Sec-A

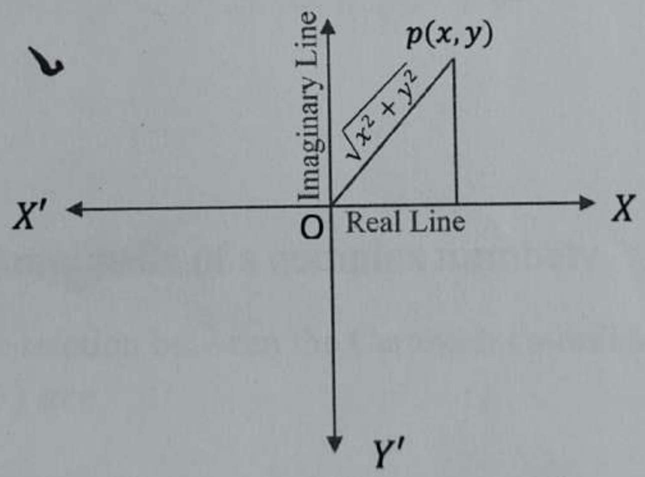
Complex number:

The number of the form $x + iy$ is called the complex number, which is interpreted as point in the complex plane. Where x and y both are real numbers. Also the complex number written as $z = (x, y)$. Here x is called the real part of z, which is denoted by $Re(z)$, i,e $Re(z) = x$ and y is called the imaginary part of z, which is denoted by $Im(z)$, i,e $Im(z) = y$. when a complex number $z = x$ displayed as point $(x, 0)$ on the real line or axis. Then the complex number z is called purely real. *Again when a complex number* $z = iy$ is displayed as point $(0, y)$ on the imaginary line or axis, then the complex number x is called purely imaginary.

Complex Plane:

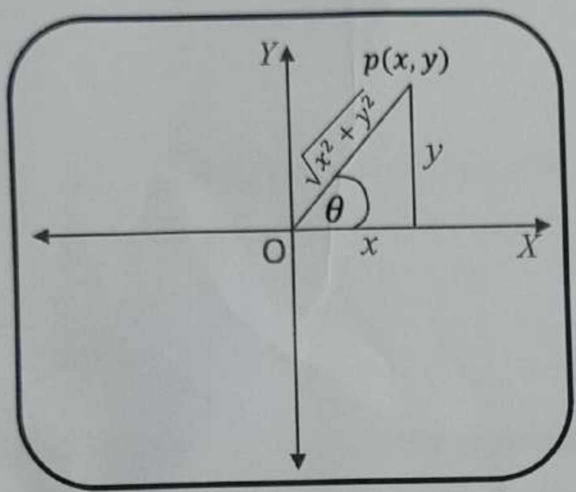
We know that a complex number $z = x + iy$ can be defined as an ordered pair (x, y), Where $(x, y) \in \mathbb{R}$[$\mathbb{R}$ *is the set of real numbers*] and it can be represented by point $p(x, y)$ with regard to two rectangular axes XOX' and

YOY'. Here O is the origin.



Which are properly called real line and imaginary line respectively. Thus a complex number Z is represented by a point P in a plane and corresponding to every point in this plane there exists a complex number. Such a plane is called complex plane or Argand plane or Argand diagram or Gaussian plane.

Modulus of a complex number:



The quantity $\sqrt{x^2+y^2}$ taken with the positive sign is defined to be modulus of the complex number $z = x + iy = (x, y)$ which is denoted by $|z|$ [*the absolute value of z*]. Thus $|z| = \sqrt{x^2+y^2}$.

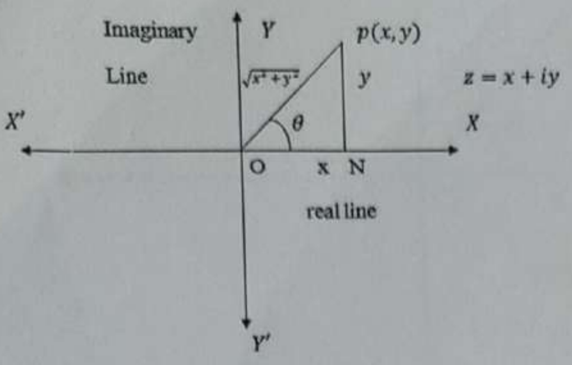
Argument or Amplitude of a complex number:

