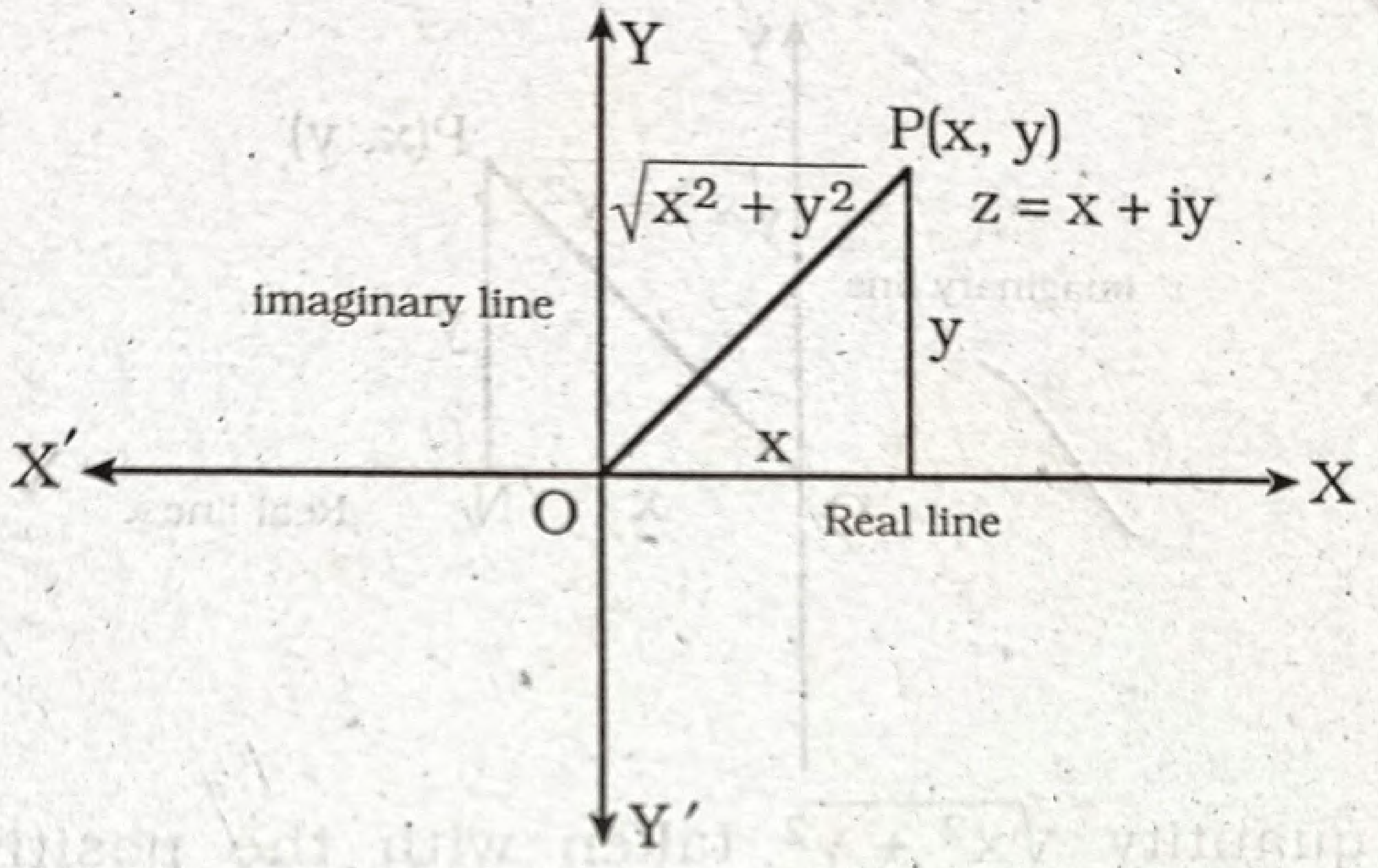
CHAPTER–1
COMPLEX NUMBERS

1.1 : *Complex Plane* : We know that a complex number $z = x + iy$ can be defined as an ordered pair (x, y), where $(x, y) \in \mathbf{R}$ [**R** is the set of real numbers] and it can be represented by point P(x, y) with regard to two rectangular axes XOX′ and YOY′. Here O is the origin.



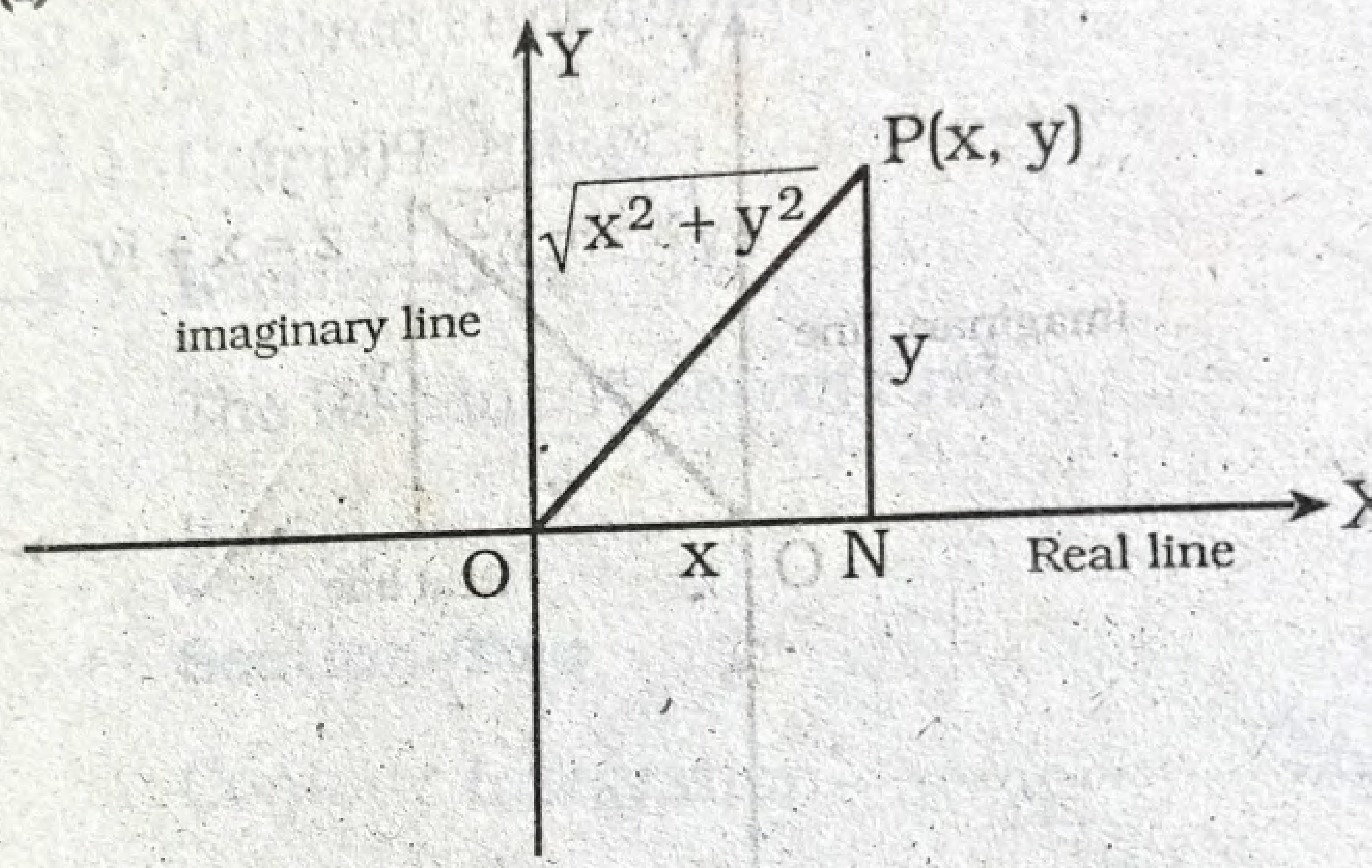
Which are properly called real line and imaginary line respectively. Thus a complex number z is represented by a point P in a plane and corresponding to every point in this plane there exists a complex number. Such a plane is called complex plane or Argand plane or Argand diagram or Gaussian plane.

1·2.1 : *Complex number* : The number of the form $x + iy$ is called the complex number, which is interpreted as point in the complex plane. Where x and y both are real numbers. Also the complex number written as $z = (x, y)$. Here x is called the real part of z, which is denoted by Re(z), i. e $Re(z) = x$, and y is called imaginary part of z, which is denoted by Im(z), i. e $Im(z) = y$. When a complex number $z = x$ is displayed as point (x, 0) on the real line or axis, then the complex number z is called purely real. Again, when a complex number $z = iy$ is displayed as point (0, y) on the imaginary line or axis, then the complex number z is called purely imaginary.

1-2.2 : *(i) Unit of real numbers :* We have stated above that x is called a purely real number. Here x is multiplied by 1. So we say that 1 is the unit of real numbers.

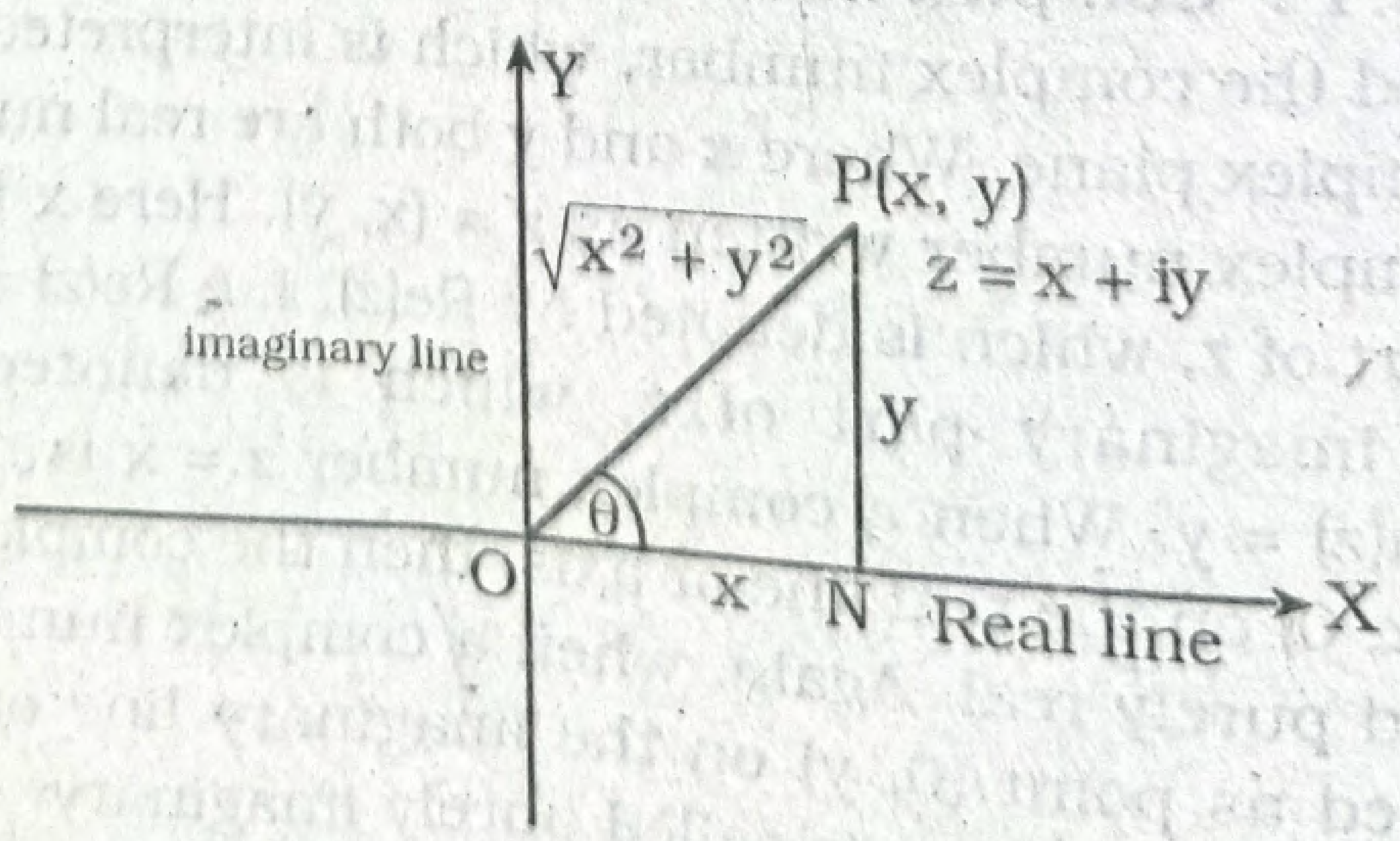
(ii) Unit of imaginary numbers : We have stated above that iy is called a purely imaginary number. Here the symbol i stands for unit of imaginary numbers, that is i is the unit of imaginary numbers.

1-2.3 : (i) Modulus of a complex number :



The quantity $\sqrt{x^2 + y^2}$ taken with the positive sign is defined to be the modulus of the complex number $z = x + iy = (x, y)$, which is denoted by $|z|$, [the absolute value of z]. Thus $|z| = \sqrt{x^2 + y^2}$.

(ii) Argument or Amplitude of a complex number : We know that the relations between the cartesian co-ordinates (x, y) and the polar co-ordinates (r, θ) are $x = r\cos\theta$ ··· (i) and $y = r\sin\theta$ ··· (ii)

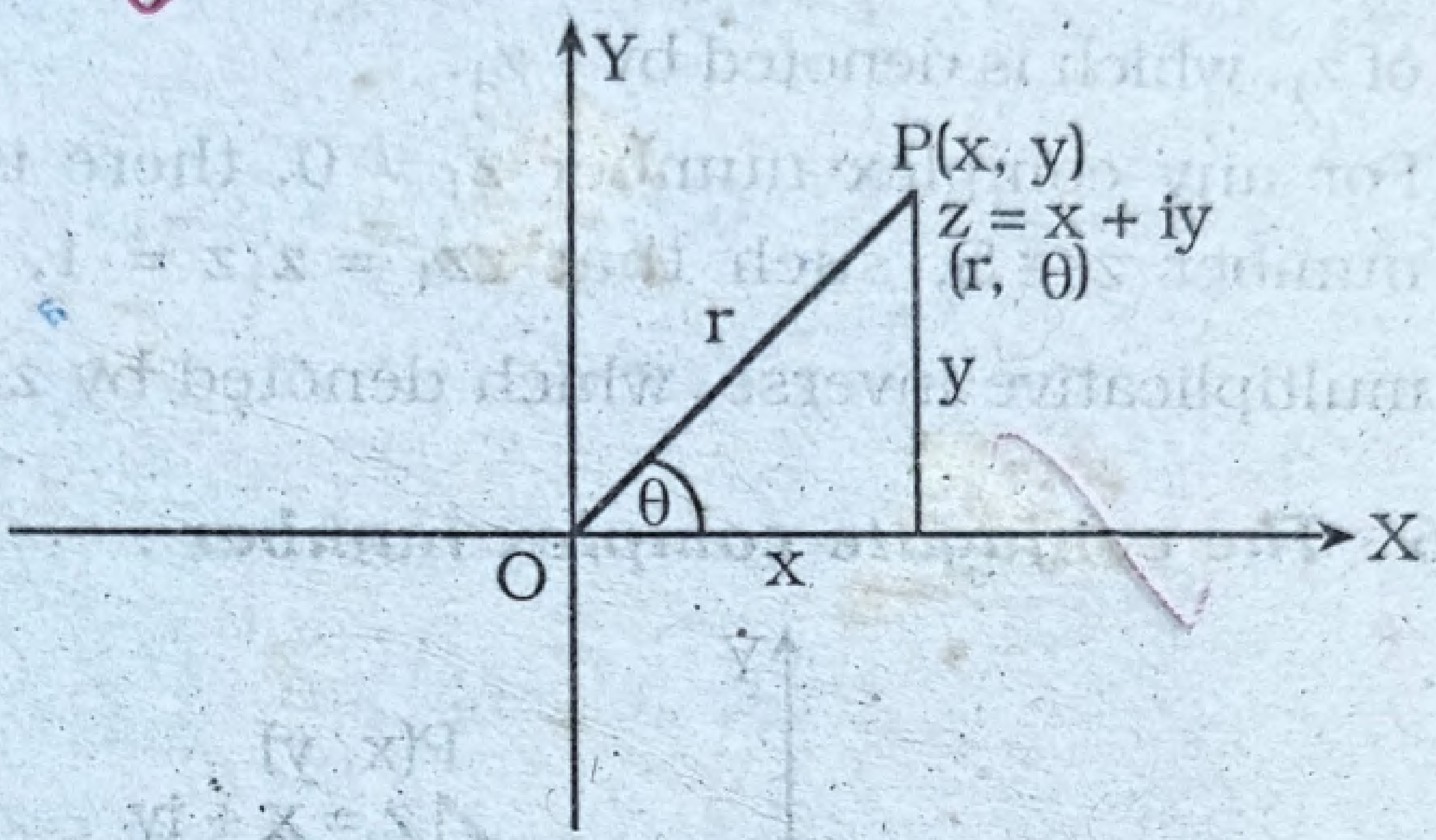


From (i) and (ii) or from adjacent fig. We have

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\frac{y}{x}.$$

The quantity $\tan^{-1}\frac{y}{x}$ is called the argument or amplitude of a complex number $z = x + iy = (x, y)$. Which is denoted by Arg(z) or Amp(z). Thus $\text{Arg}(z) = \text{Amp}(z) = \tan^{-1}\frac{y}{x}$.

1-2.4 : *The polar form of a Complex number.*



Let $z = x + iy$ be a complex number. From figure, we have $x = r\cos\theta$ and $y = r\sin\theta$ then we have

$$z = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$\Rightarrow z = re^{i\theta}$$

[$\because e^{i\theta} = \cos\theta + i\sin\theta$, which is known as Euler's formula]

Thus $z = re^{i\theta}$ is the polar form a complex number. Where (r, θ) is its polar Co-ordinates. Also $r = \sqrt{x^2 + y^2}$ is the modulus of $z = re^{i\theta}$ and $\theta = \tan^{-1}\frac{y}{x}$ is the argument of $z = r.e^{i\theta}$.

