$$\begin{cases} x & & = 3 \\ & y & = 1 \\ & & z = 2 \end{cases}$$

From the above system, we have $x = 3$, $y = 1$ and $z = 2$

$\therefore$ The required solution $(x, y, z) = (3, 1, 2)$ (Ans.)

Geometrically, the point (3, 1, 2) is the intersection of the three planes described by the three equations comprising the given system.

Summary of the Gauss-Jordan elimination method follows :

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Example-3 Solve the following system of linear equations by Gauss-Jordan elimination method. [নিচের সমীকরণ জোটটি গাউস-জর্ডান এলিমিনেশন পদ্ধতিতে সমাধান কর :]

$$\begin{cases} 3x - 2y + 8z = 9 \\ -2x + 2y + z = 3 \\ x + 2y - 3z = 8 \end{cases}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices :

$$\left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 2 & -12 & -4 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 31 & 31 \end{array}\right]$$

$$\xrightarrow{\frac{1}{31}R_3}\left[\begin{array}{ccc|c}1 & 0 & 9 & 12\\ 0 & 1 & -6 & -2\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\xrightarrow[R_2+6R_3]{R_1-9R_3}\left[\begin{array}{ccc|c}1 & 0 & 0 & 3\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & 1\end{array}\right]$$

bove system is in standard form.

he solution to the given system is given by $x = 3, y = 4$ and $z = 1$. (Ans.)

mple-4 Solve the system of linear equations given by

$$\begin{cases} \quad\;\; 2y + \;\,3z = \;\;\,7\\ 3x + 6y - 12z = -3\\ 5x - 2y + \;\,2z = -7\end{cases}$$

lution Using the Gauss–Jordan elimination method, we obtain the lowing sequence of equivalent augmented matrices :

$$\left[\begin{array}{ccc|c}0 & 2 & 3 & 7\\ 3 & 6 & -12 & -3\\ 5 & -2 & 2 & -7\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}3 & 6 & -12 & -3\\ 0 & 2 & 3 & 7\\ 5 & -2 & 2 & -7\end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1}\left[\begin{array}{ccc|c}1 & 2 & -4 & -1\\ 0 & 2 & 3 & 7\\ 5 & -2 & 2 & -7\end{array}\right]$$

$$\xrightarrow{R_3-5R_1}\left[\begin{array}{ccc|c}1 & 2 & -4 & -1\\ 0 & 2 & 3 & 7\\ 0 & -12 & 22 & -2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2}\left[\begin{array}{ccc|c}1 & 2 & -4 & -1\\ 0 & 1 & \frac{3}{2} & \frac{7}{2}\\ 0 & -12 & 22 & -2\end{array}\right]$$

$$\xrightarrow[R_3+12R_2]{R_1-2R_2}\left[\begin{array}{ccc|c}1 & 0 & -7 & -8\\ 0 & 1 & \frac{3}{2} & \frac{7}{2}\\ 0 & 0 & 40 & 40\end{array}\right]$$

$$\xrightarrow{\frac{1}{40}R_3}\left[\begin{array}{ccc|c}1 & 0 & -7 & -8\\ 0 & 1 & \frac{3}{2} & \frac{7}{2}\\ 0 & 0 & 1 & 1\end{array}\right]$$

$$\xrightarrow[R_2-\frac{3}{2}R_3]{R_1+7R_3}\left[\begin{array}{ccc|c}1 & 0 & 0 & -1\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\end{array}\right]$$

(Ans.)

... is given by $x = -1, y = 2$ and $z = 1$

- **A system of equations with an infinite number of solutions**

Example-5 Solve the system of linear equations given by

$$\begin{cases} x + 2y - 3z = -2 \\ 3x - y - 2z = 1 \\ 2x + 3y - 5z = -3 \end{cases}$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices :

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form. Corresponding system of linear equations gives

$$\begin{cases} x - z = 0 \\ y - z = -1 \end{cases}$$

a system of two equations in the three variables x, y and z.

Let's now single out one variable—say, z—and solve for x and y in terms of it.

We obtain

$$x = z$$
$$y = z - 1$$

If we assign a particular value to z-say, $z = 0$ we obtain $x = 0$ and $y = -1$, giving the solution $(0, -1, 0)$ to the system. By setting $z = 1$, we obtain the solution $(1, 0, 1)$. In general, if we set $z = t$, where t represents some real number (called a parameter), we obtain a solution given by $(t, t - 1, t)$. Since the parameter t may be any real number, we see that the system has infinitely many solutions.