We know that the relation between the Cartesian co-ordinates (x, y) and the polar co-ordinates (r, θ) *are*

$x = r\cos\theta$ ………………………………………………(i)

and

$y = r\sin\theta$ ……………………………………………………………………………. (ii)



From (i) and (ii) or from the fig. we have

$\tan\theta = \frac{y}{x}$

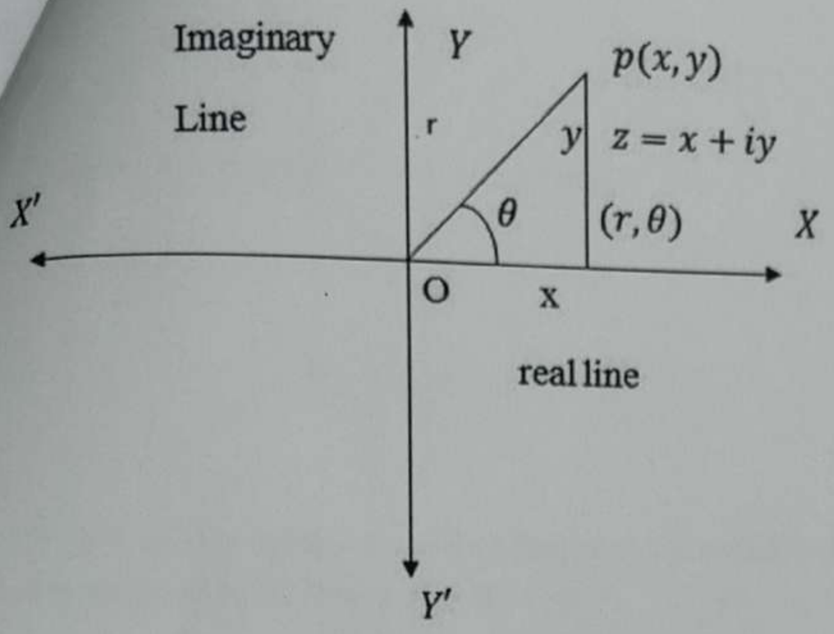
$\Rightarrow \theta = \tan^{-1}\frac{y}{x}$

The quantity $\tan^{-1}\frac{y}{x}$ is called the argument or amplitude of a complex number $z = x + iy = (x, y)$. which is denoted by $Arg(z)$ or $Amp(z)$.

Thus $Arg(z) = Amp(z) = \tan^{-1}\frac{y}{x}$.

The **polar form of a complex number*:***

Let $z = x + iy$ be a complex number



From figure we have $x = r\cos\theta$ *and* $y = r\sin\theta$ then we have

$$z = r\cos\theta + ir\sin\theta$$

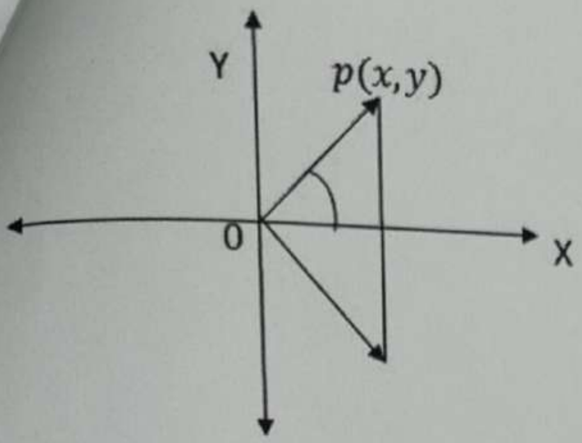
$$= r(\cos\theta + i\sin\theta)$$

$$\Rightarrow z = re^{i\theta}$$

[$\therefore e^{i\theta} = (\cos\theta + i\sin\theta, \textit{which is known as Euler's formula}$]

Thus $z = re^{i\theta}$ is the polar form a complex number. Where (r, θ) is its polar Co-ordinates. Also $r = \sqrt{x^2 + y^2}$ is the modulus of $z = re^{i\theta}$ and $\theta = \tan^{-1}\frac{y}{x}$ is the argument of $z = re^{i\theta}$.