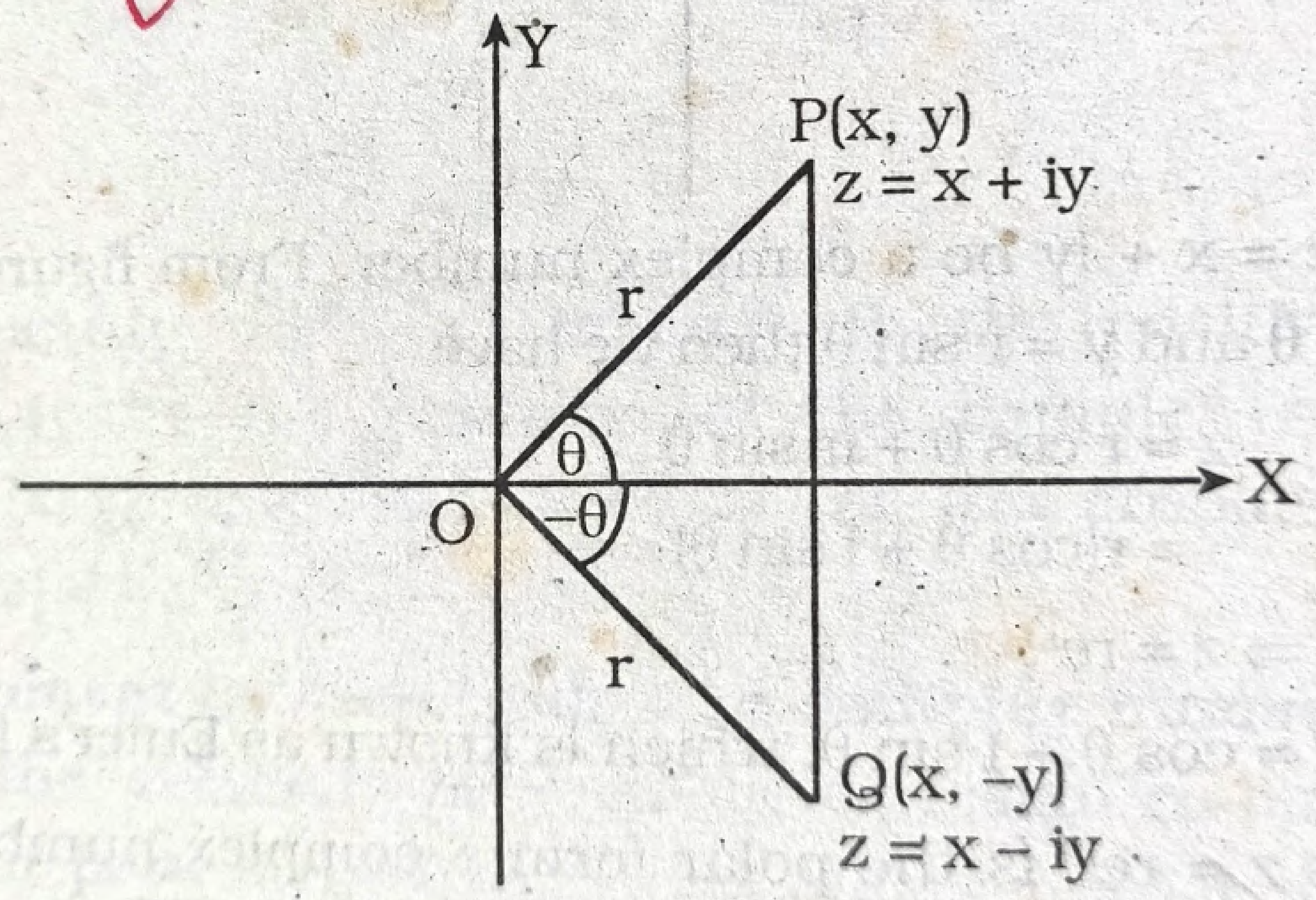
Note : The principal value of argument lies between $-\pi$ and π.

1-2.5 : *Properties of Complex numbers* : If z_1, z_2 and z_3 be the complex numbers belong to the set of complex number S, then

1. $z_1 + z_2 \in S$ and $z_1 z_2 \in S$. Closure law.
2. $z_1 + z_2 = z_2 + z_1$, Commutative law of addition.

3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, Associative law of addition.
4. $z_1 z_2 = z_2 z_1$, Commutative law of multiplication.
5. $z_1(z_2 z_3) = (z_1 z_2)\, z_3$, Associative law of multiplication.
6. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$, Distributive law.
7. (i) $z_1 = 0 = 0 + z_1 = z_1$, 0 is called the additive identity.
   (ii) $1.z_1 = z_1.1 = z_1$, 1 is called multiplicative identity.
8. For any complex number z_1, there is a unique number $z \in S$, such that $z + z_1 = 0$; z is called additive inverse of z_1, which is denoted by $-z_1$.
9. For any complex number $z_1 \neq 0$, there is a unique number $z \in S$, such that $zz_1 = z_1 z = 1$, z is called multiplicative inverse, which denoted by z_1^{-1} or $\dfrac{1}{z_1}$.

1-2.6 : The conjugate complex number :



If $z = x + iy$ is a complex number, then $x - iy$ said to be conjugate to z and is usually denoted by $\bar{z}$, that is $\bar{z} = x - iy$.

By definition, we have

if $z = x + iy = (x, y)$ then $\bar{z} = x - iy = (x, -y)$

In polar form, $z = r\cos\theta + ir\sin\theta$

$$\Rightarrow z = re^{i\theta} \qquad [\because x = r\cos\theta \text{ and } y = r\sin\theta]$$

then $\bar{z} = r\cos(-\theta) + ir\sin(-\theta)$

$$= re^{-i\theta}$$

The polar Co-ordinates of z is (r, θ), those of $\bar{z}$ is $(r, -\theta)$.

Geometrically, we see that, the conjugate of z is the reflection or image of z about the real line.

The moduli of both z and $\bar{z}$ are same, which is $r = \sqrt{x^2 + y^2}$

But the argument of z is θ and that of $\bar{z}$ is − θ.

Thus $|z| = |\bar{z}|$ and $\text{Arg}(\bar{z}) = -\text{Arg}(z)$

***1-2.7* : If z_1 and z_2 be two complex numbers, then prove that—**

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (ii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(iii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$ (iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(v) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

(vi) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

Proof : Let $z_1 = x_1 + iy_1 \Rightarrow \overline{z_1} = x_1 - iy_1$

and $z_2 = x_2 + iy_2 \Rightarrow \overline{z_2} = x_2 - iy_2$

(i) L. H. S $= \overline{z_1 + z_2}$

$= \overline{(x_1 + iy_1) + (x_2 + iy_2)}$

$= \overline{(x_1 + x_2) + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2)$

$= (x_1 - iy_1) + (x_2 - iy_2)$

$= \overline{z_1} + \overline{z_2}$ = R. H. S **(Proved)**

(ii) L. H. S. $= \overline{(x_1 + iy_1) - (x_2 + iy_2)}$

$= \overline{(x_1 - x_2) + i(y_1 - y_2)}$

$= (x_1 - x_2) - i(y_1 - y_2)$

$= (x_1 - iy_1) - (x_2 - iy_2)$

$= \overline{z_1} - \overline{z_2}$ = R. H. S. **(Proved)**

(iii) L. H. S $= \overline{z_1 z_2}$

$= \overline{(x_1 + iy_1)(x_2 + iy_2)}$

$= \overline{x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2}$

$= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$

$$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$$
$$= x_1x_2 - ix_1y_2 - ix_2y_1 - y_1y_2$$
$$= x_1(x_2 - iy_2) - iy_1(x_2 - iy_2)$$
$$= (x_1 - iy_1)(x_2 - iy_2)$$
$$= \overline{z_1}\,\overline{z_2} = \text{R. H. S.}$$ **(Proved)**

Alternative method :

In polar form, let $z_1 = r_1 e^{i\theta_1} \Rightarrow \overline{z_1} = r_1 e^{-i\theta_1}$

and $z_2 = r_2 e^{i\theta_2} \Rightarrow \overline{z_2} = r_2 e^{-i\theta_2}$

$$\text{L. H. S} = \overline{z_1 z_2}$$
$$= \overline{r_1 e^{i\theta} . r_2 e^{i\theta_2}}$$
$$= \overline{r_1 r_2 e^{i(\theta_1 + \theta_2)}}$$
$$= r_1 r_2 e^{-i(\theta_1 + \theta_2)}$$
$$= r_1 r_2 e^{-i\theta_1} . e^{-i\theta_2}$$
$$= r_1 e^{-i\theta_1} . r_2 e^{-i\theta_2}$$
$$= \overline{z_1} \cdot \overline{z_2} = \text{R. H. S.}$$ **(Proved)**

(iv) $\text{L. H. S} = \overline{\left(\frac{z_1}{z_2}\right)}$

$$= \overline{\left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)}$$
$$= \overline{\left\{\frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}\right\}}$$
$$= \overline{\left\{\frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{x_2^2 - i^2y_2^2}\right\}}$$
$$= \overline{\left\{\frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)}\right\}}$$
$$= \left\{\frac{(x_1x_2 + y_1y_2) - i(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)}\right\}$$
$$= \frac{x_1x_2 + ix_1y_2 - ix_2y_1 + y_1y_2}{(x_2 + iy_2)(x_2 - iy_2)}$$
$$= \frac{x_1(x_2 + iy_2) - iy_1(x_2 + iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$$= \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$

$$= \frac{x_1 - iy_1}{x_2 - iy_2} = \frac{\overline{z_1}}{\overline{z_2}} = \text{R. H. S} \quad \textbf{(Proved)}$$

Alternative method :

Let $z_1 = r_1 e^{i\theta_1} \Rightarrow \overline{z_1} = r_1 e^{-i\theta_1}$ and $z_2 = r_2 e^{i\theta_2} \Rightarrow \overline{z_2} = r_2 e^{-i\theta_2}$

$$\text{L. H. S} = \overline{\left(\frac{z_1}{z_2}\right)}$$

$$= \overline{\left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)}$$

$$= \overline{\left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}}$$

$$= \frac{r_1}{r_2} e^{-i(\theta_1 - \theta_2)}$$

$$= \frac{r_1}{r_2} . e^{-i\theta_1} . e^{i\theta_2}$$

$$= \frac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}}$$

$$= \frac{\overline{z_1}}{\overline{z_2}} = \text{R. H. S} \quad \textbf{(Proved)}$$

(v) Let $z_1 = r_1 e^{i\theta_1} \quad \therefore \text{Arg}(z_1) = \theta_1$

and $z_2 = r_2 e^{i\theta_2} \quad \therefore \text{Arg}(z_2) = \theta_2$.

Now $z_1 z_2 = r_1 e^{i\theta_1} . r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$\therefore \text{Arg}(z_1 z_2) = \theta_1 + \theta_2$

$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$. **(Proved)**

(vi) Let $z_1 = r_1 e^{i\theta_1} \Rightarrow \text{Arg}(z_1) = \theta_1$

and $z_2 = r_2 e^{i\theta_2} \Rightarrow \text{Arg}(z_2) = \theta_2$

Now $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) . e^{i(\theta_1 - \theta_2)}$

$\therefore \text{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$. **(Proved)**

$\Rightarrow 2[|x|^2 + |y|^2] \geq [|x| + |y|]^2$

$\Rightarrow |x| + |y| \leq \sqrt{2}\sqrt{|x|^2 + |y|^2}$

$\Rightarrow |x| + |y| \leq \sqrt{2}\sqrt{x^2 + y^2}$

$\Rightarrow |x| + |y| \leq \sqrt{2}\,|z|$ $\quad [\because |z| = |x + iy| = \sqrt{x^2 + y^2}]$

$\Rightarrow \{|x| + |y|\}/\sqrt{2} \leq |z|$ ··· (ii)

Combing (i) and (ii) we get

$\{|x| + |y|\}/\sqrt{2} \leq |z| \leq |x| + |y|$. **(Proved)**

1-2.33 : The followings should be kept in mind :

(i) If the principal argument of a complex number is 0 or π, then the complex number is purely real.

(ii) If the principal argument of a complex number is $\pi/2$ or $-\pi/2$ then the complex number is purely imaginary.

(iii) If $Re(z) = |z|$, then z is real.

(iv) If $Im(z) = |z|$, then z is purely imaginary.

(v) If z is negative real number, then the principal argument $[Arg(z)] = \pi$. [by convention]

(vi) $arg(z) = Arg(z) + 2n\pi$, $n \in I$ [the set of integers]

(vii) The argument of 0 is undefined.

(viii) $|z| = 1$ is the equation of unit circle whose centre is at (0, 0) and radius is 1.

WORKOUT EXAMPLES

1. Find the modulus and Principal argument of the following complex numbers :

(i) $2 + i$ (ii) $\pm i$ (iii) ± 1 (iv) $\dfrac{2 - i}{2 + i}$

(v) $5 - 5i$ (vi) $(2 + 3i)^2$ (vii) $\left(\dfrac{1 + i}{1 - i}\right)^2$ (viii) $\dfrac{\sqrt{3} + i}{\sqrt{3} - i}$

(ix) $\left(\dfrac{1 + \sqrt{3}i}{1 - \sqrt{3}i}\right)^2$ (x) $(-3 + 5i)^2$.

Solution : **(i)** Let $z = 2 + i$

$$\therefore |z| = |2+i| = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$

$\therefore$ Modulus of $(2 + i)$ is $\sqrt{5}$.

Argument of $z = 2 + i$ is $\tan^{-1}\frac{1}{2}$.

(ii) Let $z = \pm i$

$$\therefore |z| = |\pm i| = |0 \pm i| = \sqrt{0^2 + (\pm 1)^2} = \sqrt{0+1} = 1$$

$\therefore$ Modulus of $z = \pm i$ is 1.

And argument of $z = \pm i$ is $\tan^{-1}\left(\frac{\pm 1}{0}\right)$

$$= \pm \tan^{-1} \infty = \pm \frac{\pi}{2},$$

(iii) Let $z = \pm 1$

$$\Rightarrow |z| = |\pm 1 + 0i| = \sqrt{(\pm 1)^2 + 0^2} = 1$$

$\therefore$ Modulus of $z = \pm 1$ is 1 and Argument is $\tan^{-1}\left(\frac{0}{\pm 1}\right) = 0$ and π.

(iv) Let $z = \frac{3-i}{2+i} = \frac{(3-i)(2-i)}{(2+i)(2-i)} = \frac{6-5i+i^2}{4-i^2} = \frac{5-5i}{5}$

$$= 1 - i \quad \therefore |z| = |1 - i| = \sqrt{1 + (-1)^2} = \sqrt{2}$$

$\therefore$ Modulus of z is $\sqrt{2}$

And the argument of z is $\tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$.

(v) Let $z = 5 - 5i$

$$\therefore |z| = |5 - 5i| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$$

$\therefore$ Modulus of $z = 5 - 5i$ is $5\sqrt{2}$

And the argument of $z = 5 - 5i$ is $\tan^{-1}\left(\frac{-5}{5}\right)$

$$= \tan^{-1}(-1)$$

$$= -\frac{\pi}{4}$$

(vi) Let $z = (2 + 3i)^2$

$$= 4 + 12i + 9i^2$$

$$= 4 - 9 + 12i$$

$$= -5 + 12i$$

$\therefore |z| = |-5 + 12i| = \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144}$

$$= \sqrt{169} = 13$$

$\therefore$ Modulus of $z = (2 + 3i)^2$ is 13

And the argument of z is $\tan^{-1}\left(\frac{-12}{5}\right)$.

(vii) Let $z = \left(\frac{1 + i}{1 - i}\right)^2$

$$= \frac{1 + 2i + i^2}{1 - 2i + i^2}$$

$$= \frac{1 + 2i - 1}{1 - 2i - 1}$$

$$= \frac{2i}{-2i} = -1.$$

$\therefore |z| = \left|\left(\frac{1 + i}{1 - i}\right)^2\right| = |-1| = |-1 + 0i| = \sqrt{(-1)^2 + 0^2} = 1$

$\therefore$ Modulus of $z = \left(\frac{1 + i}{1 - i}\right)^2$ is 1.

And the argument of z is $\tan^{-1}\left(\frac{0}{-1}\right) = \pi$.

(viii) Let $z = \frac{\sqrt{3} + i}{\sqrt{3} - i}$

$$= \frac{(\sqrt{3} + i)(\sqrt{3} + i)}{(\sqrt{3} + i)(\sqrt{3} - i)}$$

$$= \frac{(\sqrt{3})^2 + 2\sqrt{3}i + i^2}{(\sqrt{3})^2 - i^2}$$

$$= \frac{3 + 2\sqrt{3}i - 1}{3 + 1}$$

$$= \frac{2 + 2\sqrt{3}i}{4}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$\therefore |z| = \left|\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\left(\frac{1}{4} + \frac{3}{4}\right)}$$

$$= \sqrt{1} = 1$$

$\therefore$ Modulus of z is 1.

And the argument of z is $\tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$$= \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{3}$$

(ix) Let $z = \left(\frac{1 + \sqrt{3}i}{1 - \sqrt{3}i}\right)^2$

$$= \frac{1 + 2\sqrt{3}i + 3i^2}{1 - 2\sqrt{3}i + 3i^2}$$

$$= \frac{1 - 3 + 2\sqrt{3}i}{1 - 3 - 2\sqrt{3}i}$$

$$= \frac{-2 + 2\sqrt{3}i}{-2 - 2\sqrt{3}i}$$

$$= \frac{-2(1 - \sqrt{3}i)}{-2(1 + \sqrt{3}i)}$$

$$= \frac{(1 - \sqrt{3}i)(1 - \sqrt{3}i)}{(1 + \sqrt{3}i)(1 - \sqrt{3}i)}$$

$$= \frac{1 - 2\sqrt{3}i + (\sqrt{3})^2 i^2}{1 - 3i^2}$$

$$= \frac{1 - 2\sqrt{3}i - 3}{1 + 3}$$

$$= \frac{-2 - 2\sqrt{3}i}{4}$$

$$= -\frac{1}{2} + \left(\frac{-\sqrt{3}}{2}\right)i$$

$$\therefore |z| = \left|-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right|$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= \sqrt{1} = 1$$

$\therefore$ Modulus of z is 1.

And the argument is $\tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$

$$= \tan^{-1}(\sqrt{3})$$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right)$$

$$= \tan^{-1}\tan\left(\pi + \frac{\pi}{3}\right)$$

$$= \frac{4\pi}{3}.$$

(x) Let $z = (-3 + 5i)^2$

$$= 9 - 30i + 25i^2$$

$$= 9 - 25 - 30i$$

$$= -16 - 30i$$

$$\therefore |z| = |-16 - 30i|$$

$$= \sqrt{(-16)^2 + (-30)^2}$$

$$= \sqrt{4 \times 8^2 + 4 \times 15^2}$$

$$= 2\sqrt{64 + 225}$$

$$= 2\sqrt{289}$$

$$= 2 \times 17 = 34$$

$\therefore$ Modulus of z is 34.