> **Note** : Geometrically, the solutions of Example-5 lie on the straight line in three-dimensional space given by the intersection of the three planes determined by the three equations in the system.

Example-6 Solve the system of linear equations given by

$$\begin{cases} x + y + z = 1 \\ 3x - y - z = 4 \\ x + 5y + 5z = -1 \end{cases}$$

Solution Using the Gauss-Jordan elimination method, we obtain the following sequence of equivalent augmented matrices :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Observe that row 3 in the last matrix reads $0x + 0y + 0z = -1$ that is, $0 = -1$!

We therefore conclude that the system is inconsistent and has no solution. Geometrically, we have a situation in which two of the planes intersect in a straight line but the third plane is parallel to this line of intersection of the two planes and does not intersect it. Consequently, there is no point of intersection of the three planes.

1.13 Consistency of a system of linear equations

সরল সমীকরণ জোটের সামঞ্জস্যতা

(a) Let us consider the following system of non-homogeneous linear equations:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \cdots\cdots\cdots (1)$$

The above system can be written in the following matrix form :

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$$

or, $Ax = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$.

$$\begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or, } \begin{bmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 10y + 6z = 0 \text{ or, } x + 5y + 3z = 0$$

Let $x = a, y = b$, then $3z = -a - 5b$ or $z = \frac{-a}{3} - \frac{5b}{3}$

∴ The required solution is $x = a, y = b, z = \frac{-a}{3} - \frac{5b}{3}$ (Ans.)

4.14 Applications of system of linear equations in real life problem
বাস্তব জীবনে সরল সমীকরণ জোটের ব্যবহার

Example 5 A medicine company "Square group of Bangladesh limited" wishes to produce three types of medicine: type X, Y and Z. To manufacture a type X medicine requires 2 minutes each on machines I and II and 3 minutes on machine III. A type of Y medicine requires 2 minutes on machine I, 3 minutes on machine II and 4 minutes on machine III. A type Z medicine requires 3 minutes on machine I, 4 minutes on machine II and 3 minutes on machine III. There are 3.5 hours available on machine 1, 4.5 hours available on machine II and 5 hours available on machine III. How many medicine of each type should company make in order to use all the available time?

Solution Here, 3.5 hours = 210 minutes, 4.5 hours = 270 minutes and 5 hours = 300 minutes.

Let x, y, z be the number of medicines of types X, Y and Z respectively. The we have the following system of linear equations:

$$\begin{cases} 2x + 2y = 3z = 210 \\ 2x + 3y + 4z = 270 \\ 3x + 4y + 3z = 300 \end{cases}$$

The augmented matrix of the above system is

$$\begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 2 & 3 & 4 & : & 270 \\ 3 & 4 & 3 & : & 300 \end{bmatrix}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 2 & -3 & : & -30 \end{bmatrix} \begin{bmatrix} R_2' = R_2 - R_1 \\ R_3' = 2R_3 - 3R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 0 & -5 & : & -150 \end{bmatrix} [R_3' = R_3 - 2R_2]$$

$$\sim \begin{bmatrix} 2 & 2 & 3 & : & 210 \\ 0 & 1 & 1 & : & 60 \\ 0 & 0 & 1 & : & 30 \end{bmatrix} \left[R_3' = \left(\frac{-1}{5}\right) R_3\right]$$

$$\sim \begin{bmatrix} 2 & 2 & 0 & : & 210 \\ 0 & 1 & 0 & : & 30 \\ 0 & 0 & 1 & : & 30 \end{bmatrix} \begin{bmatrix} R_1' = R_1 - 3R_3 \\ R_2' = R_2 - R_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & : & 60 \\ 0 & 1 & 0 & : & 30 \\ 0 & 0 & 1 & : & 30 \end{bmatrix} \left[R_1' = \frac{1}{2} R_1\right]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 30 \\ 0 & 1 & 0 & : & 30 \\ 0 & 0 & 1 & : & 30 \end{bmatrix} [R_1' = R_1 - R_2]$$

corresponding system of linear equation is

$$\begin{cases} x & = 30 \\ y & = 30 \\ z & = 30. \end{cases}$$

nce the solution of the above system is $x = y = z = 30$

us the number of each type of medicine is 30. (Ans.)