The conjugate complex number:



If $z = x + iy$ is a complex number then $x - iy$ is said to be conjugate to z and is usually denoted by $\bar{z}$, that is $\bar{z} = x - iy$.

By definition, we have

If $z = x + iy = (x, y)$ then $\bar{z} = x - iy = (x, -y)$

In polar form,

$z = r\cos\theta + ir\sin\theta$

$\Rightarrow\ z = re^{i\theta}$ $\qquad$ [$\therefore$ $x = r\cos\theta$ and $y = ir\sin\theta$]

Then,

$\bar{z} = r\cos(-\theta) + ir\sin(-\theta)$

$= re^{-i\theta}$

Therefore the polar Co-ordinates of z is(r, θ), and those of $\bar{z}$ is $(r, -\theta)$.

Geometrically, we see that the conjugate of z is the reflection or image of z about the real line.

The module of both z and $\bar{z}$ are same, which is $r = \sqrt{x^2 + y^2}$

But the argument of z is θ and that $\bar{z}$ is$-\theta$.

Thus $|z| = |\bar{z}\,|$ and $Arg(\bar{z}) = -Arg(z)$.

Problem: If z_1 and z_2 be two complex numbers, then prove that—

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
(ii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
(iii) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

Proof:

Let $z_1 = x_1 + iy_1 \Rightarrow \bar{z}_1 = x_1 - iy_1$

and

$$z_2 = x_2 + iy_2 \Rightarrow \bar{z}_2 = x_2 - iy_2$$

(i)

$$L.H.S = \overline{z_1 + z_2}$$
$$= \overline{(x_1 + iy_1) + (x_2 + iy_2)}$$
$$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$$
$$= (x_1 + x_2) - i(y_1 + y_2)$$
$$= (x_1 - iy_1) + (x_2 - iy_2)$$
$$= \bar{z}_1 + \bar{z}_2$$
$$= R.H.S$$

$So\ L.H.S = R.H.S$ (Proved)

(ii)

$$L.H.S = \overline{(x_1 + iy_1) - (x_2 + iy_2)}$$
$$= \overline{(x_1 - x_2) + i(y_1 - y_2)}$$
$$= (x_1 - x_2) - i(y_1 - y_2)$$
$$= (x_1 - iy_1) - (x_2 - iy_2)$$

$= \overline{z_1} - \overline{z_2}$

$= R.H.S$

So $L.H.S = R.H.S$ (Proved)

(iii)

$L.H.S = \overline{z_1 . z_2}$

$= \overline{(x_1 + iy_1)(x_2 + iy_2)}$

$= \overline{x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1 y_2}$

$= \overline{(x_1x_2 - y_1 y_2) + i(x_1y_2 + x_2y_1)}$

$= (x_1x_2 - y_1 y_2) - i(x_1y_2 + x_2y_1)$

$= x_1x_2 - ix_1y_2 - ix_2y_1 - y_1 y_2$

$= x_1(x_2 - iy_2) - iy_1 (x_2 - iy_2)$

$= (x_1 - iy_1)(x_2 - iy_2)$

$= \overline{z_1} . \overline{z_2}$

$= R.H.S$

So $L.H.S = R.H.S$ (Proved)

(iv)

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Proof:

Let, $z_1 = x_1 + iy_1 \Rightarrow \overline{z_1} = x_1 - iy_1$

And $z_2 = x_2 + iy_2 \Rightarrow \overline{z_2} = x_2 - iy_2$.

L.H.S$=\left(\overline{\frac{z_1}{z_2}}\right)$

$=\left(\overline{\frac{x_1+iy_1}{x_2+iy_2}}\right)$

$=\left\{\overline{\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}}\right\}$

$=\left\{\overline{\frac{x_1x_2-ix_1y_2+ix_2y_1-i^2y_1y_2}{x^2{}_2-i^2y^2{}_2}}\right\}$

$=\left\{\overline{\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{(x^2{}_2+y^2{}_2)}}\right\}$