(vii) Given that $Im(z^2) = 4$

Let $z = x + iy$

$\therefore Im(x + iy)^2 = 4$

$\Rightarrow Im(x^2 + 2ixy + i^2y^2) = 4$

$\Rightarrow Im(x^2 - y^2 + i.2xy) = 4$

$\Rightarrow 2xy = 4$

$\Rightarrow xy = 2$

which is the equation of a hyperbola. **(Proved)**

(viii) Given that $Re(z^2) = 4$

Let $z = x + iy$

$\therefore Re(x + iy)^2 = 4$

$\Rightarrow Re(x^2 + i^2y^2 + i.2xy) = 4$

$\Rightarrow Re(x^2 - y^2 + i.2xy) = 4$

$\Rightarrow x^2 - y^2 = 4$

$\Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1$

which is the equation of a rectangular hyperbola.

(ix) Prove that $|z + 2i| + |z - 2i| = 6$

Let $z = x + iy$

$\therefore |x + iy + 2i| + |x + iy - 2i| = 6$

$\Rightarrow |x + i(y + 2)| + |x + i(y - 2)| = 6$

$\Rightarrow \sqrt{x^2 + (y + 2)^2} + \sqrt{x^2 + (y - 2)^2} = 6$

$\Rightarrow \sqrt{x^2 + (y + 2)^2} = 6 - \sqrt{x^2 + (y - 2)^2}$

$\Rightarrow x^2 + (y + 2)^2 = 36 - 12\sqrt{x^2 + (y - 2)^2} + x^2 + (y - 2)^2$

[by squaring]

$\Rightarrow x^2 - x^2 + (y + 2)^2 - (y - 2)^2 - 36 = - 12\sqrt{x^2 + (y - 2)^2}$

$\Rightarrow 4.2.y - 36 = - 12\sqrt{x^2 + (y - 2)^2}$

$\Rightarrow 3\sqrt{x^2 + (y - 2)^2} = 9 - 2y$

$\Rightarrow 9[x^2 + (y - 2)^2] = (9 - 2y)^2$ [by squaring]

$\Rightarrow 9x^2 + 9y^2 - 36y + 36 = 81 - 36y + 4y^2$

$\Rightarrow 9x^2 + 5y^2 = 81 - 36$

$\Rightarrow 9x^2 + 5y^2 = 45$

$\Rightarrow \frac{x^2}{5} + \frac{y^2}{9} = 1$

which is the equation of an ellipse. **(Proved)**

(x) Given that $|z - 4i| + |z + 4i| = 10$

Let $z = x + iy$

$\therefore |x + iy - 4i| + |x + iy + 4i| = 10$

$\Rightarrow |x + i(y - 4)| + |x + i(y + 4)| = 10$

$\Rightarrow \sqrt{x^2 + (y - 4)^2} + \sqrt{x^2 + (y + 4)^2} = 10$

$\Rightarrow \sqrt{x^2 + (y + 4)^2} = 10 - \sqrt{x^2 + (y - 4)^2}$

$\Rightarrow x^2 + (y + 4)^2 = 100 - 20\sqrt{x^2 + (y - 4)^2} + x^2 + (y - 4)^2$

[by squaring]

$\Rightarrow (y + 4)^2 - (y - 4)^2 - 100 = -20\sqrt{x^2 + (y - 4)^2}$

$\Rightarrow 4.4.y - 100 = -20\sqrt{x^2 + (y - 4)^2}$

$\Rightarrow 4y - 25 = -5\sqrt{x^2 + (y - 4)^2}$

$\Rightarrow 16y^2 - 200y + 625 = 25(x^2 + y^2 - 8y + 16)$ [by squaring]

$\Rightarrow 25x^2 + 25y^2 - 16y^2 = 625 - 400$

$\Rightarrow 25x^2 + 9y^2 = 225$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

which is the equation of an ellipse.

(xi) Given that $|z - 3| - |z + 3| = 4$

Let $z = x + iy$

$\therefore |x + iy - 3| - |x + iy + 3| = 4$

$\Rightarrow |(x - 3) + iy| - |(x + 3) + iy| = 4$

$\Rightarrow \sqrt{(x - 3)^2 + y^2} - \sqrt{(x + 3)^2 + y^2} = 4$

$\Rightarrow \sqrt{(x - 3)^2 + y^2} = 4 + \sqrt{(x + 3)^2 + y^2}$

$\Rightarrow (x - 3)^2 + y^2 = 16 + 8\sqrt{(x + 3)^2 + y^2} + (x + 3)^2 + y^2$

[by squaring]

$\Rightarrow -8\sqrt{(x + 3)^2 + y^2} = (x + 3)^2 - (x - 3)^2 + 16$

$\Rightarrow -8\sqrt{(x + 3)^2 + y^2} = 4.3.x + 16$

$\Rightarrow -2\sqrt{(x + 3)^2 + y^2} = (3x + 4)$

$\Rightarrow 4\{(x+3)^2 + y^2\} = (3x+4)^2$ [by squaring]

$\Rightarrow 4x^2 + 24x + 36 + 4y^2 = 9x^2 + 24x + 16$

$\Rightarrow 9x^2 - 4x^2 - 4y^2 = 20$

$\Rightarrow 5x^2 - 4y^2 = 20$

$\Rightarrow \frac{x^2}{4} - \frac{y^2}{5} = 1$

which is the equation of a hyperbola. **(Proved)**

4.(i) Find an equation for a circle of radius 4 with centre at (–2, 1).

(ii) Find an equation for a circle of radius 2 with centre at (– 3, 4).

(iii) Find an equation for a circle of radius 3 with cantre at (2, –1).

(iv) Find an equation for a circle of radius 3 with centre at (3, –4).

(v) Find an equation for a circle of radius 3 with centre at (2, 3).

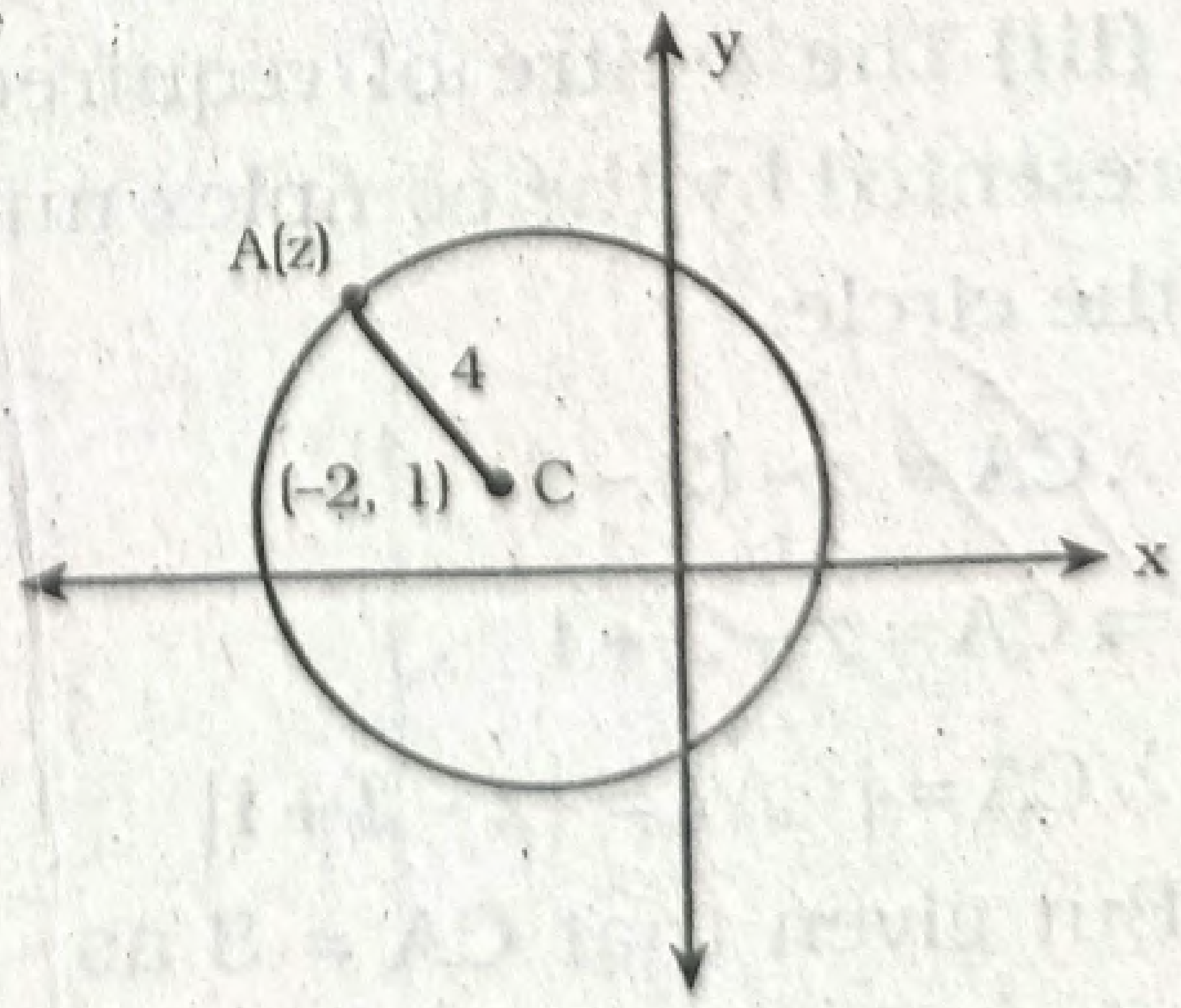
(vi) Find an equation for a circle of radius $\sqrt{13}$ with centre at (–3, –2).

Solution : **(i)** The centre of required circle C(–2, 1) which can be represented by the complex number –2 + i. Let A(z) be any point on the circle.

$\therefore \overrightarrow{CA} = z - (-2 + i)$

$\overrightarrow{CA} = z + 2 - i$

$\Rightarrow CA = |\overrightarrow{CA}| = |z + 2 - i|$



But given that CA = 4, as shown in figure–

$\therefore |z + 2 - i| = 4$, which is the required equation of circle in complex form.

Let $z = x + iy$

$\therefore |x + iy + 2 - i| = 4$

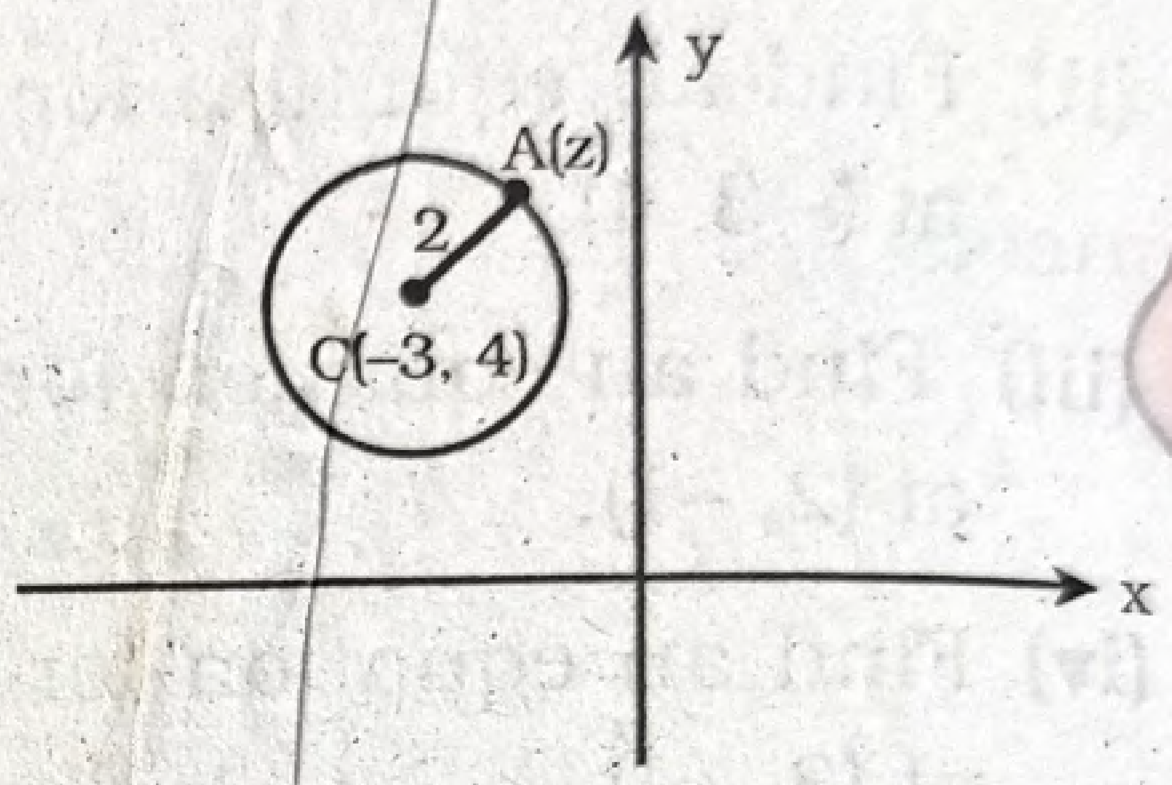
$\Rightarrow |(x + 2) + i(y - 1)| = 4$

$\Rightarrow \sqrt{(x + 2)^2 + (y - 1)^2} = 4$

$\Rightarrow (x + 2)^2 + (y - 1)^2 = 16$

which is the required equation of a circle in rectangular form.

(ii) The centre of required circle C(−3, 4) which can be represented by the complex number $-3 + 4i$. Let A(z) be any point on the circle.



$\therefore \overrightarrow{CA} = z - (-3 + 4i)$

$\Rightarrow \overrightarrow{CA} = z + 3 - 4i$

$\Rightarrow CA = |\overrightarrow{CA}| = |z + 3 - 4i|$

But given that CA = 2 as shown in figure–

$\therefore |z + 3 - 4i| = 2$

Which is the required equation of circle in complex form.

Let $z = x + iy$

$\therefore |x + iy + 3 - 4i| = 2$

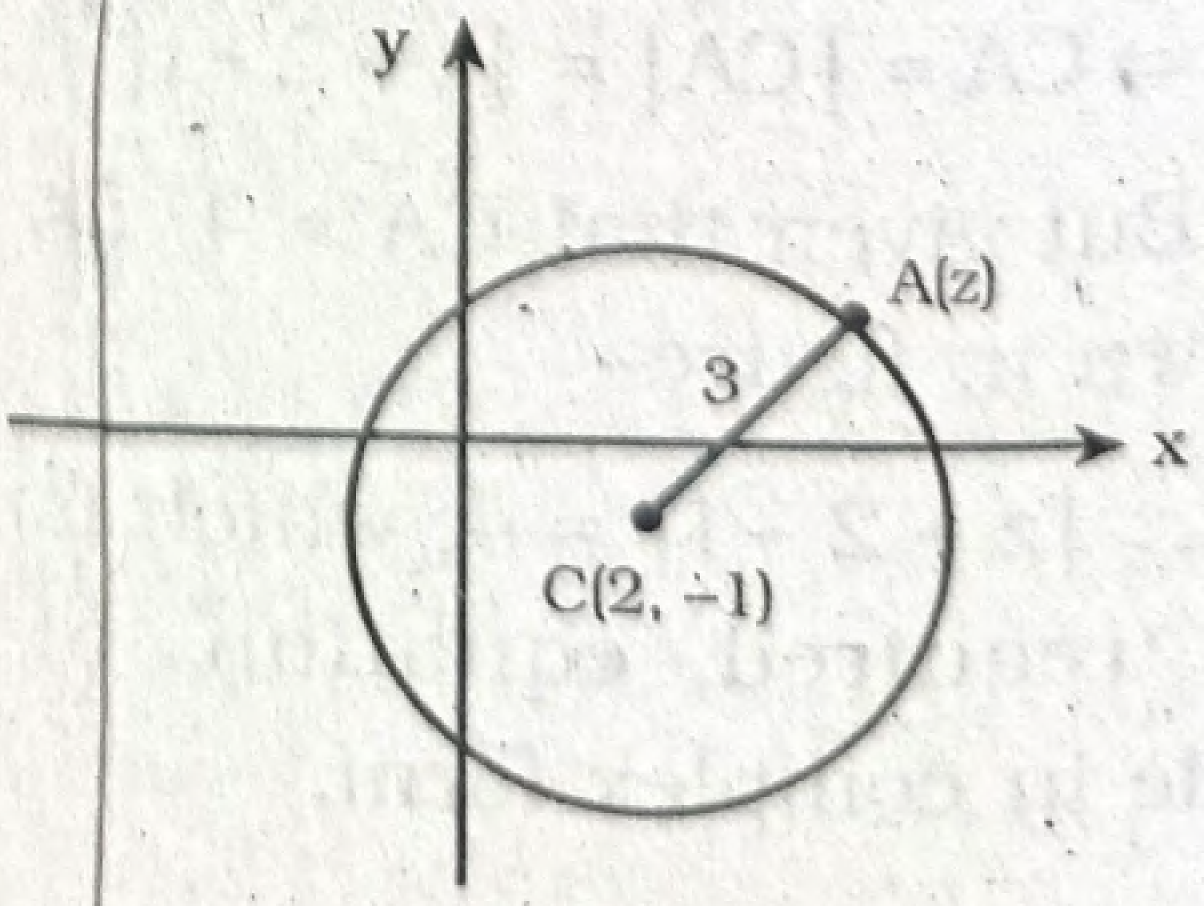
$\Rightarrow |(x + 3) + i(y - 4)| = 2$

$\Rightarrow \sqrt{(x + 3)^2 + (y - 4)^2} = 2$

$\Rightarrow (x + 3)^2 + (y - 4)^2 = 2^2 = 4$

which is the required equation of circle in rectangular form.

(iii) The centre of required circle C(2, −1) which can be represented by the complex number $2 - i$. Let A(z) be any point on the circle.



$\therefore \overrightarrow{CA} = z - (2 - i)$

$\Rightarrow \overrightarrow{CA} = z - 2 + i$

$\therefore CA = |\overrightarrow{CA}| = |z - 2 + i|$

But given that CA = 3 as shown in figure–

$\therefore |z - 2 + i| = 3$

which is the required equation of circle in complex form.