xample-56 Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph sses through the points (1, 4), (2, 0) and (3, 12). [বহুপদী $p(x) = a_0 + a_1x + a_2x^2$ য় কর যার লেখ (1, 4), (2, 0) এবং (3, 12) বিন্দু দিয়ে গমন করে ।]

olution Given polynomial $p(x) = a_0 + a_1x + a_2x^2$ .... (1)

bstituting $x = 1, 2$ and 3 into $p(x)$ and equating the results to the respective y-lues produces the system of linear equations in the variables a_0, a_1 and a_2 own below :

$$\begin{cases} p(1) = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4 \\ p(2) = a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0 \\ p(3) = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12 \end{cases}$$

educing this system to echelon form by the elementary operations.

$$\sim \begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ a_1 + 5a_2 = 12 \end{cases} \begin{bmatrix} L_2' = L_2 - L_1 \\ L_3' = L_3 - L_2 \end{bmatrix}$$

$$\sim \begin{cases} a_0 + a_1 + a_2 = 4 \\ a_1 + 3a_2 = -4 \\ 2a_2 = 16 \end{cases} [L_3' = L_3 - L_2]$$

By back substitution method from 3rd equation, we have $a_2 = 8$

From 2nd equation, we get $a_1 + 24 = -4$ $\quad \therefore a_1 = -28$

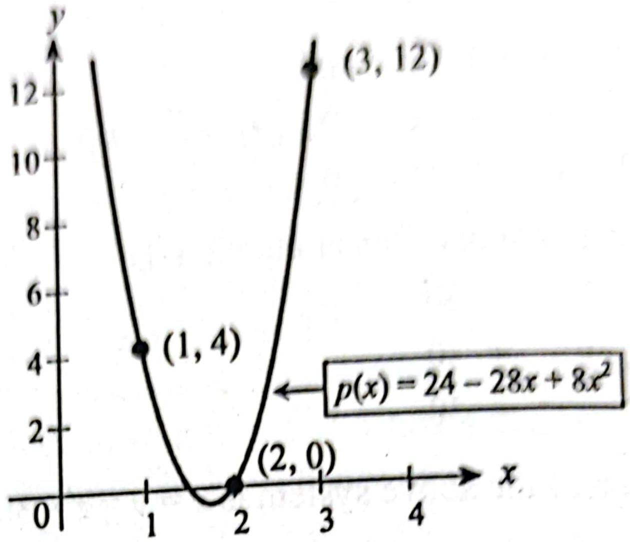
and from 1st equation, we get $a_0 - 28 + 8 = 4$ $\therefore a_0 = 24$

Hence the solution of this system is $a_0 = 24$, $a_1 = -28$ and $a_2 = 8$

So the polynomial function is

$$p(x) = 24 - 28x + 8x^2$$

The graph of p is shown in the following figure :



Example-57 Find a polynomial that fits the points (–2, 3), (–1, 5), (0, 1), (1, 4) and (2, 10) [একটি বহুপদী নির্ণয় কর যা (–2, 3), (–1, 5), (0, 1), (1, 4) এবং (2, 10) বিন্দুগুলি দ্বারা গঠিত।]

Solution We have provided five points, So we choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \quad \text{.................... (1)}$$

Substituting the given points into $p(x)$ produces the system of linear equations listed below :