$=\left\{\frac{(x_1x_2+y_1y_2)-i(x_2y_1-x_1y_2)}{(x^2{}_2+y^2{}_2)}\right\}$

$=\frac{x_1x_2+ix_1y_2-ix_2y_1+y_1y_2}{(x_2+iy_2)(x_2-iy_2)}$

$=\frac{x_1(x_2+iy_2)-iy_1(x_2+iy_2)}{(x_2+iy_2)(x_2-iy_2)}$

$=\frac{(x_1-iy_1)(x_2+iy_2)}{(x_2+iy_2)(x_2-iy_2)}$

$=\frac{(x_1-iy_1)}{(x_2-iy_2)}$

$=\frac{\overline{z_1}}{\overline{z_2}}$

=R.H.S (Proved)

Alternative method:

Let, $z_1 = r_1e^{i\theta_1} => \overline{z_1} = r_1e^{-i\theta_1}$

And $z_2 = r_2e^{i\theta_2} => \overline{z_2} = r_2e^{-i\theta_2}$

L.H.S$=\left(\overline{\frac{z_1}{z_2}}\right)$

$=\overline{\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)}$

$=\overline{(\frac{r_1}{r_2})e^{i(\theta_1-\theta_2)}}$

$=\left(\frac{r_1}{r_2}\right)e^{-i(\theta_1-\theta_2)}$

$=\left(\frac{r_1}{r_2}\right)\cdot e^{-i\theta_1}\cdot e^{i\theta_2}$

$=\frac{r_1e^{-i\theta_1}}{r_2e^{-i\theta_2}}$

$=\frac{\overline{z_1}}{\overline{z_2}}$

=R.H.S (Proved)

(v)

$$\boldsymbol{Arg(z_1z_2) = Arg(z_1) + Arg(z_2)}$$

Proof:

Let, $z_1 = r_1e^{i\theta_1}$ $\therefore Arg(z_1) = \theta_1$

And $z_2 = r_2e^{i\theta_2}$ $\therefore Arg(z_2) = \theta_2$

Now $z_1z_2 = r_1e^{i\theta_1}\cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$

$\therefore Arg(z_1z_2) = \theta_1 + \theta_2$

$Arg(z_1z_2) = Arg(z_1) + Arg(z_2)$ (Proved)

(vi)

$z = (x + iy)$

$\bar{z} = x - iy$

And $Im(z) = y$

Subtracting (ii) from (i),

$z - \bar{z} = 2iy$

$2i\, Im(z)$

(Proved)

(vi)

Let, $z = (x + iy)$

$\therefore |z| = \sqrt{x^2 + y^2}$ ……………………………………………………. (1)

and $Re(z) = x$ ………………………………………………….. (2)

If $y = 0$ then $z = \sqrt{x^2 + 0} = x$ …………………………. (3)

and if, $y \neq 0$ then $|\bar{z}| > x$ ……………………………………………….. (4)

From, (2) & (4), $|z| \geq x$

$|z| \geq Re|z|$

(Proved)

Problem:

Prove that the modulus of sum or difference of two complex numbers is always less than or equal to the sum of their moduli

Or

If z_1 and z_2 be complex numbers then prove that

$)|z_1 + z_2| \leq |z_1| + |z_2|$

$(ii)|z_1 - z_2| \leq |z_1| + |z_2|$

Proof:

Let $z_1$and z_2 be complex number.

We know that,

$|z|^2 = z\bar{z} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

If $z = z_1 + z_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

then $\bar{z} = \overline{z_1 + z_2} = \bar{z_1} + \bar{z_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$

Using (2) & (3) in (1) we get

$|z_1 + z_2|^2$

$= (z_1 + z_2)(\bar{z_1} + \bar{z_2})$

$= z_1\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_1} + z_2\bar{z_2}$

$= |z_1|^2 + z_1\bar{z_2} + \bar{z_1}\bar{\bar{z_2}} + |z_2|^2 \ by(1)$

$= |z_1|^2 + z_1\bar{z_2} + \overline{z_1\bar{z_2}} + |z_2|^2$

$= |z_1|^2 + 2Re(z_1\bar{z_2}) + |z_2|^2 \qquad [z + \bar{z} = 2Re(z)]$

$\leq |z_1|^2 + 2|z_1\bar{z_2}| + |z_2|^2 \qquad [\because |z| \geq Re(z)]$

$= |z_1|^2 + 2(|z_1||\bar{z_2}|) + |z_2|^2 \qquad [\because |z_1 z_2| = |z_1 z_2|, |\bar{z}| = |z|]$