Let $z = x + iy$

$\therefore |x + iy - 2 - 3i| = 3$

$\Rightarrow |(x - 2) + i(y - 3)| = 3$

$\Rightarrow \sqrt{(x - 2)^2 + (y - 3)^2} = 3$

$\Rightarrow (x - 2)^2 + (y - 3)^2 = 9$

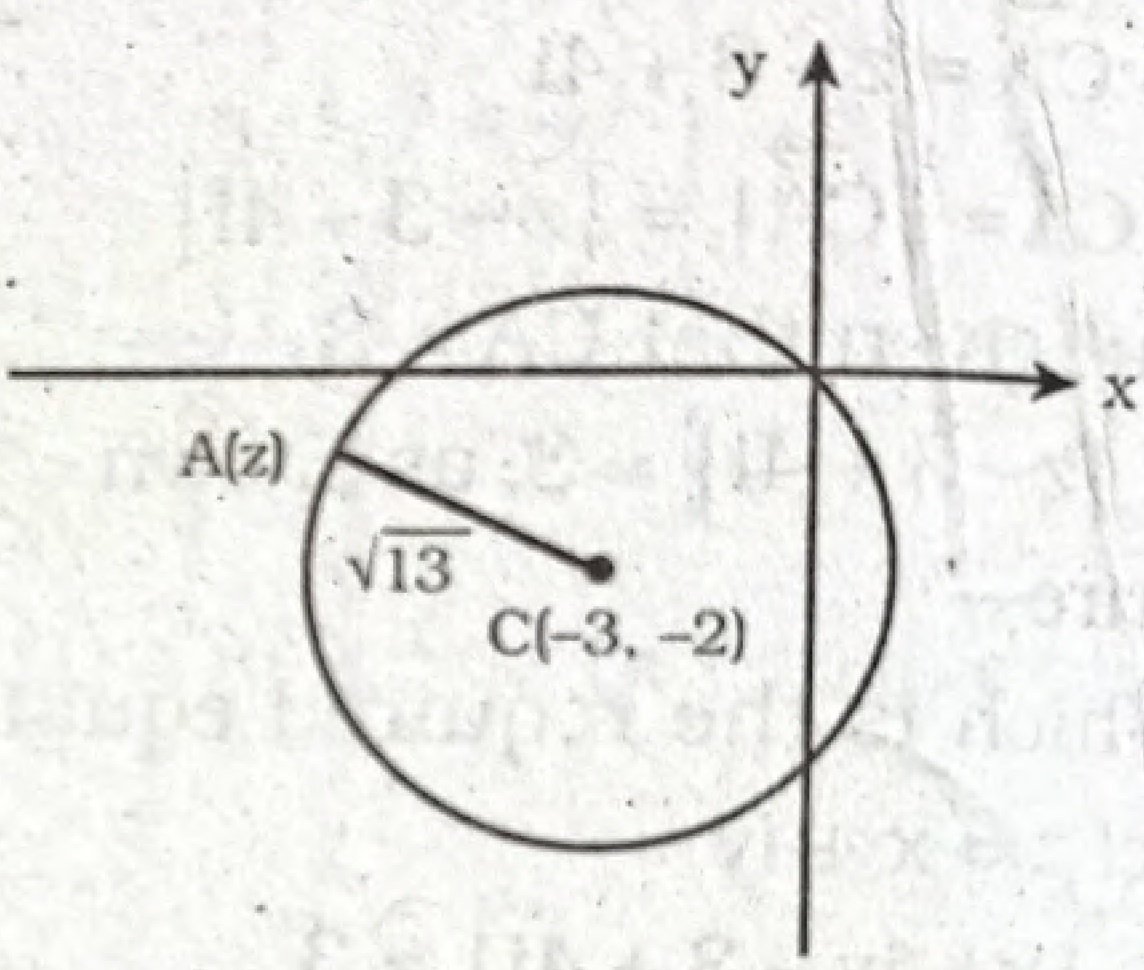
which is the required equation of circle in rectangular form.

(vi) The centre of required circle C(–3, –2) which can be represented by the complex number – 3 – 2i. Let A(z) be any point on the circle.

$\therefore \overrightarrow{CA} = z - (-3 - 2i)$

$\Rightarrow \overrightarrow{CA} = z + 3 + 2i$

$\Rightarrow CA = |\overrightarrow{CA}| = |z + 3 + 2i|$



But given that $CA = \sqrt{13}$ as shown in figure.

$\therefore |z + 3 + 2i| = \sqrt{13}$

$\Rightarrow$ which is the required equation of the circle in the complex form. Let $z = x + iy$.

$\therefore |x + iy + 3 + 2i| = \sqrt{13}$

$\Rightarrow |(x + 3) + i(y + 2)| = \sqrt{13}$

$\Rightarrow \sqrt{(x + 3)^2 + (y + 2)^2} = \sqrt{13}$

$\Rightarrow (x + 3)^2 + (y + 2)^2 = 13$

which is the required equation of the circle in the rectangular form.

5. *Sketch the region in z-plane represented by the set of points :*

(i) $Re(\bar{z} - 1) = 2$

(ii) $|z - 1| + |z + 1| = 4$

(iii) $z^2 + \bar{z}^2 = 2$

(iv) $|z + i| - |z - i = 3$

(v) $\left|\frac{z - 3}{z + 3}\right| = 2$ **[D. U. M. Sc (CP) '80]**

(vi) $\text{Im}(z) = 1$ (vii) $|z - i| = 2$

(viii) $\text{Im}(z^2) = 4$ (ix) $|z - 4| = |z|$

(x) $\text{Re}\left(\frac{1}{z}\right) = 1$ (xi) $|z - 2 + i| = 1$

(xii) $|z + 3i| = 4$ (xiii) $\text{Re}(z^2) = 1$

(xiv) $|z - i| = |z + i|$ (xv) $\left|\frac{2z-3}{2z+3}\right| = 1$

[N. U. H. 2003, 2006]

(xvi) $\text{Re}(z) + \text{Im}(z) = 0.$

Solution : (i) The given expression is

$\text{Re}(\bar{z} - 1) = 2$

Let $z = x + iy$

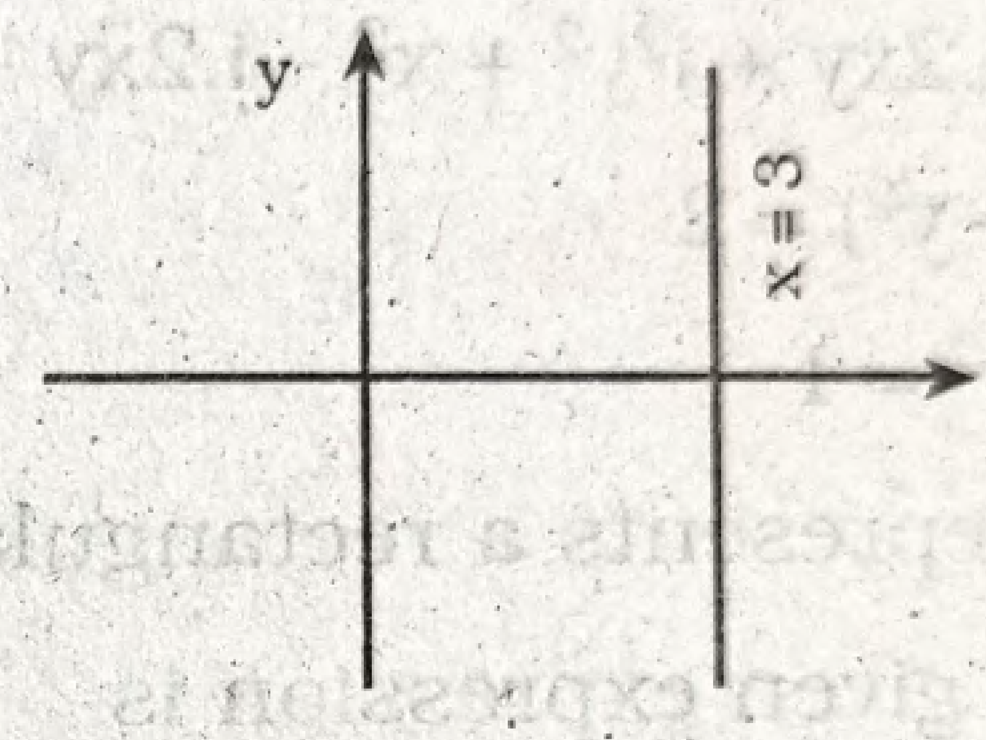
$\therefore \bar{z} = x - iy$

$\therefore \text{Re}(x - iy - 1) = 2$

$\Rightarrow \text{Re}[(x - 1) - iy] = 2$

$\Rightarrow x - 1 = 2$

$\Rightarrow x = 3$



which represents a straight line parallel to y-axis.

(ii) The given expression is $|z - 1| + |z + 1| = 4$

Let $z = x + iy$

$\therefore |x + iy - 1| + |x + iy + 1| = 4$

$\Rightarrow |(x - 1) + iy| + |(x + 1) + iy| = 4$

$\Rightarrow \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$

$\Rightarrow (x + 1)^2 + y^2 = 16 - 8\sqrt{(x-1)^2 + y^2} + (x - 1)^2 + y^2$

$\Rightarrow (x + 1)^2 - (x - 1)^2 = 16 - 8\sqrt{(x-1)^2 + y^2}$

$\Rightarrow 4x = 4\,[4 - 2\sqrt{(x-1)^2 + y^2}]$

$\Rightarrow x = 4 - 2\sqrt{(x-1)^2 + y^2}$

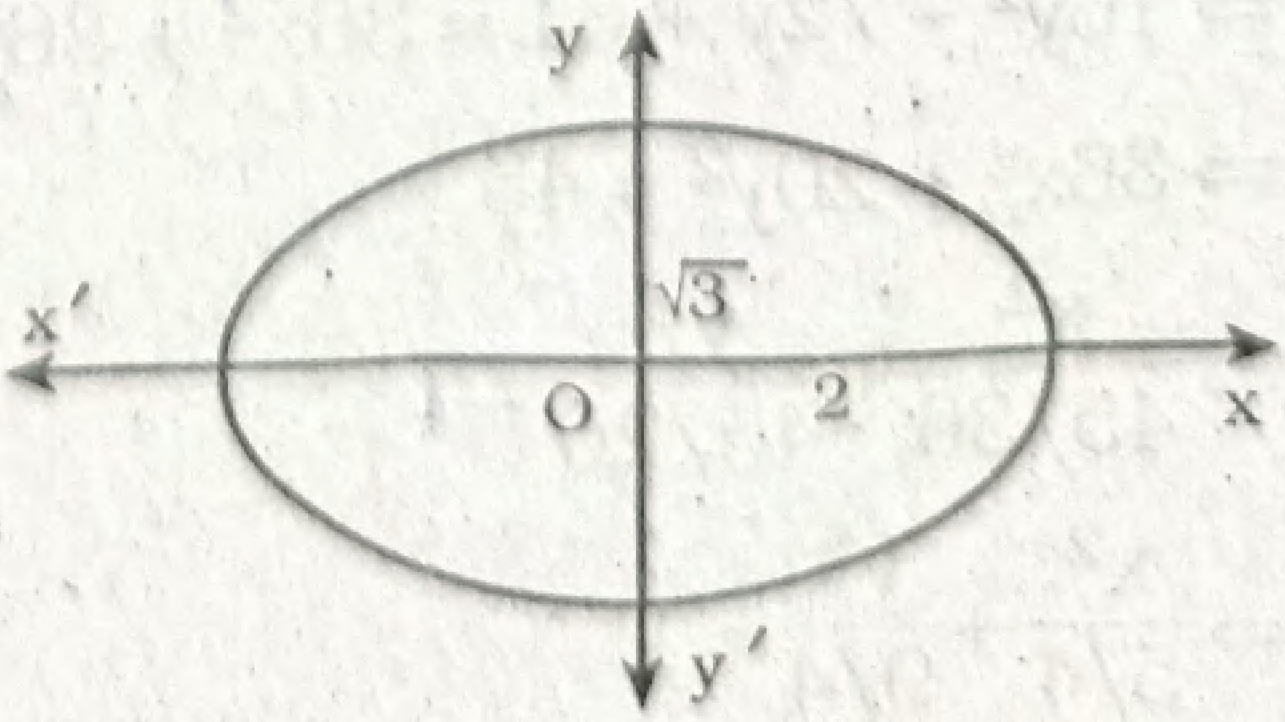
$\Rightarrow (x - 4)^2 = 4\{(x - 1)^2 + y^2\}$

$\Rightarrow x^2 - 8x + 16$

$= 4x^2 - 8x + 4 + 4y^2$

$\Rightarrow 4x^2 + 4y^2 - x^2 = 12$

$\Rightarrow 3x^2 + 4y^2 = 12$



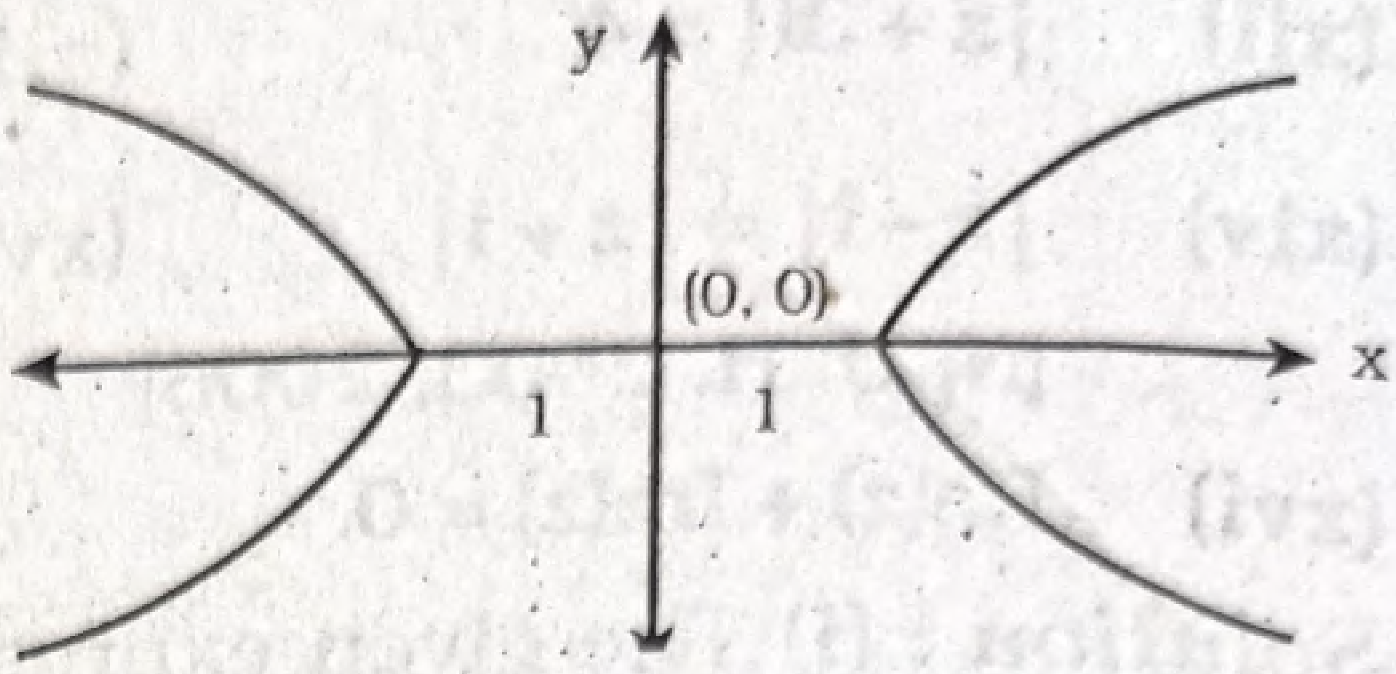
$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1$$

which represents an ellipse.

(iii) The given expression is $z^2 + \bar{z}^2 = 2 \cdots (1)$

Let $z = x + iy$

$\Rightarrow \bar{z} = x - iy$



Then equation (1) becomes–

$$\Rightarrow (x + iy)^2 + (x - iy)^2 = 2$$

$$\Rightarrow x^2 + i.2xy + i^2y^2 + x^2 - i.2xy + i^2y^2 = 2$$

$$\Rightarrow 2(x^2 - y^2) = 2$$

$$\Rightarrow x^2 - y^2 = 1$$

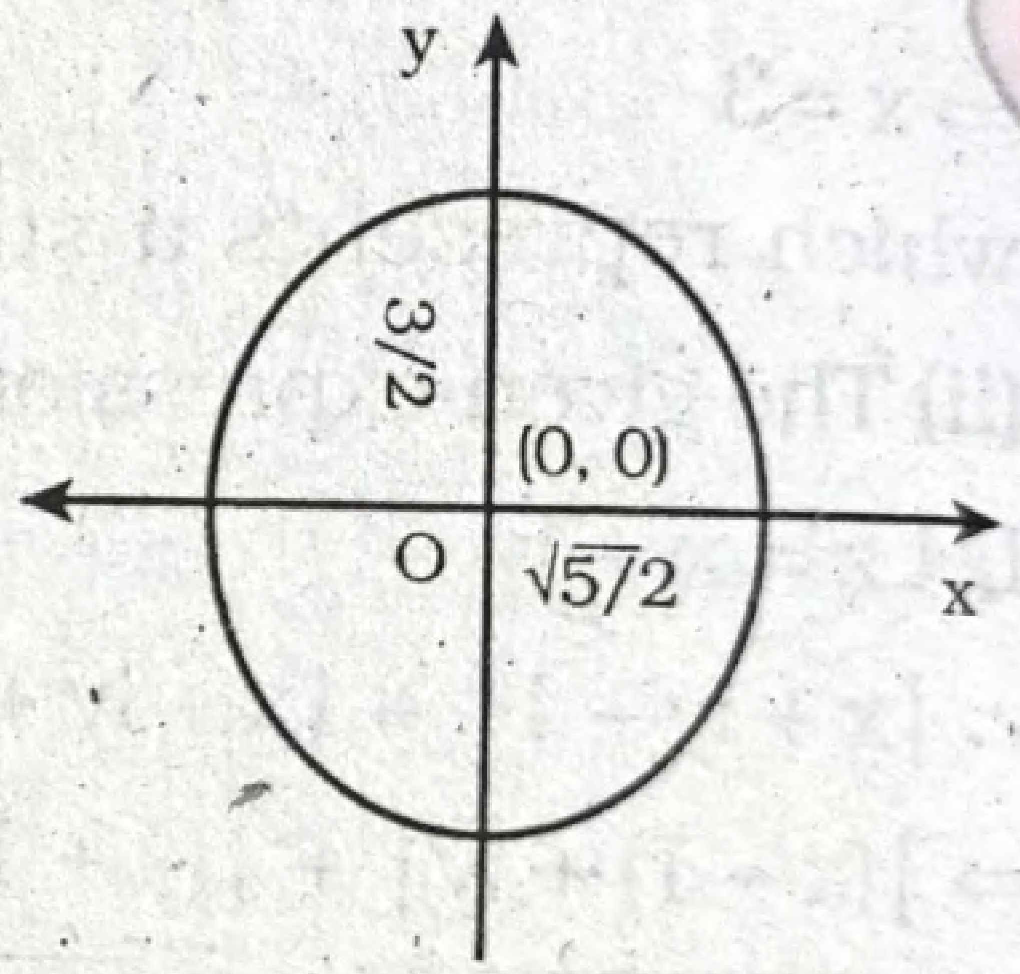
which represents a rectangular hyperbola.