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

The solution of these equations is

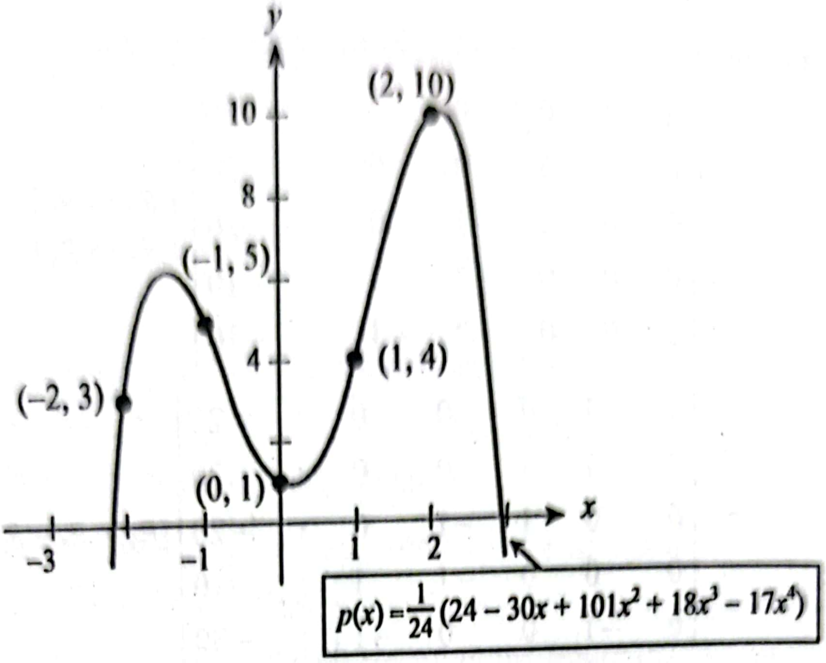
$$a_0 = 1,\ a_1 = -\frac{30}{24},\ a_2 = \frac{101}{24},\ a_3 = \frac{18}{24};\ a_4 = -\frac{17}{24}$$

which means the polynomial function is

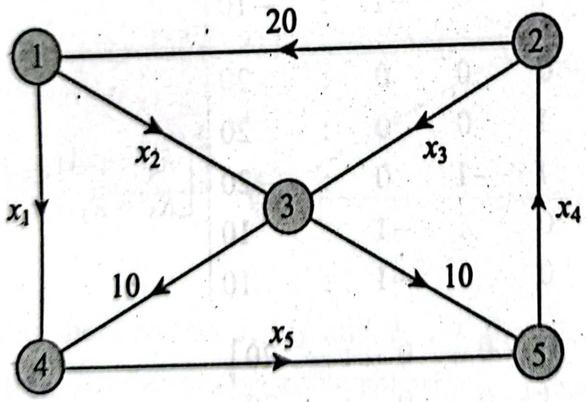
$$p(x) = 1 - \frac{30}{24}x + \frac{101}{24}x^2 + \frac{18}{24}x^3 - \frac{17}{24}x^4$$

$$= \frac{1}{24}(24 - 30x + 101x^2 + 18x^3 - 17x^4)$$

...aph of $p(x)$ is shown in the following figure :



...mple-58 Set up a system of linear equations to represent the network ...n in the following figure and solve the system. [নিম্নের চিত্রে প্রদর্শিত নেটওয়ার্ক ...রল সমীকরণ জোট গঠন কর এবং জোটটির সমাধান কর :]



...ution

...ı of the network's five junctions gives rise to a linear equation, as shown below :

$x_1 + x_2$	$= 20$	Junction 1
$x_3 - x_4$	$= -20$	Junction 2
$x_2 + x_3$	$= 20$	Junction 3
$x_1 \quad - x_5$	$= -10$	Junction 4
$-x_4 + x_5$	$= -10$	Junction 5

...augmented matrix for this system is

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & : & -10 \\ 0 & 1 & 0 & 0 & 1 & : & 30 \\ 0 & 0 & 1 & 0 & -1 & : & -10 \\ 0 & 0 & 0 & 1 & -1 & : & 10 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} [R_1' = R_1 - R_2]$$

orresponding system of equations are as follows :

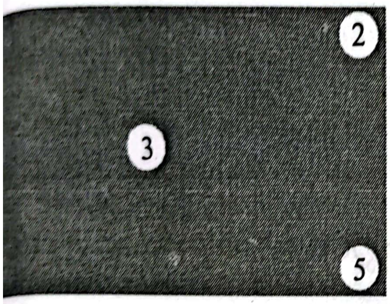
$$\begin{cases} x_1 \qquad\qquad - x_5 = -10 \\ \quad x_2 \qquad\quad\; x_5 = 30 \\ \qquad x_3 \quad - x_5 = -10 \\ \qquad\quad x_4 - x_5 = 10 \end{cases}$$

bove system is in echelon form having 4 equations in 5 unknowns. So it
$-4) = 1$ free variable, which is x_5.