$= |z_1|^2 + 2(|z_1||z_2|) + |z_2|^2$

$= \{|z_1| + |z_2|\}^2$

$|z_1 + z_2|^2 \leq \{|z_1| + |z_2|\}^2$

$|z_1 + z_2| \leq |z_1| + |z_2|$

placing $z_2 by\ (-z_2)$ we get, $|z_1 - z_2| \leq |z_1| + |z_2|$
(Proved)

Problem: Perform the indicated operations both analytically and graphically:

$(a)(3+4i)+(5+2i)$

$(b)(6-2i)-(2-5i)$

$(c)(-3+5i)+(4+2i)+(5-3i)+(-4-6i)$

Solution: (a)

Analytically,

$(3+4i)+(5+2i)=3+5+4i+2i$

$=8+6i$

Graphically

Represent the two complex numbers by points P_1 and P_2 respectively as in fig-(1).

Complete the parallelogram OP_1PP_2 with OP_1 and OP_2 as adjacent sides, point P represent the sum, $8+6i$, of the two given complex numbers.

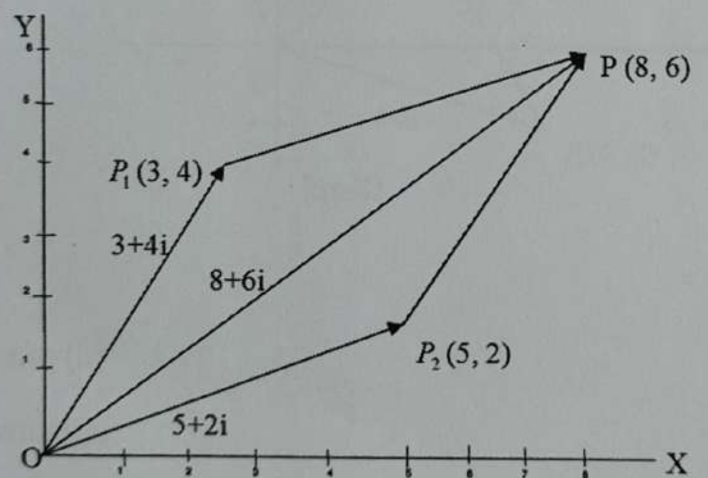


Fig-(1)

Solution: (b)

Analytically,

$(6-2i) - (2-5i) = 6-2-2i+5i$

$= 4+3i$

Graphically,

$(6-2i) - (2-5i) = (6-2i) + (-2+5i)$

Represent the two complex numbers by points P_1 and P_2 respectively as in fig-(2).

Complete the parallelogram with OP_1 and OP_2 as adjacent sides, then point P represent the sum, $4+3i$, of the two given complex numbers.

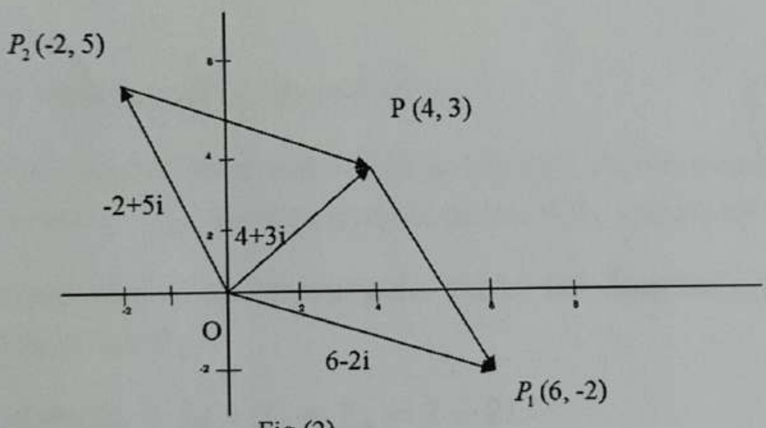


Fig-(2)

Solution: (b)