(iv) The given expression is

$$|z + i| - |z - i| = 3$$

$$\Rightarrow |x + iy + i| - |x + iy - i| = 3,$$

where $z = x + iy$



$$\Rightarrow \sqrt{x^2 + (y + 1)^2} = 3 + \sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow x^2 + (y + 1)^2 = 9 + 6\sqrt{x^2 + (y - 1)^2} + x^2 + (y - 1)^2$$

$$\Rightarrow (y + 1)^2 - (y - 1)^2 = 9 + 6\sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow 4y - 9 = 6\sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow (4y - 9)^2 = 36[x^2 + (y - 1)^2]$$

$$\Rightarrow 16y^2 - 72y + 81 = 36x^2 + 36y^2 - 72y + 36$$

$$\Rightarrow 36x^2 + 20y^2 = 45$$

$$\Rightarrow \frac{x^2}{45/36} + \frac{y^2}{45/20} = 1$$

$$\Rightarrow \frac{x^2}{5/4} + \frac{y^2}{9/4} = 1$$

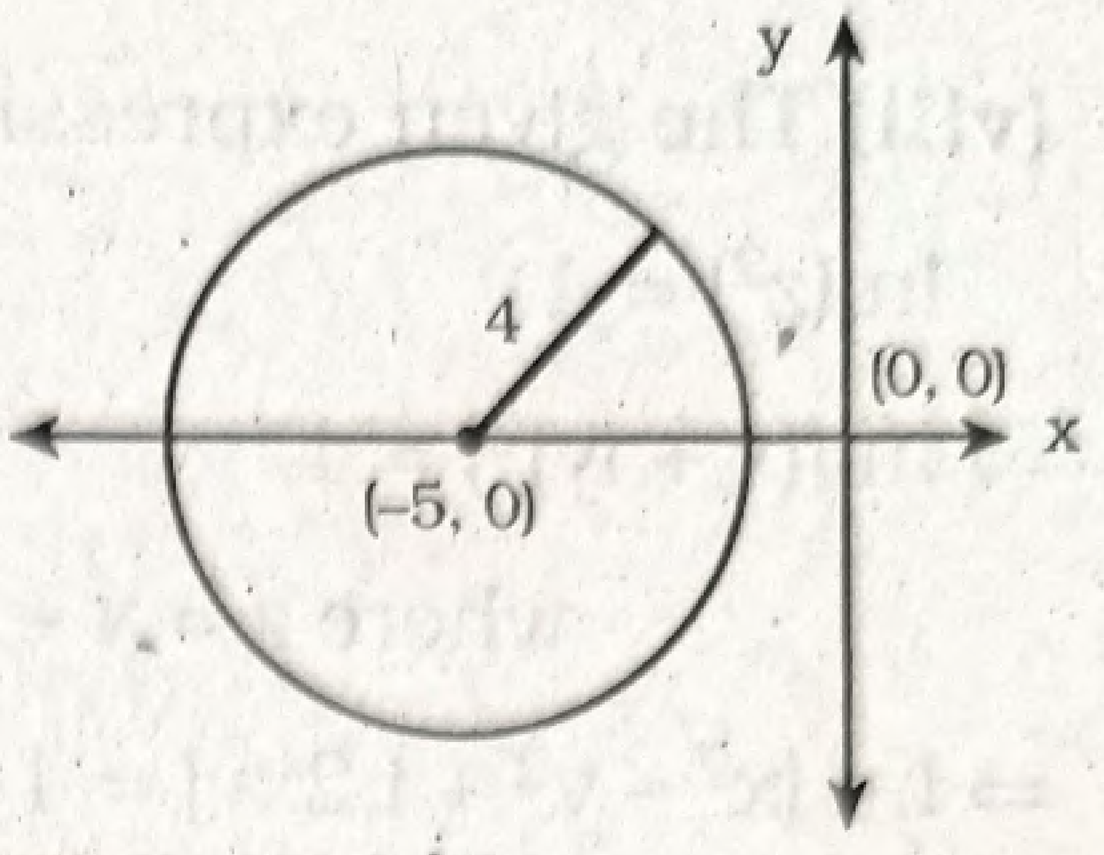
which represents an ellipse.

(v) The given expression is

$$\left|\frac{z-3}{z+3}\right| = 2$$

$$\Rightarrow \frac{|z-3|}{|z+3|} = 2 \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$\Rightarrow |z-3| = 2|z+3|$$



$$\Rightarrow |x + iy - 3| = 2|x + iy + 3|, \text{ where } z = x + iy$$

$$\Rightarrow |(x-3) + iy| = 2|(x+3) + iy|$$

$$\Rightarrow \sqrt{(x-3)^2 + y^2} = 2\sqrt{(x+3)^2 + y^2}$$

$$\Rightarrow (x-3)^2 + y^2 = 4\,[(x+3)^2 + y^2]$$

$$\Rightarrow 4(x+3)^2 - (x-3)^2 = y^2 - 4y^2$$

$$\Rightarrow 4(x^2 + 6x + 9) - (x^2 - 6x + 9) = -3y^2$$

$$\Rightarrow 3x^2 + 30x + 27 = -3y^2$$

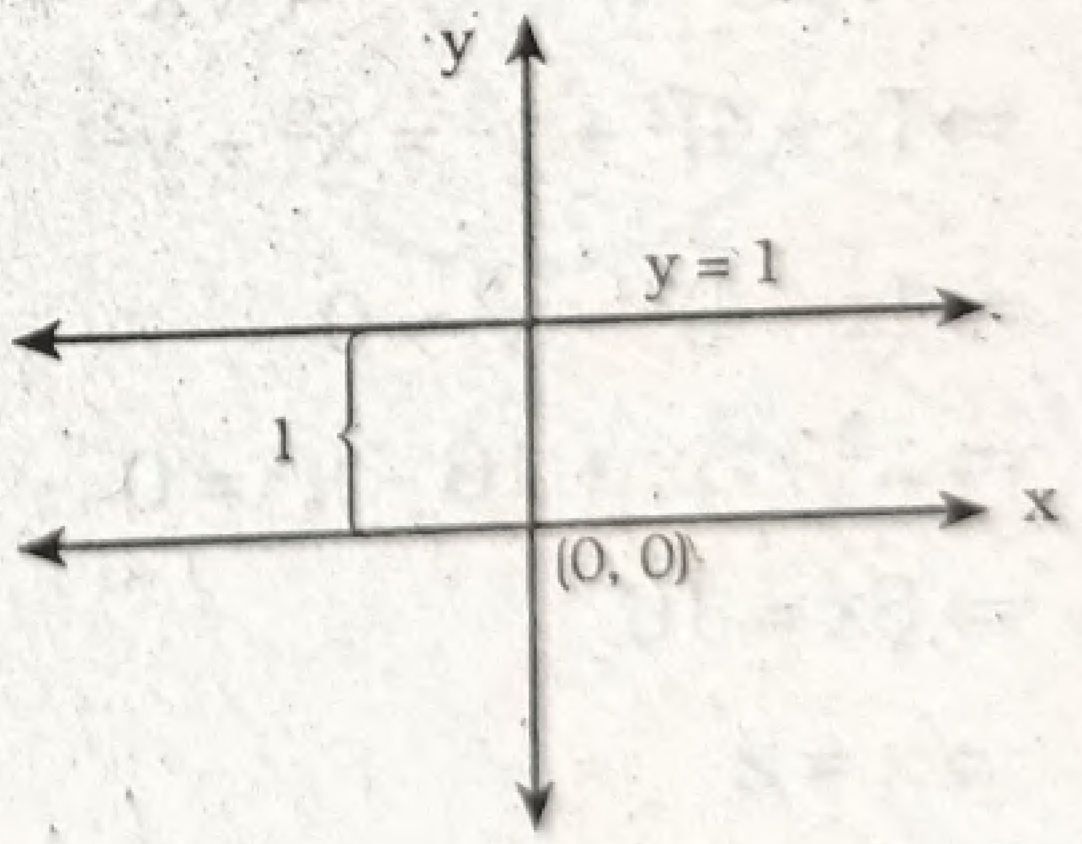
$$\Rightarrow x^2 + 10x + 9 + y^2 = 0$$

$$\Rightarrow x^2 + y^2 + 10x + 9 = 0$$

which represents a circle whose centre is at (–5, 0) and radius is 4.

(vi) The given expression is

$\text{Im}(z) = 1$



$\Rightarrow \text{Im}(x + iy) = 1$ where $z = x + iy$

$\Rightarrow y = 1$

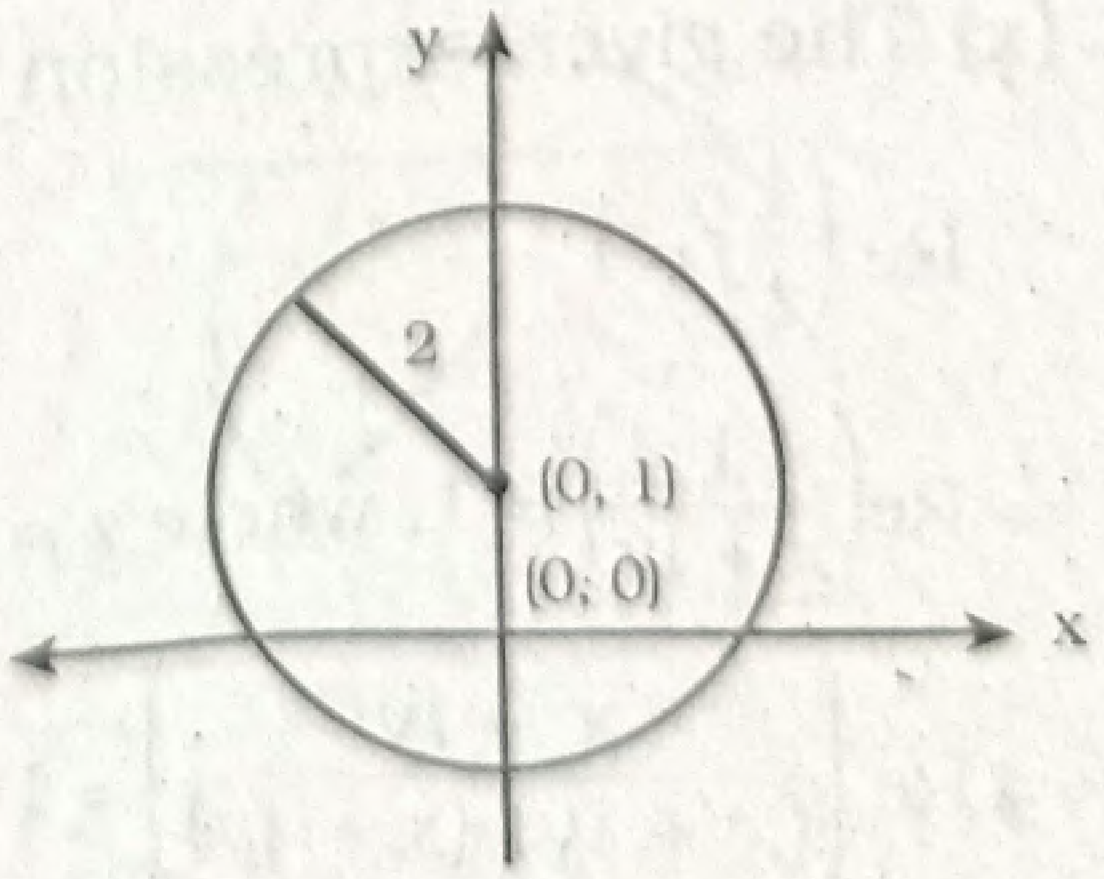
which represents the straight line parallel to x-axis.

(vii) The given expression is

$$|z - i| = 2$$

$$\Rightarrow |x + iy - i| = 2,$$

where $z = x + iy$



$$\Rightarrow |x + i(y-1)| = 2$$

$$\Rightarrow \sqrt{x^2 + (y-1)^2} = 2$$

$$\Rightarrow x^2 + (y-1)^2 = 2^2$$

$$\Rightarrow x^2 + (y-1)^2 = 4$$

which represents a circle whose centre is at (0, 1) and radius is 2.

(viii) The given expression is

$$\text{Im}(z^2) = 4$$

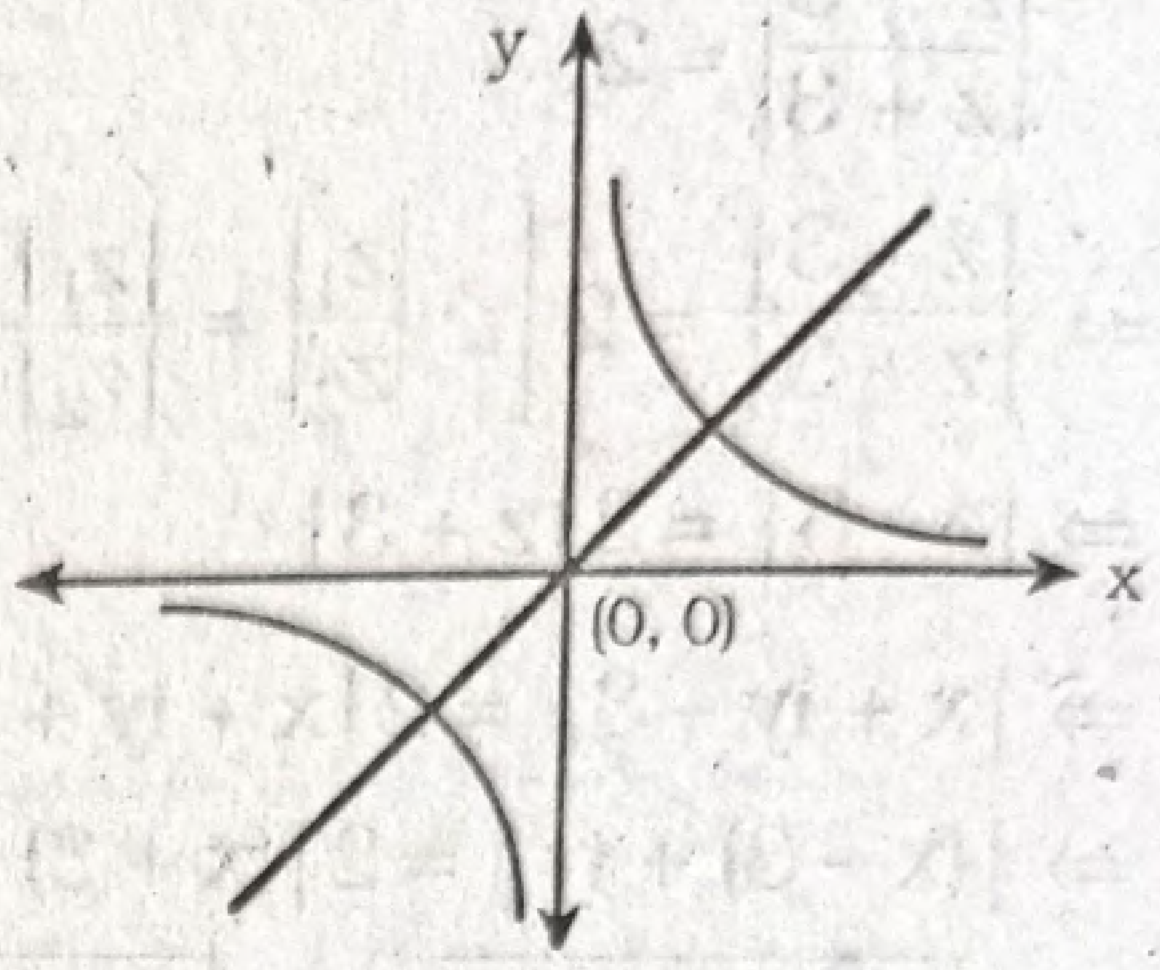
$$\Rightarrow \text{Im}[(x + iy)^2] = 4,$$

where $z = x + iy$

$$\Rightarrow \text{Im}\{x^2 - y^2 + i.2xy\} = 4$$

$$\Rightarrow 2xy = 4$$

$$\Rightarrow xy = 2$$



which represents a hyperbola lies in the 1st and 3rd quadrants. The straight line $y = x$ is its principal axis.