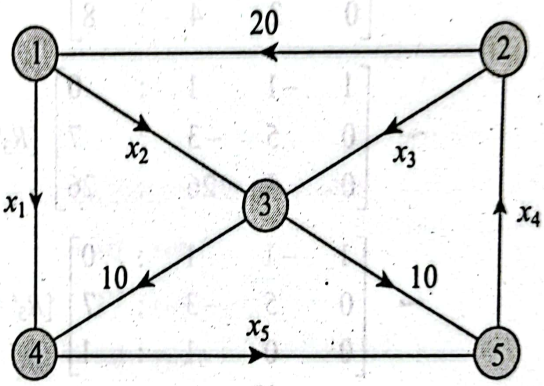
$_5 = t$, then by back substitution method, we have

$+10,\ x_3 = t - 10,\ x_2 = 30 - t,\ x_1 = t - 10$, where t is a real number.

is system has an infinite number of solutions. For a particular solution let t
then the following $x_1 = 0$ & $x_3 = 0$ which shown in the figure (a). Again
g $t = 20$ would produce the network shown in the figure (b)

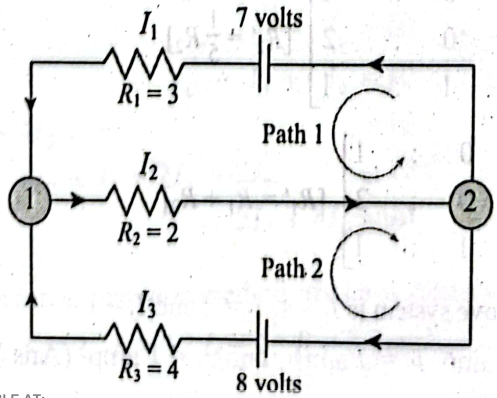


(a)



(b)

mple-59 Determine the currents I_1, I_2 and I_3 for the electrical network
n in the following figure. [নিম্নের তড়িৎ বর্তনীর জন্য তড়িৎ I_1, I_2 এবং I_3 নির্ণয় কর :]



Solution Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \qquad \textit{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$R_1I_1 + R_2I_2 = 3I_1 + 2I_2 = 7 \qquad \text{Path 1}$$

$$R_2I_2 + R_3I_3 = 2I_2 + 4I_3 = 8 \qquad \text{Path 2}$$

So, we have the following system of three linear equations in the I_1, I_2 and I_3.

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ 3I_1 + 2I_2 \qquad = 7 \\ \qquad 2I_2 + 4I_3 = 8 \end{cases}$$

The augmented matrix of the above system is

$$\begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 3 & 2 & 0 & : & 7 \\ 0 & 2 & 4 & : & 8 \end{bmatrix}$$

Reducing the system to echelon form by the elementary row operations

$$\sim \begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 2 & 4 & : & 8 \end{bmatrix} [R_2' = R_2 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 0 & 26 & : & 26 \end{bmatrix} [R_3' = 5R_3 - 2R_2]$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & : & 0 \\ 0 & 5 & -3 & : & 7 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} [R_3' = \frac{1}{26}R_3]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & : & -1 \\ 0 & 5 & 0 & : & 10 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \begin{bmatrix} R_1' = R_1 - R_3 \\ R_2' = R_2 + 3R_3 \end{bmatrix}$$

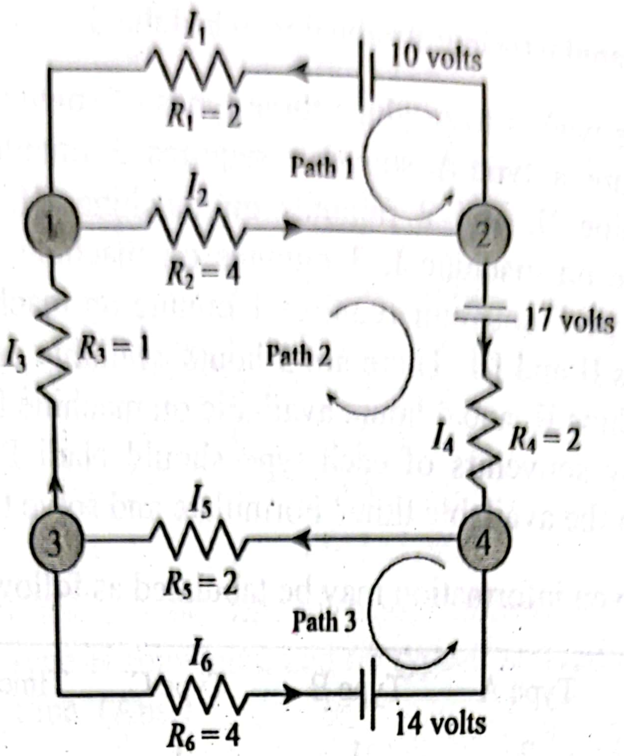
$$\sim \begin{bmatrix} 1 & -1 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} [R_2' = \frac{1}{5}R_2]$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} [R_1' = R_1 + R_2]$$