Analytically,

$(-3+5i) + (4+2i) + (5-3i) + (-4-6i)$

$= (-3+4+5-4) + (5i+2i-3i-6i)$

$= 2 - 2i$

Graphically,

Represent the number to be added by Z_1, Z_2, Z_3, Z_4 respectively

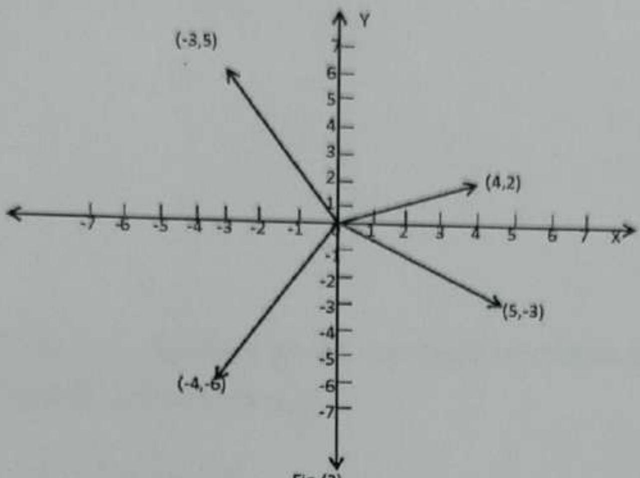


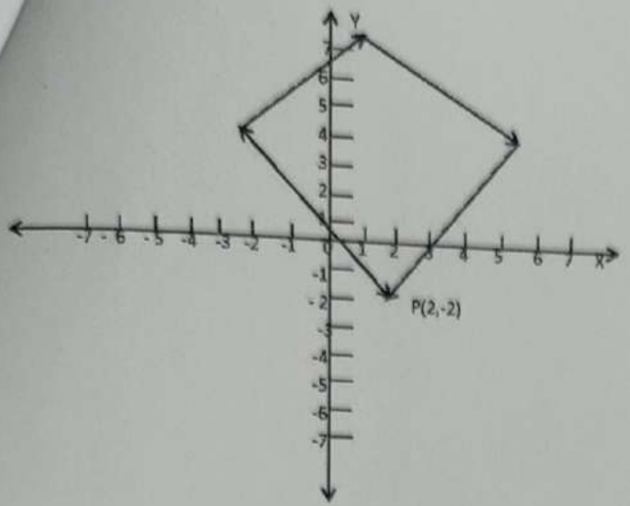
Fig-(3)

These are shown graphically in Fig-(3).

To find the required sum proceed as in Fig-(4) . At the terminal point of Z_1 construct vector Z_2. At the terminal point of Z_3 construct vector Z_4.

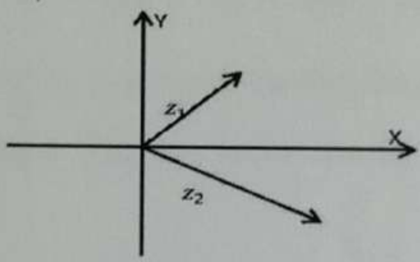
The resultant sum of constructing the vector OP from the initial point of Z_1 to the terminal point of Z_4.

That is , OP= $Z_1 + Z_2 + Z_3 + Z_4 = 2 - 2i$.



Problem: If Z_1 and Z_2 are two given complex numbers as in the figure below the construct graphically the followings:

$(a)\ 3Z_1 - 2Z_2,\ (b) \frac{1}{2}Z_2 + \frac{5}{3}Z_1$



Solution: (a) From the figure $OA = 3Z_1$ is a vector having length 3 times vector $Z_1$and the same direction and $OB = -2Z_2$ is a vector having length 2 times vector $Z_2$and opposite direction.