(ix) The given expression is

$$|z - 4| = |z|$$

$$\Rightarrow |x + iy - 4| = |x + iy|,$$

where $z = x + iy$

$$\Rightarrow |(x - 4) + iy| = |x + iy|$$

$$\Rightarrow \sqrt{(x - 4)^2 + y^2} = \sqrt{x^2 + y^2}$$

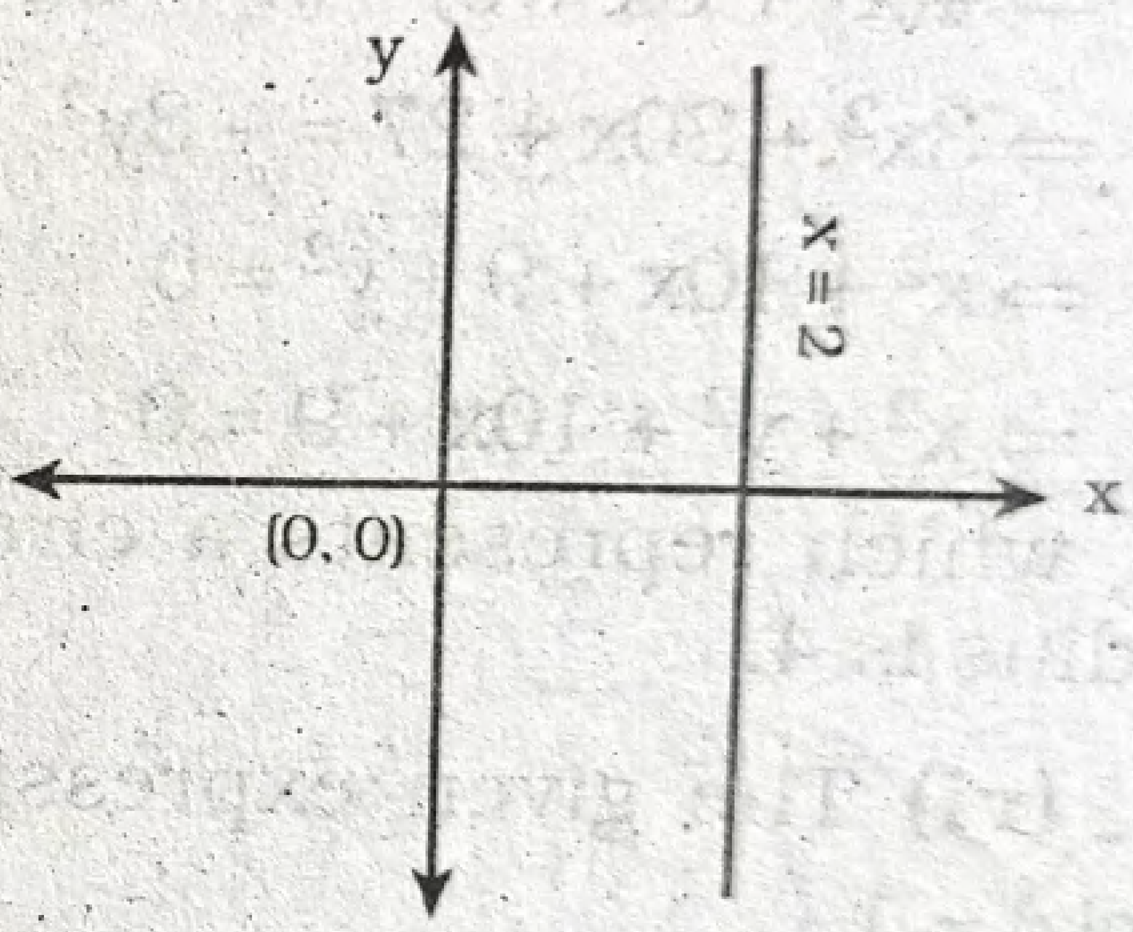
$$\Rightarrow (x - 4)^2 + y^2 = x^2 + y^2$$

$$\Rightarrow (x - 4)^2 - x^2 = 0$$

$$\Rightarrow x^2 - 8x + 16 - x^2 = 0$$

$$\Rightarrow 8x = 16$$

$$\Rightarrow x = 2$$



which represents a straight line parallel to y-axis.

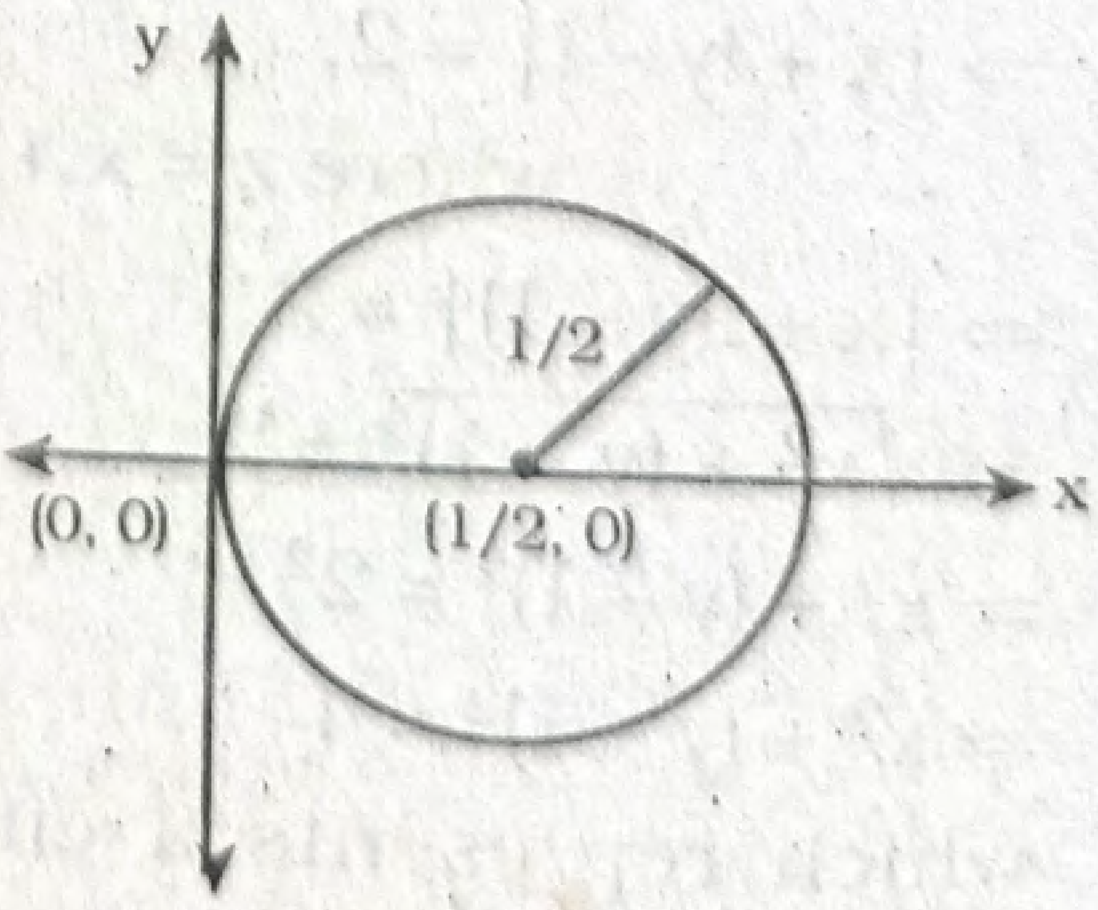
(x) The given expression is

$$\text{Re}\left(\frac{1}{z}\right) = 1$$

$$\Rightarrow \text{Re}\left(\frac{1}{x + iy}\right) = 1, \text{ where } z = x + iy$$

$$\Rightarrow \text{Re}\left[\frac{x - iy}{(x + iy)(x - iy)}\right] = 1$$

$$\Rightarrow \text{Re}\left[\frac{x - iy}{x^2 + y^2}\right] = 1$$



$$\Rightarrow \text{Re}\left[\frac{x}{x^2+y^2} - i.\frac{y}{x^2+y^2}\right] = 1$$

$$\Rightarrow \frac{x}{x^2+y^2} = 1$$

$$\Rightarrow x^2 + y^2 = x$$

$$\Rightarrow x^2 - x + y^2 = 0$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

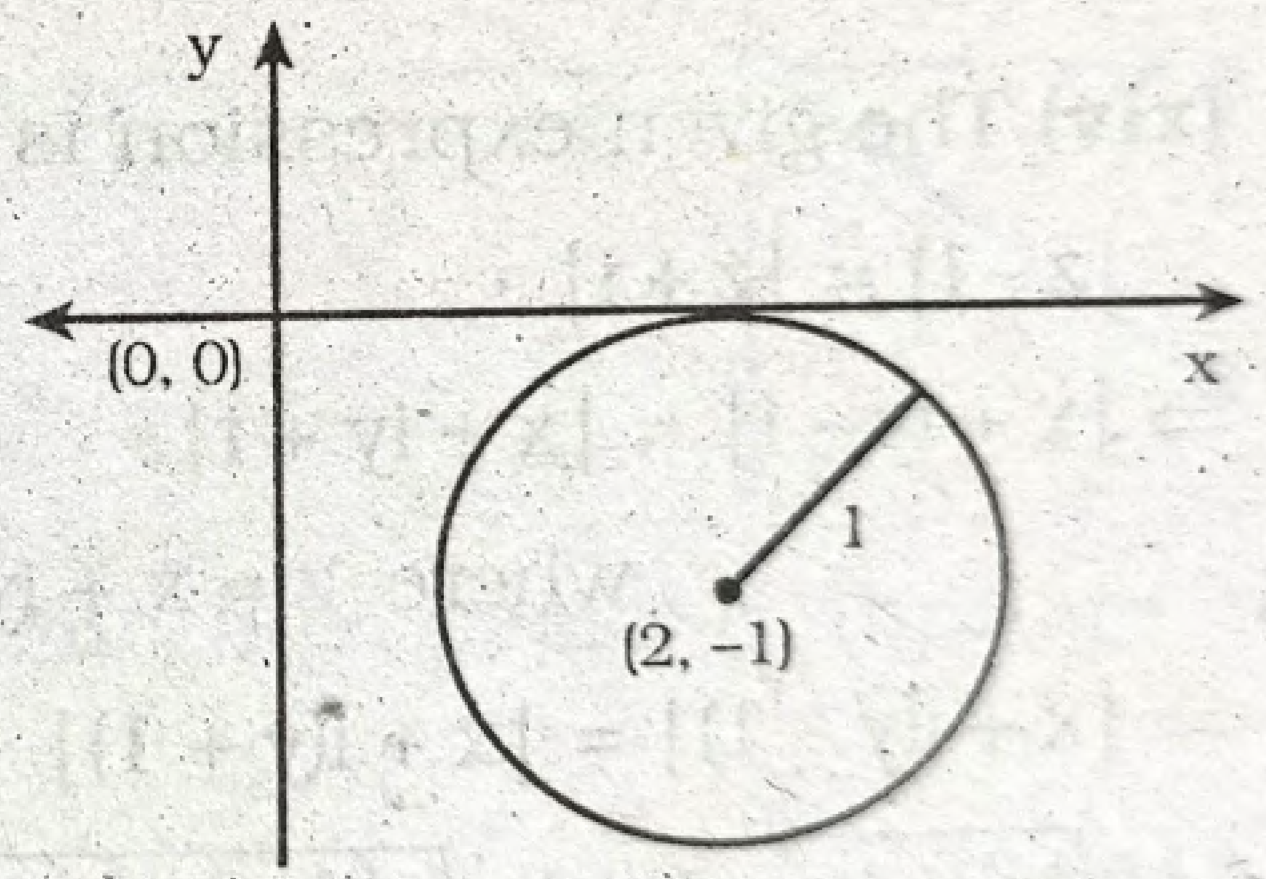
which represents a circle whose centre is at $\left(\frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$.

(xi) The given expression is

$$|z - 2 + i| = 1$$

$$\Rightarrow |x + iy - 2 + i| = 1,$$

where $z = x + iy$



$$\Rightarrow |(x-2) + i(y+1)| = 1$$

$$\Rightarrow \sqrt{(x-2)^2 + (y+1)^2} = 1$$

$$\Rightarrow (x-2)^2 + (y+1)^2 = 1$$

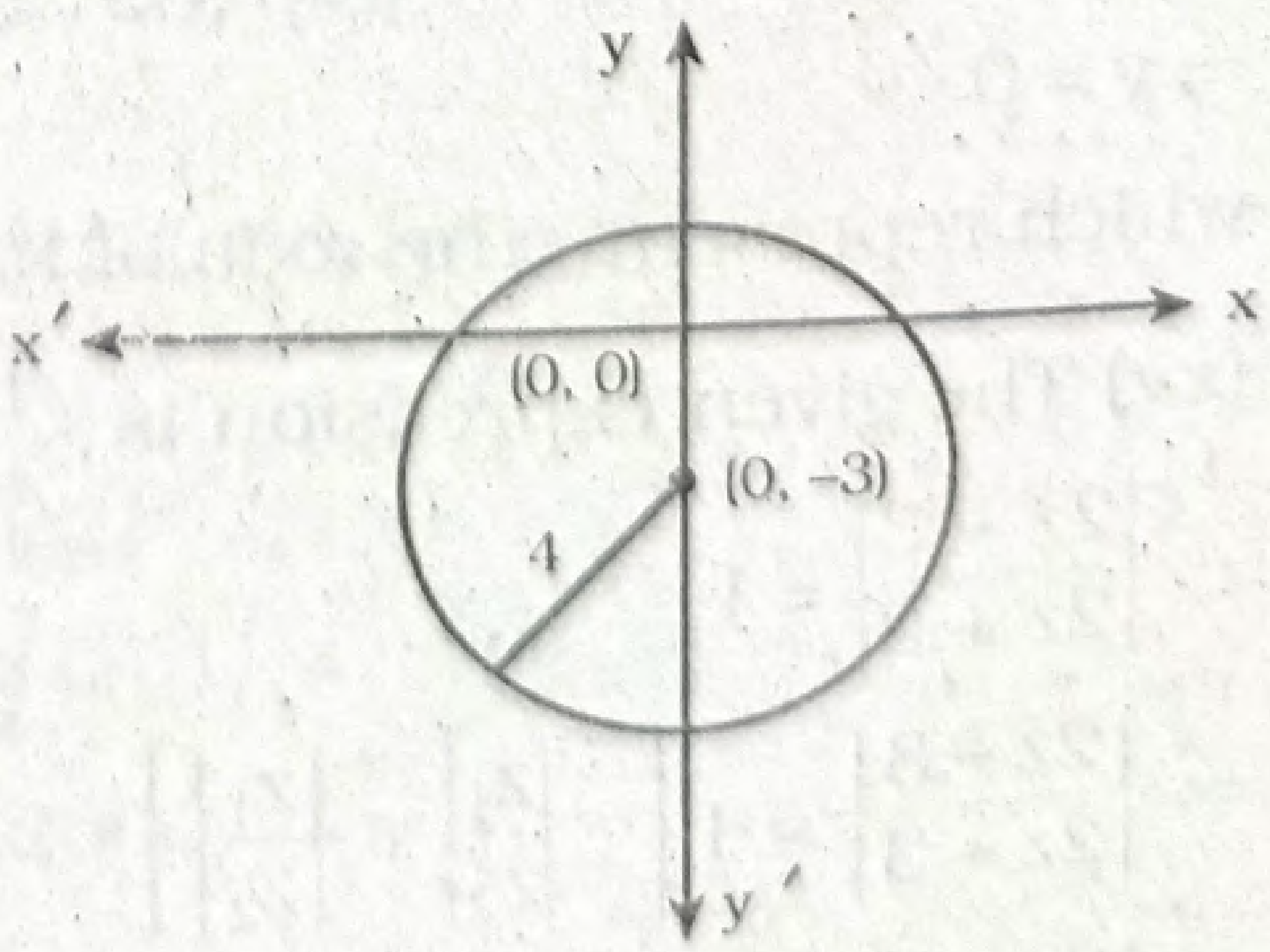
which represents a circle whose centre is at (2, −1) and radius is 1.

(xiv) The given expression is

$$|z + 3i| = 4$$

$$\Rightarrow |x + iy + 3i| = 4,$$

where $z = x + iy$



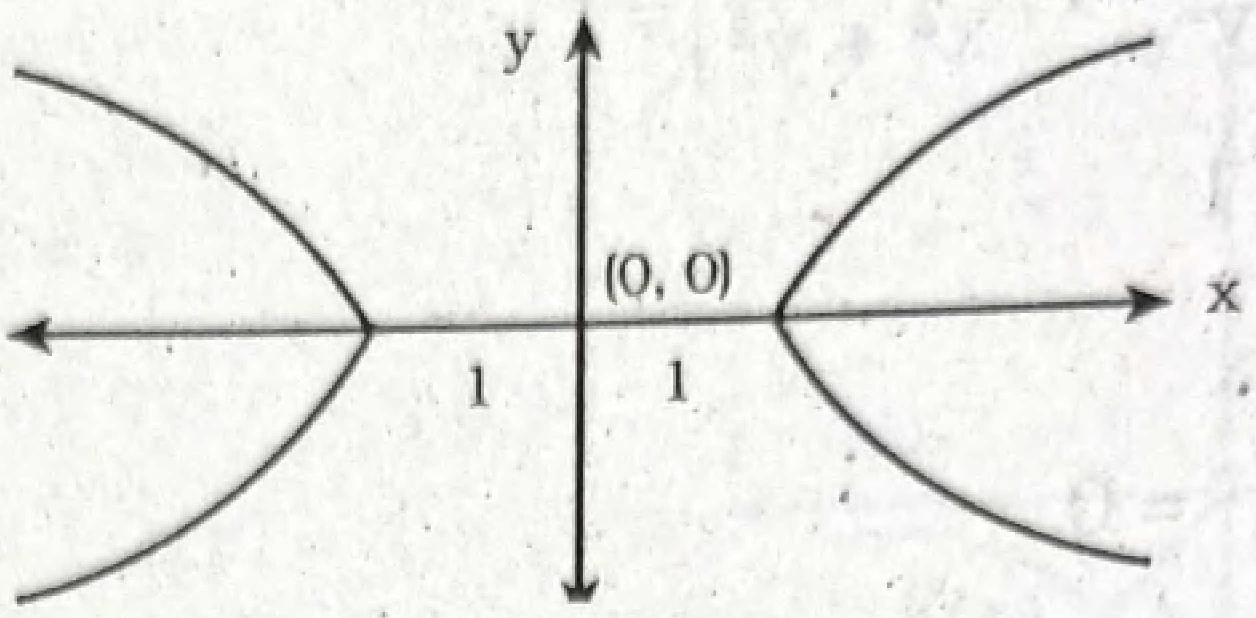
$$\Rightarrow |x + i(y+3)| = 4$$

$$\Rightarrow \sqrt{x^2 + (y+3)^2} = 4$$

$$\Rightarrow x^2 + (y+3)^2 = 4^2$$

which represents a circle whose centre is at (0, −3) and radius is 4.