The solution of the above system is $I_1 = 1$, $I_2 = 2$ and $I_3 = 1$

Which means $I_1 = 1$ amp, $I_2 = 2$ amps and $I_3 = 1$ amp. (Ans.)

Example-60 Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network shown in the following figure. [নিম্নের তড়িৎ বর্তনীর জন্য তড়িৎ I_1, I_2, I_3, I_4, I_5 এবং I_6 নির্ণয় কর :]



Solution Applying Kirchhoff's first law to the four junctions produces

$$I_1 + I_3 = I_2 \quad \text{Junction 1}$$
$$I_1 + I_4 = I_2 \quad \text{Junction 2}$$
$$I_3 + I_6 = I_5 \quad \text{Junction 3}$$
$$I_4 + I_6 = I_5 \quad \text{Junction 4}$$

and applying Kirchhoff's second law to the three paths produces

$$2I_1 + 4I_2 = 10 \quad \text{Path 1}$$
$$4I_2 + I_3 + 2I_4 + 2I_5 = 17 \quad \text{Path 2}$$
$$2I_5 + 4I_6 = 14 \quad \text{Path 3}$$

Now we have the following system of seven linear equations in the variables I_1, I_2, I_3, I_4, I_5, and I_6.

$$\begin{cases} I_1 - I_2 + I_3 = 10 \\ I_2 - I_2 + I_4 = 10 \\ I_3 - I_5 + I_6 = 10 \\ I_4 - I_5 + I_6 = 10 \\ 2I_1 + 4I_2 = 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 = 17 \\ 2I_5 + 4I_6 = 14 \end{cases}$$

Using Gauss-Jordan elimination method, we have solution of the above system is

$$I_1 = 1,\ I_2 = 2,\ I_3 = 1,\ I_4 = 1,\ I_5 = 3 \text{ and } I_6 = 2$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp, $I_4 = 1$ amp, $I_5 = 3$ amps, and $I_6 = 2$ amps. (Ans.)

Example-61 **[Manufacturing: Production Scheduling]**

Nadi Publications wishes to produce three types of souvenirs: types A, B, and C. To manufacture a type-A souvenir requires 2 minutes on machine I, 1 minute on machine II, and 2 minutes on machine III. A type-B souvenir requires 1 minute on machine I, 3 minutes on machine II, and 1 minute on machine III. A type-C souvenir requires 1 minute on machine I and 2 minutes each on machines II and III. There are 3 hours available on machine I, 5 hours available on machine II, and 4 hours available on machine III for processing the order. How many souvenirs of each type should Nadi Publications make in order to use all of the available time? Formulate and solve the problem.

Solution The given information may be tabulated as follows :

	Type A	Type B	Type C	Time Available (min)
Machine I	2	1	1	180
Machine II	1	3	2	300
Machine III	2	1	2	240

We have to determine the number of each of three types of souvenirs to be made. So, let x, y and z denote the respective numbers of type-A, type-B, and type-C souvenirs to be made. The total amount of time that machine I is used is given by $2x + y + z$ minutes and must equal 180 minutes. This leads to the equation :

$$2x + y + z = 180 \quad \text{[Time spent on machine I]}$$

Similar considerations on the use of machines II and III lead to the following equations

$$x + 3y + 2z = 300 \quad \text{[Time spent on machine II]}$$

$$2x + y + 2z = 240 \quad \text{[Time spent on machine III]}$$

Since the variables x, y and z must satisfy simultaneously the three conditions represented by the three equations, the solution to the problem is found by solving the following system of linear equations:

$$\begin{cases} 2x + y + z = 180 \\ x + 3y + 2z = 300 \\ 2x + y + 2z = 240 \end{cases}$$

