{15}\begin{bmatrix} -60 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

which is the diagonal matrix of the eigenvalues of the matrix A. Hence P is the required matrix that diagonalizes the

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -1 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ -6 & 0 & 0 & -6 \\ 8 & 4 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$ = D which is the diagonal matrix of the eigenvalues of the matrix A.

Hence P is the required matrix that diagonalizes the given matrix A.

9.5 Cayley – Hamilton Theorem

Every square matrix satisfies its own characteristic equation i. e. if the characteristic equation of the nth order matrix A is $f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{n-1} \lambda + a_n = 0$ then

Cayley-Hamilton theorem states that

$f(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_{n-1} A + a_n I = O$ where

I is the nth order unit matrix and O is the nth order zero matrix.

Proof : The characteristic equation for the matrix A is $|\lambda I - A| = 0$, which can be written as

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_{n-1} \lambda + a_n = 0 \qquad (1)$$

where a_i, i = 1, 2, ..., n are scalars.

This is a scalar equation and can be multiplied by any vector X say, to give $\lambda^n X + a_1 \lambda^{n-1} X + \ldots + a_{n-1} \lambda X + a_n X = 0$ (2)

Now if X is the eigenvector corresponding to the eigenvalue λ which satisfies the equation (1), then by definition we can write $AX = \lambda X$

$$A^2 X = AAX = A\lambda X = \lambda AX = \lambda^2 X$$

$$A^3 X = AA^2 X = A\lambda^2 X = \lambda^2 AX = \lambda^3 X$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$A^n X = \lambda^n X,$$

Using these above relations we can re-write the equation (2) as $A^n X + a_1 A^{n-1} X + \ldots + a_{n-1} AX + a_n IX = 0$ (3)

This will be true for all eigenvectors, But since they are of order n, are linearly independent and are n in numbers, any other n-order vector can be expressed as a linear combination. Therefore, equation (3) holds for all n- order vectors and so is true in general.

Hence $A^n + a_1 A^{n-1} + \ldots + a_{n-1} A + a_n I = 0$ i. e. A satisfies its own characteristic equation,

Cayley-Hamilton theorem can be applied to find the inverse of a non-singular matrix. When A is non-singular, then A^{-1} exists and in the expansion of $|\lambda I - A|$ the constant term a_n is different from zero, Hence we can multiply the Cayley–Hamilton equation $A^n + a_1 A^{n-1} + \ldots + a_{n-1} A + a_n I = 0$ by A^{-1} and we get $A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I + a_n A^{-1} = 0$.

Whence solving for A^{-1} we find

$$A^{-1} = -\frac{1}{a_n} [A^{n-1} + a_1 A^{n-2} + \ldots + a_{n-1} I].$$

Example 10. Find the charateristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and verify Cayley-Hamilton theorem for it.

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{bmatrix}$$

The determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -3 \\ -2 & \lambda+1 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)(\lambda^2-1-1) + 2(-2\lambda+2-3) - 3(2+3\lambda+3)$$

$$= (\lambda-1)(\lambda^2-2) - 4\lambda - 2 - 9\lambda - 15$$

$$= \lambda^3 - 2\lambda - \lambda^2 + 2 - 10\lambda - 17$$

$$= \lambda^3 - \lambda^2 - 15\lambda - 15.$$

Therefore, the characteristic equation of A is

$$\lambda^3 - \lambda^2 - 15\lambda - 15 = 0.$$

Now in order to verify Cayley-Hamilton theorem we have to show that $A^3 - A^2 - 15A - 15I = 0$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$$

$$A^3 - A^2 - 15A - 15I =$$

$$\begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 44-14-15-15 & 33-3-30-0 & 53-8-45-0 \\ 33-3-30-0 & 6-6+15-15 & 21-6-15-0 \\ 53-8-45-0 & 21-6-15-0 & 41-11-15-15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \text{i. e. } A^3 - A^2 - 15A - 15I = 0.$$

Hence the Cayley-Hamilton theorem is verified.

Example 11. Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & -2 \\ -3 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{bmatrix}$$

Now the determinant of the matrix $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & -2 \\ -3 & \lambda-1 & 0 \\ -1 & -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda - 1)^3 + 2\{-3(\lambda - 1)\} - 2(3 + \lambda - 1)$$

$$= (\lambda - 1)^3 - 6\lambda + 6 - 4 - 2\lambda$$

$$= \lambda^3 - 3\lambda^2 + 3\lambda - 1 - 8\lambda + 2$$

$$= \lambda^3 - 3\lambda^2 - 5\lambda + 1.$$

Therefore, the characteristic equation of A is

$$\lambda^3 - 3\lambda^2 - 5\lambda + 1 = 0,$$

Now using Cayley-Hamilton theorem we get

$$A^3 - 3A^2 - 5A + I = 0$$

Multiplying the above Cayley-Hamilton equation on both sides by A^{-1} we have $A^2 - 3A - 5I + A^{-1} = 0$

or, $A^{-1} = 3A + 5I - A^2$.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$A^{-1} = 3A + 5I - A^2 = 3\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 4 \\ 6 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+5-9 & 6+0-6 & 6+0-4 \\ 9+0-6 & 3+5-7 & 0+0-6 \\ 3+0-5 & 3+0-4 & 3+5-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$

Hence $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$.

9.6 Minimal (minimum) Polynomial of a matrix.

Let A be an n-square matrix over a field F. By Cayley-Hamilton theorem we know that every square matrix satisfies its own characteristic equation. So there are non-zero polynomials $f(\lambda)$ for which $f(A) = 0$; for example, the characteristic polynomial of A. Among these polynomials we take the lowest degree polynomials and from them we select the monic polynomial which is called the **minimal polynomial** of A or the **minimum polynomial** of A.

Example 12. Find the minimum polynomial $m(\lambda)$ of the

matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Solution : The characteristic polynomial of A is

$$\Delta(\lambda) = |\lambda I - A| = \left| \lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \right|$$