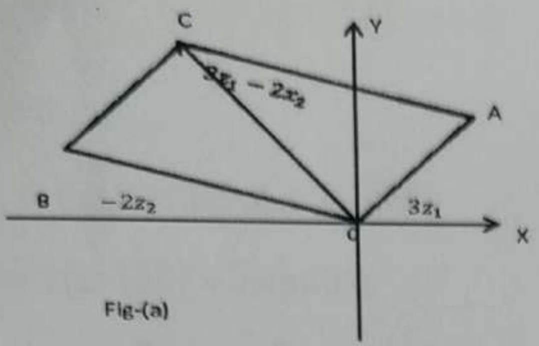
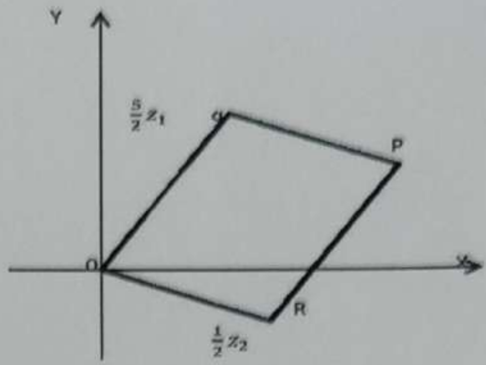
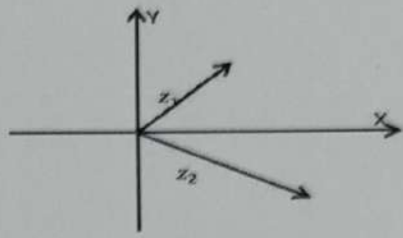


Fig-(a)





Then the required vector $OC = OA + OB = 3Z_1 - 2Z_2$

(b) From the figure $OR = \frac{1}{2}Z_2$ is a vector having length $\frac{1}{2}$ times vector Z_2 and the same direction, and $OQ = \frac{5}{3}Z_1$ is a vector having length $\frac{5}{3}$ times vector Z_1 and the same direction.

Hence , the required vector $OP = \frac{1}{2}Z_2 + \frac{5}{3}Z_1$.

Problem: Express $2 + 2\sqrt{3}i$ in polar form .

Solution:

Modulus or absolute value of the given complex number is,

$r = |2 + 2\sqrt{3i}| = \sqrt{4 + 12} = 4$

Amplitude or argument ,

$\theta = \tan^{-1}\frac{2\sqrt{3}}{2} = \tan^{-1}\sqrt{3} = 60°$

Then.

$2 + 2\sqrt{3}i = r(cos\theta + sin\theta)$

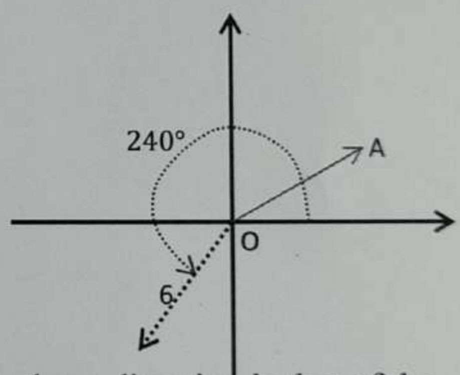
$= 4(cos60° + isin60°)$

$= 4(\cos\frac{\pi}{3} + isin\frac{\pi}{3})$

$= 4e^{\frac{i\pi}{3}}$

Problem:

Represent Graphically $6(cos240° + isin240°)$



Solution:

$6(cos240° + isin240°) = 6e^{\frac{4i\pi}{3}}$

If we start with vector OA whose magnitude is 6 and whose direction is that of the positive X axsis ,we can obtain OP by rotating OA counter-clockwise through an angle of $240°$

Problem:

Find the modulus and principal argument of the following complex number.

$(i)\ 2 + i \qquad (ii) \left(\frac{1+i}{1-i}\right)^2$

Solution: (i)

Let , $z = 2 + i$

Modulus or absolute value of z,

$$r = |2 + i| = \sqrt{4+1} = \sqrt{5}$$

Amplitude or argument of z ,

$$\theta = \tan^{-1}\frac{1}{2}$$

(ii)

$$Z = \left(\frac{1+i}{1-i}\right)^2 = \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^2 = \left(\frac{(1+i)^2}{1^2 - i^2}\right)^2$$

$$= \left\{\frac{1+i^2+2i}{1+1}\right\}^2 = \left(\frac{2i}{2}\right)^2 = i^2 = -1 = -1 + 0.i$$

Modulus or absolute value of z,

$$r = \sqrt{0^2 + (-1)^2} = 1$$

Amplitude or argument of z ,

$$\theta = \tan^{-1}\frac{0}{-1} = 0$$

Equation of a circle of a complex number:

If a complex number has center (h, k) and radius r then the equation of a circle of a complex number is

$$|Z - h - ik| = r$$

By putting $Z = x + iy$ then we get

$$|x + iy - h - ik| = r$$