Solving the foregoing system of linear equations by the Gauss-Jordan elimination method, we obtain the following sequence of equivalent augmented matrices :

$$\left[\begin{array}{cc|c} 1 & 1 & 180 \\ 3 & 2 & 300 \\ 1 & 2 & 240 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 300 \\ 2 & 1 & 1 & 180 \\ 2 & 1 & 2 & 240 \end{array}\right]$$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 300 \\ 0 & -5 & -3 & -420 \\ 0 & -5 & -2 & -360 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 300 \\ 0 & 1 & \frac{3}{5} & 84 \\ 0 & -5 & -2 & -360 \end{array}\right]$$

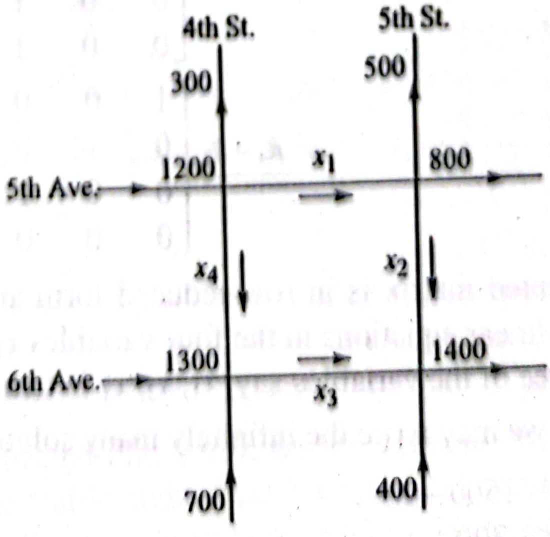
$$\xrightarrow[R_3 + 5R_2]{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 48 \\ 0 & 1 & \frac{3}{5} & 84 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$$\xrightarrow[R_2 - \frac{3}{5}R_3]{R_1 - \frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 36 \\ 0 & 1 & 0 & 48 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

ns, $x = 36$, $y = 48$, and $z = 60$; that is, Nadi Publications should make 36 pe-A souvenirs, 48 type-B souvenirs, and 60 type-C souvenirs in order to use available machine time. (Ans.)

xample-62 [Traffic Control]

he following figure shows the flow of downtown traffic in a certain city ring the rush hours on a typical weekday. The arrows indicate the direction traffic flow on each one-way road, and the average number of vehicles per our entering and leaving each intersection appears beside each road. 5th venue and 6th Avenue can each handle up to 2000 vehicles per hour without ausing congestion, whereas the maximum capacity of both 4th Street and 5th treet is 1000 vehicles per hour. The flow of traffic is controlled by traffic ghts installed at each of the four intersections.



a. Write a general expression involving the rates of flow x_1, x_2, x_3, x_4 and suggest two possible flow patterns that will ensure no traffic congestion.

b. Suppose the part of 4th Street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic.

Solution

a. To avoid congestion, all traffic entering an intersection must also leave that intersection. Applying this condition to each of the four intersections in a clockwise direction beginning with the 5th Avenue and 4th Street intersection, we obtain the following equations :

$$1500 = x_1 + x_4$$
$$1300 = x_1 + x_2$$
$$1800 = x_2 + x_3$$
$$2000 = x_3 + x_4$$

This system of four linear equations in the four variables x_1, x_2, x_3, x_4 may be rewritten in the more standard form

$$\begin{aligned} x_1 \qquad\qquad + x_4 &= 1500 \\ x_1 + x_2 \qquad\qquad &= 1300 \\ x_2 + x_3 \qquad &= 1800 \\ x_3 + x_4 &= 2000 \end{aligned}$$

Using the Gauss-Jordan elimination method to solve the system, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1300 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 1 & 1 & 0 & 1800 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 1 & 1 & 2000 \end{array}\right]$$

$$\xrightarrow{R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form and is equivalent to a system of three linear equations in the four variables x_1, x_2, x_3, x_4. Thus, we may express three of the variables-say, x_1, x_2, x_3 in terms of x_4. Setting $x_4 = t$ (t a parameter), we may write the infinitely many solutions of the system as

$$x_1 = 1500 - t$$
$$x_2 = -200 - t$$
$$x_3 = 2000 - t$$
$$x_4 = t$$

Observe that for a meaningful solution we must have $200 \le t \le 1000$ since x_1, x_2, x_3, and x_4 must all be nonnegative and the maximum capacity of a street is 1000. For example, picking $t = 300$ gives the flow pattern

$$x_1 = 1200 \qquad x_2 = 100 \qquad x_3 = 1700 \qquad x_4 = 300$$

Selecting $t = 500$ gives the flow pattern

$$x_1 = 1000 \qquad x_2 = 300 \qquad x_3 = 1500 \qquad x_4 = 500 \text{ (Ans.)}$$