(xiii) The given expression is



$\text{Re}(z^2) = 1$

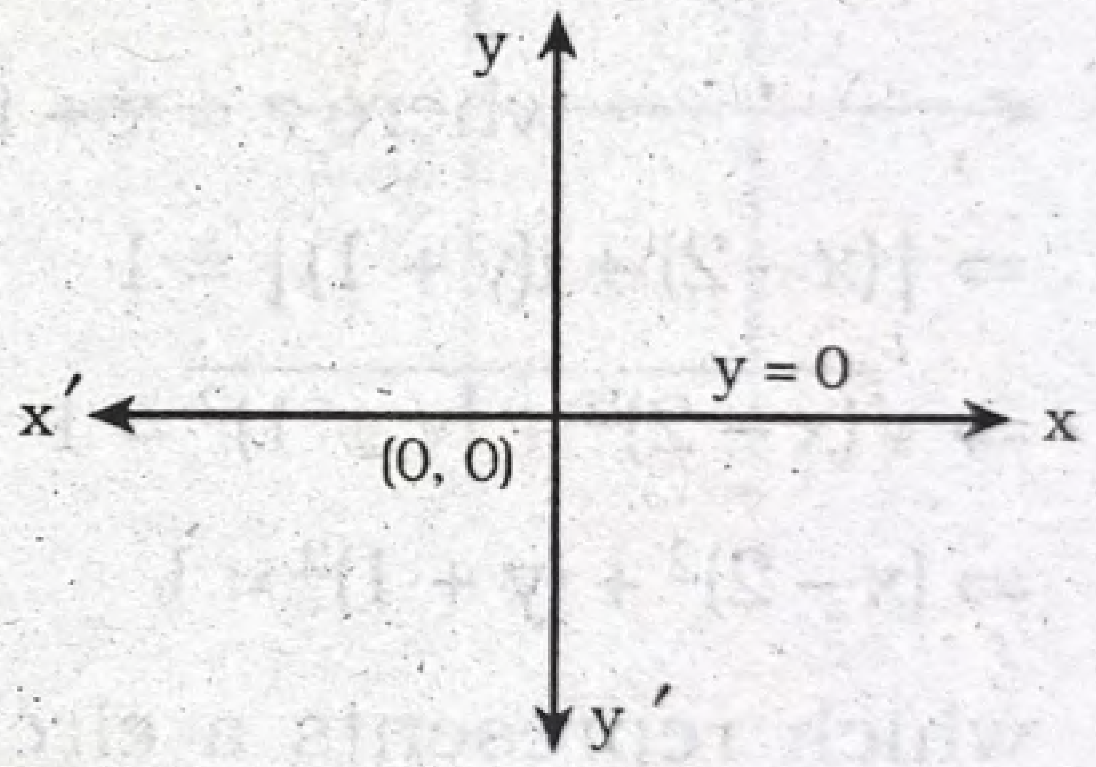
$\Rightarrow \text{Re}[(x + iy)^2] = 1$, where $z = x + iy$

$\Rightarrow \text{Re}[x^2 - y^2 + i.2xy] = 1$

$\Rightarrow x^2 - y^2 = 1$

which represents a rectangular hyperbola, whose principal axis is the axis of x.

(xiv) The given expression is



$|z - i| = |z + i|$

$\Rightarrow |x + iy - i| = |x + iy + i|$,

where $z = x + iy$

$\Rightarrow |x + i(y - 1)| = |x + i(y + 1)|$

$\Rightarrow \sqrt{x^2 + (y - 1)^2} = \sqrt{x^2 + (y + 1)^2}$

$\Rightarrow x^2 + (y - 1)^2 = x^2 + (y + 1)^2$

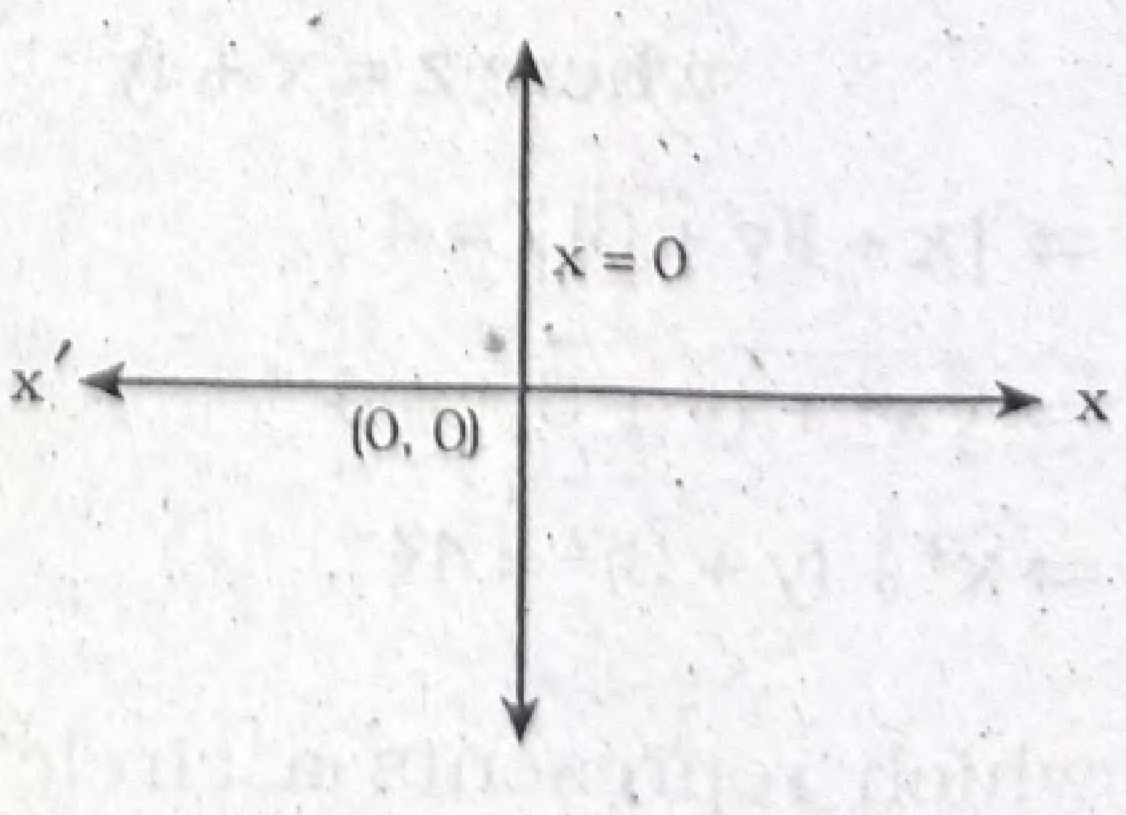
$\Rightarrow (y + 1)^2 - (y - 1)^2 = 0$

$\Rightarrow 4y = 0$

$\Rightarrow y = 0$

which represents the axis of x.

(xv) The given expression is



$\left|\frac{2z - 3}{2z + 3}\right| = 1$

$\Rightarrow \frac{|2z - 3|}{|2z + 3|} = 1 \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$

$\Rightarrow |2z - 3| = |2z + 3|$

$\Rightarrow |2(x + iy) - 3| = |2(x + iy) + 3|$, where $z = x + iy$

$\Rightarrow |(2x-3)+i.2y| = |(2x+3)+i.2y|$

$\Rightarrow \sqrt{(2x-3)^2+4y^2} = \sqrt{(2x+3)^2+4y^2}$

$\Rightarrow (2x-3)^2+4y^2 = (2x+3)^2+4y^2$

$\Rightarrow (2x+3)^2-(2x-3)^2 = 0$

$\Rightarrow 4.2x.3 = 0$

$\Rightarrow 24x = 0$

$\Rightarrow x = 0$

which represents the axis of y.

(xvi) The given expression is

$Re(z) + Im(z) = 0 \cdots (1)$

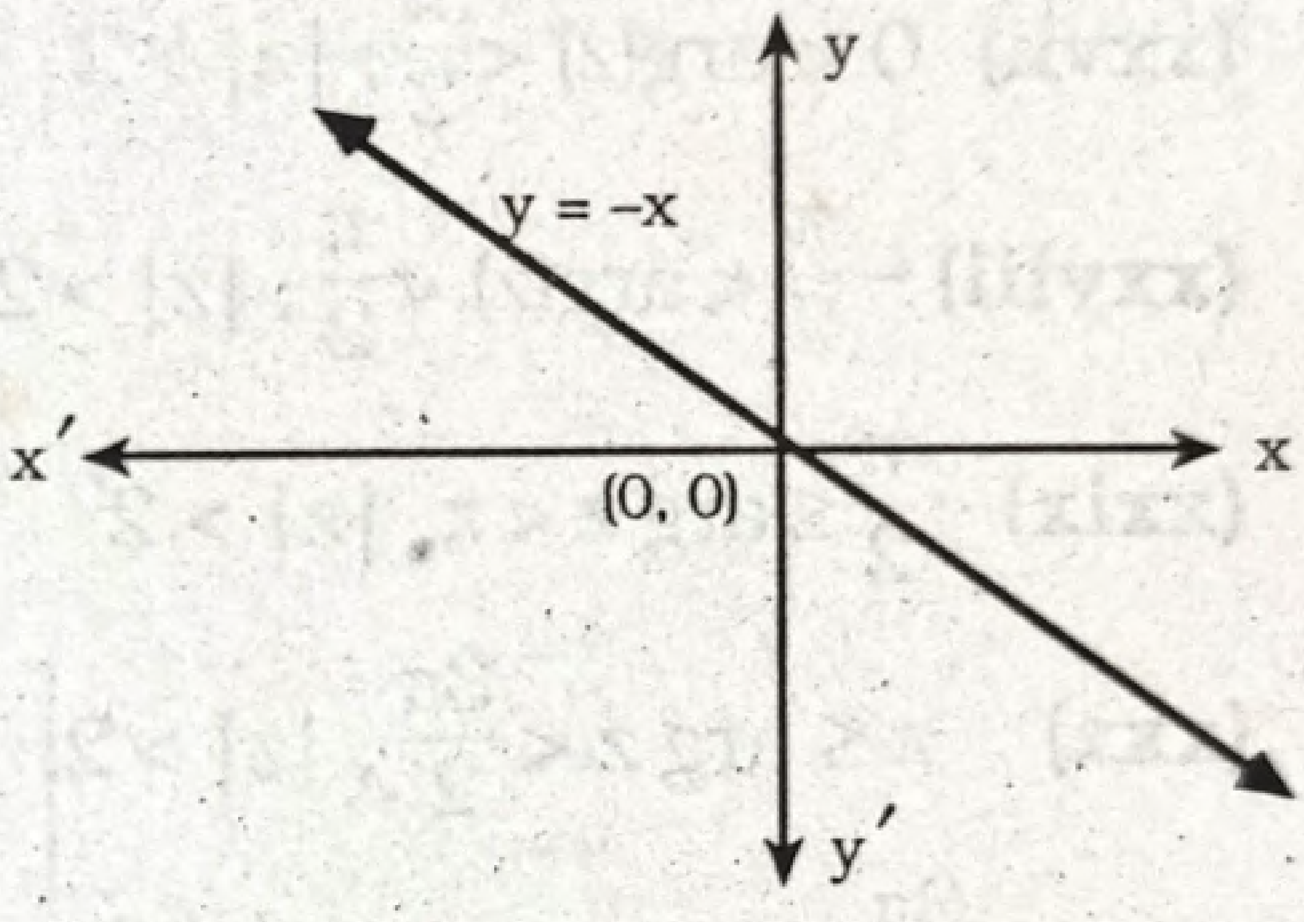
Let $z = x + iy$

$\Rightarrow Re(z) = x$ and $Im(z) = y$

$\therefore$ Then (1) becomes

$x + y = 0$

$\Rightarrow y = -x$



which represents a straight line passing through the origin and whose slope is –1.

6. Describe geometrically the set of points z satisfying the following conditions.

(i) $Re(z) \geq 0$ **(ii)** $Im(z) \leq 0$

(iii) $Re(z) > 1$ **(iv)** $Im(z) > 1$

(v) $|z-4| \geq |z|$ **[D. U. H. '88, '98]**

(vi) $|z| \geq |z-2i|$

(vii) $|z-i| \leq |z+i|$ **[N. U. M. Sc. (p) 2002]**

(viii) $|z-2| \leq |z+2|$ **(ix)** $|z+3i| \geq 4$ **[D. U. H. '89]**

(x) $|z+3| \leq 4$ **(xi)** $|z+2+i| \leq 1$

(xii) $|2z+3| > 4$ **(xiii)** $|z-1| \geq 3$

(xiv) $Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$ **[N. U. H. 2000]**

(xv) $Im\left(\frac{1}{z}\right) \geq \frac{1}{2}$ **(xvi)** $Re\left(\frac{1}{z}\right) < 1$ **[N. U. H. 2001]**

(xvii) $|z+1-i| \leq |z-1+i|$ **[N. U. H. 2002]**

(xviii) $1 < |z-2i| < 1$ [D. U. H. '90; N. U. H. '98]

(xix) $1 < |z+i| \leq 2$ [N. U. H–09, D. U. H. '89, M. Sc (p) '89]

(xx) $0 \leq |2z-1| \leq 1$ [N. U. H. 2007]

(xxi) $Re(z^2) > 1$ (xxii) $Im\,(z^2) \leq 1$

(xxiii) $|z+2-3i| + |z-2+3i| < 10$ [D. U. H. '89]

(xxiv) $0 \leq Re\,(iz) < 1$

(xxv) $-\pi < \arg(z) < \pi,\ |z| > 2$ [D. U. H. '88]

(xxvi) $|z-2| - |z+2| > 3$ [D. U. H. '90]

(xxvii) $0 < \arg(z) < \frac{\pi}{2},\ |z| > 2$

(xxviii) $-\frac{\pi}{2} < \arg(z) < \frac{\pi}{2},\ |z| > 2$

(xxix) $\frac{\pi}{2} \leq \arg z < \pi,\ |z| > 2$

(xxx) $\pi \leq \arg z < \frac{3\pi}{2},\ |z| > 2.$

(xxxi) $\frac{3\pi}{2} \leq \arg z < 2\pi;\ |z| > 2.$

(xxxii) $0 < \arg z < 2\pi;\ |z| > 2.$

(xxxiii) $0 < \arg z < \pi;\ |z| > 2$

(xxxiv) $\frac{\pi}{2} < \arg z < \frac{3\pi}{2};\ |z| > 2$

(xxxv) $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2};\ |z| > 2.$

Solution : **(i)** The given expression is

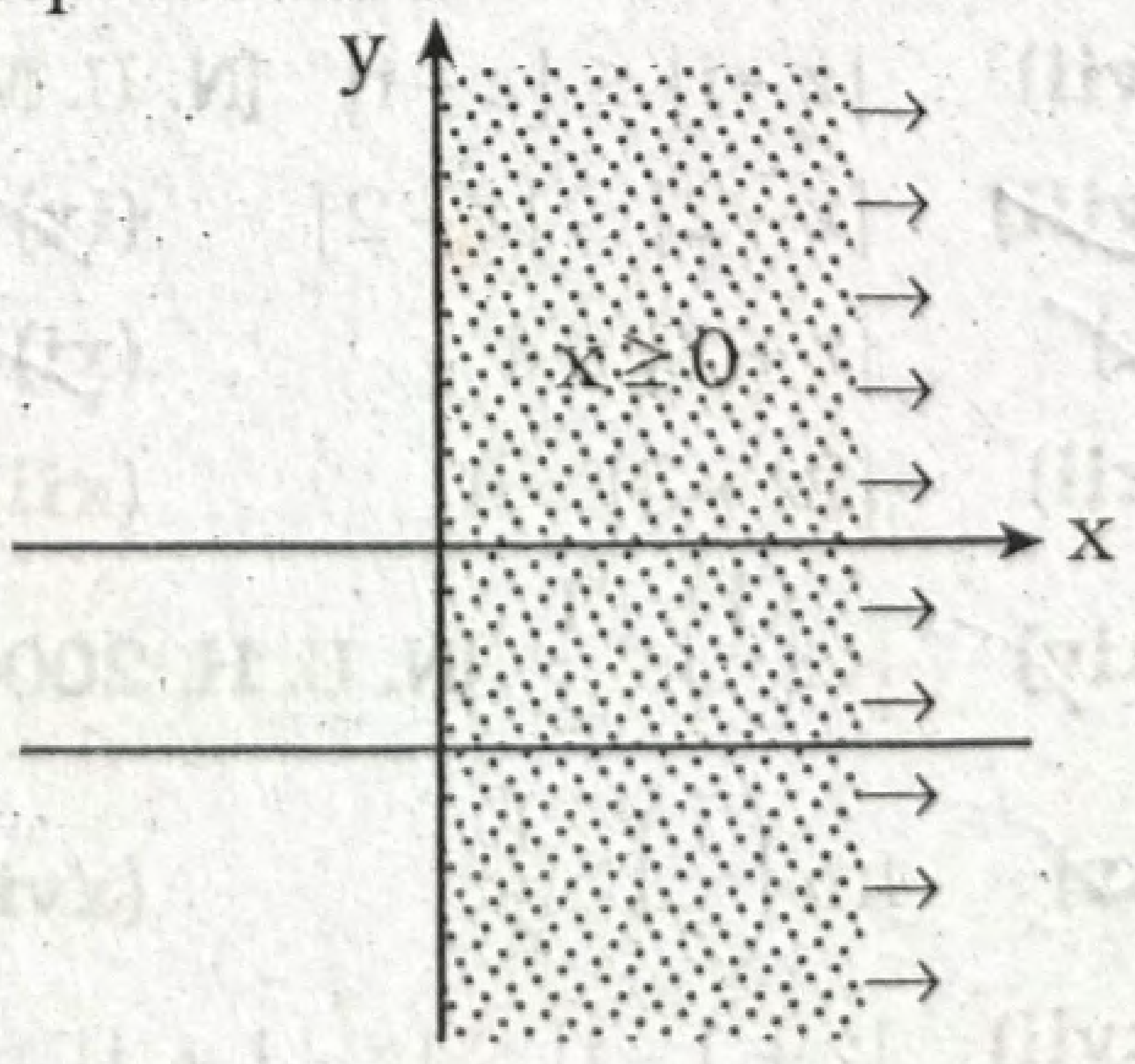
$Re(z) \geq 0$

$\Rightarrow Re(x+iy) \geq 0,$

where $z = x + iy$

$\Rightarrow x \geq 0$

which represents the right hand sides of y-axis including the y-axis.