In this case, x_4 must not exceed 300. Again, using the results of part (a), we find, upon setting $x_4 = t = 300$, the flow pattern

$$x_1 = 1200 \qquad x_2 = 100 \qquad x_3 = 1700 \qquad x_4 = 300$$

obtained earlier. Picking $t = 250$ gives the flow pattern

$$x_1 = 1250 \qquad x_2 = 50 \qquad x_3 = 1750 \qquad x_4 = 250 \text{ (Ans.)}$$

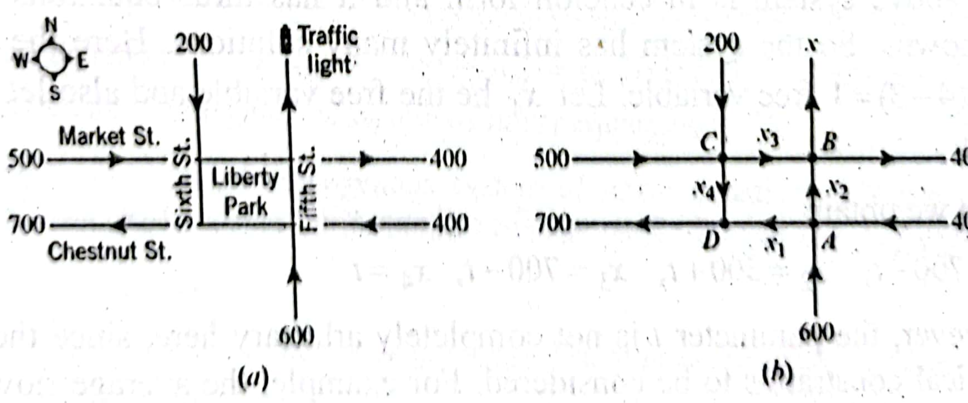
67-3

Example-68 **[Design of Traffic Patterns]** *[DUH 2016]*

The network in the following Figure shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(a) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(b) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?



Solution

(a) According to indicated Figure (*b*), we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Since flows in and out are equal, we have

$x+700+400=1,700$ or, $x=600$ (Ans.)

(*b*) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection Point	Flow In		Flow Out
A	$400+600$	=	x_1+x_2
B	x_2+x_3	=	$400+x$
C	$500+200$	=	x_3+x_4
D	x_1+x_4	=	700

Thus, with $x=600$, as computed in part (a), we obtain the following linear system:

$$\begin{cases} x_1 + x_2 \qquad\qquad\quad = 1,000 \\ \qquad x_2 + x_3 \qquad\quad = 1,000 \\ \qquad\qquad x_3 + x_4 = 700 \\ x_1 \qquad\qquad\;\; + x_4 = 700 \end{cases}$$

Reduce the system to echelon form by means of elementory operations.

$$\begin{cases} x_1 + x_2 \qquad\qquad\quad = 1,000 \\ \qquad x_2 + x_3 \qquad\quad = 1,000 \\ \qquad\qquad x_3 + x_4 = 700 \\ \qquad\qquad x_3 + x_4 = 700 \end{cases} \qquad [L_4' = L_4 + L_3 - L_1]$$

$$\begin{cases} x_1 + x_2 \qquad\qquad\quad = 1,000 \\ \qquad x_2 + x_3 \qquad\quad = 1,000 \\ \qquad\qquad x_3 + x_4 = 700 \end{cases}$$

The above system is in echelon form and it has three equations in four unknowns. So the system has infinitely many solutions. Here the system has $(4-3)=1$ free variable. Let x_4 be the free variable and also let $x_4=t$, $t \in \mathbb{R}$.

Then we obtain

$x_1=700-t,\quad x_2=300+t,\quad x_3=700-t,\quad x_4=t$ ... (1)

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \le t \le 700$, which implies that the average flow rates along the streets will fall in the ranges

$0 \le x_1 \le 700,\quad 300 \le x_2 \le 1,000,\quad 0 \le x_3 \le 700,\quad 0 \le x_4 \le 700$ (Ans.)