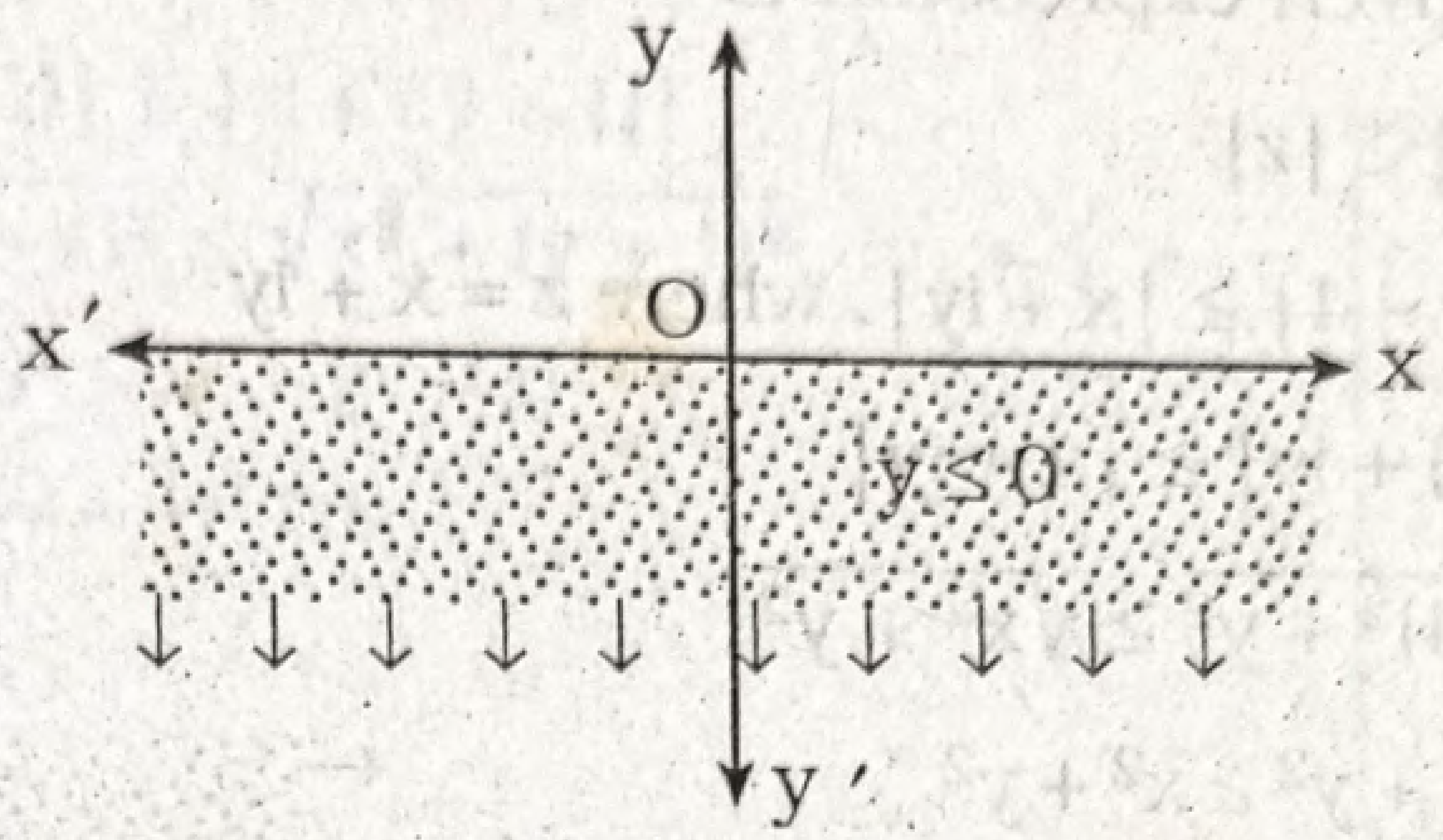


(ii) The given expression is

$\text{Im}(z) \le 0$

$\Rightarrow \text{Im}(x + iy) \le 0$, where $z = x + iy$

$\Rightarrow y \le 0$



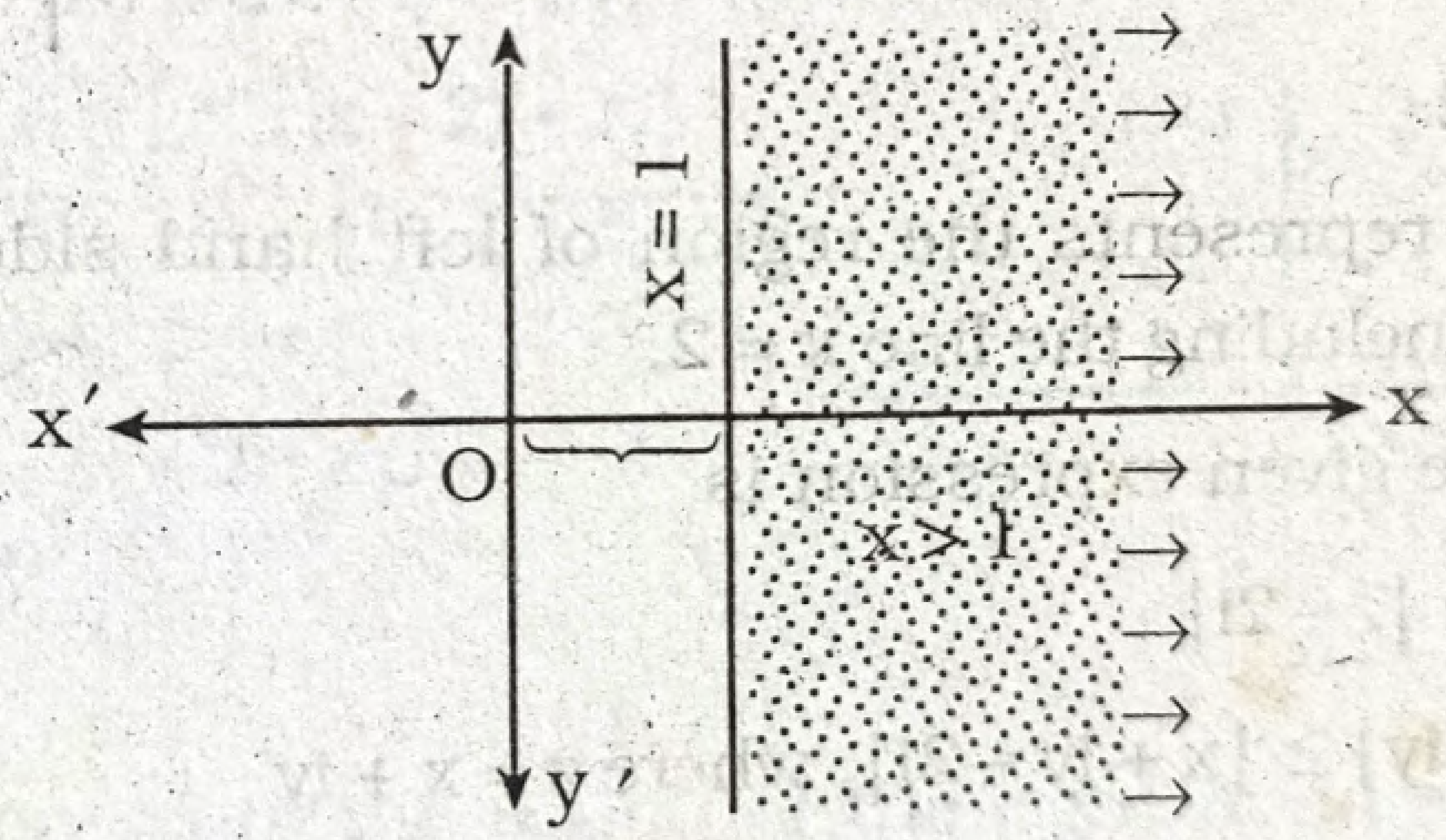
which represents the lower parts of the x-axis including x-axis.

(iii) The given expressiion is

$\text{Re}(z) > 1$

$\Rightarrow \text{Re}(x + iy) > 1$, where $z = x + iy$

$\Rightarrow x > 1$



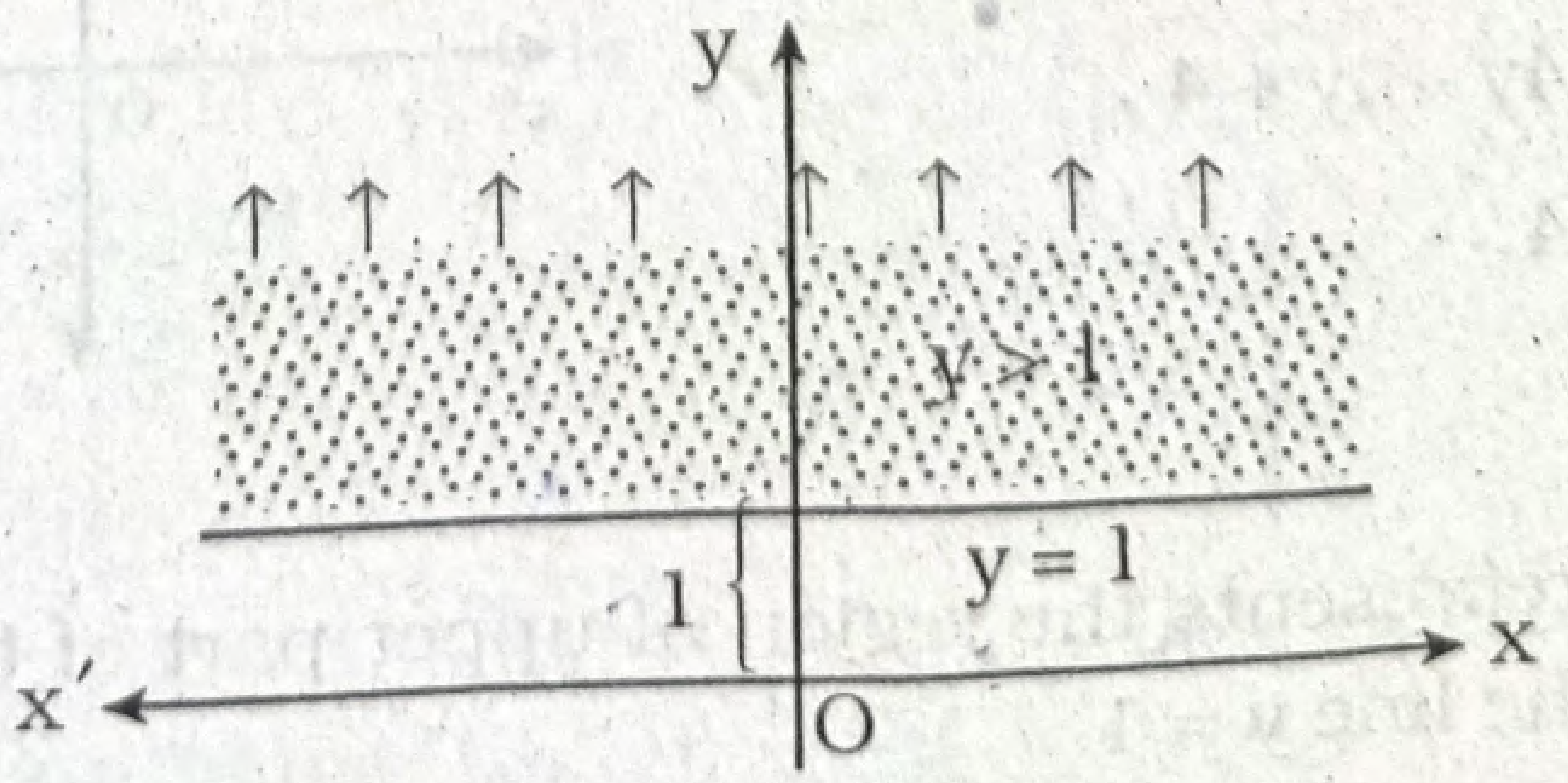
which represents the region of the right hand sides of the line $x = 1$.

(iv) The given expression is

$\text{Im}(z) > 1$

$\Rightarrow \text{Im}(x + iy) > 1$, where $z = x + iy$

$y > 1$



which represents the region of the upper part of the line y = 1

(v) The given expression is

$$|z-4| \geq |z|$$

$\Rightarrow |x + iy - 4| \geq |x + iy|$, where $z = x + iy$

$\Rightarrow |(x-4) + iy| \geq |x + iy|$

$\Rightarrow \sqrt{(x-4)^2 + y^2} \geq \sqrt{x^2 + y^2}$

$\Rightarrow (x-4)^2 + y^2 \geq x^2 + y^2$

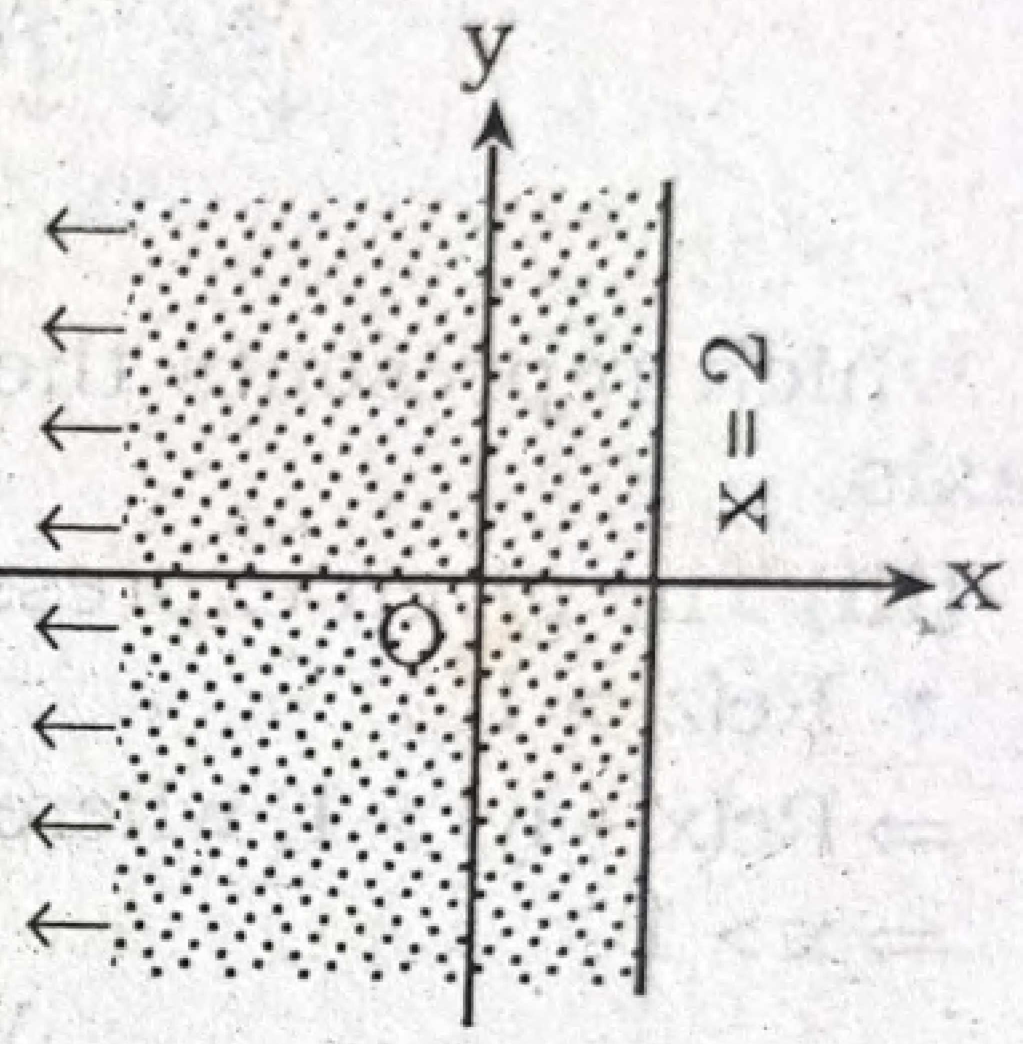
$\Rightarrow x^2 - 8x + 16 + y^2 - x^2 - y^2 \geq 0$

$\Rightarrow -8x + 16 \geq 0$

$\Rightarrow 8x - 8x + 16 \geq 8x$

$\Rightarrow 16 \geq 8x$

$\Rightarrow x \leq 2$



which represents the region of left hand side of the line x = 2 and including the line x = 2.

(vi) The given expression is

$$|z| \geq |z-2i|$$

$\Rightarrow |x + iy| \geq |x + iy - 2i|$, where $z = x + iy$

$\Rightarrow |x + iy| \geq |x + i(y-2)|$

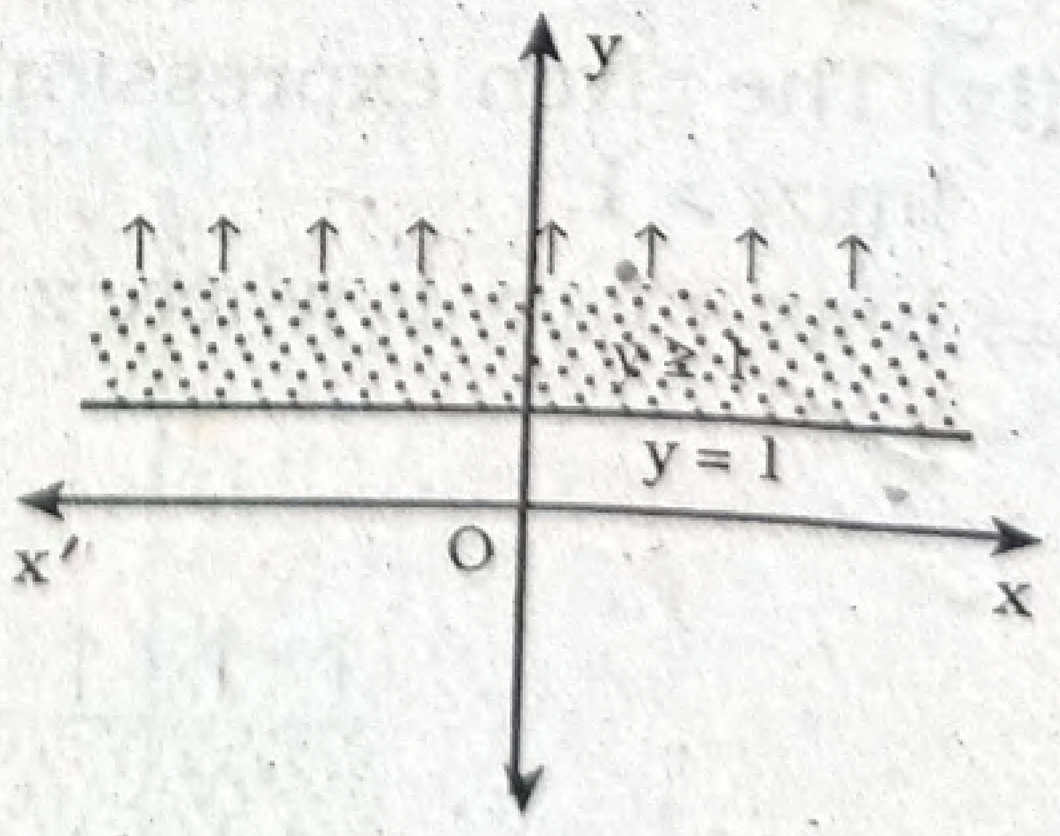
$\Rightarrow \sqrt{x^2 + y^2} \leq \sqrt{x^2 + (y-2)^2}$

$\Rightarrow x^2 + y^2 \geq x^2 + y^2 - 4y + 4$

$\Rightarrow 0 \geq -4y + 4$

$\Rightarrow 4y \geq 4y - 4y + 4$

$\Rightarrow 4y \geq 4$



$\Rightarrow y \geq 1$

which represents the region of upper part of the line y = 1 including the line y = 1.

(vii) The given expression is

$|z - i| \leq |z + i|$

$\Rightarrow |x + iy - i| \leq |x + iy + i|$, where $z = x + iy$

$\Rightarrow |x + i(y - 1)| \leq |x + i(y + 1)|$

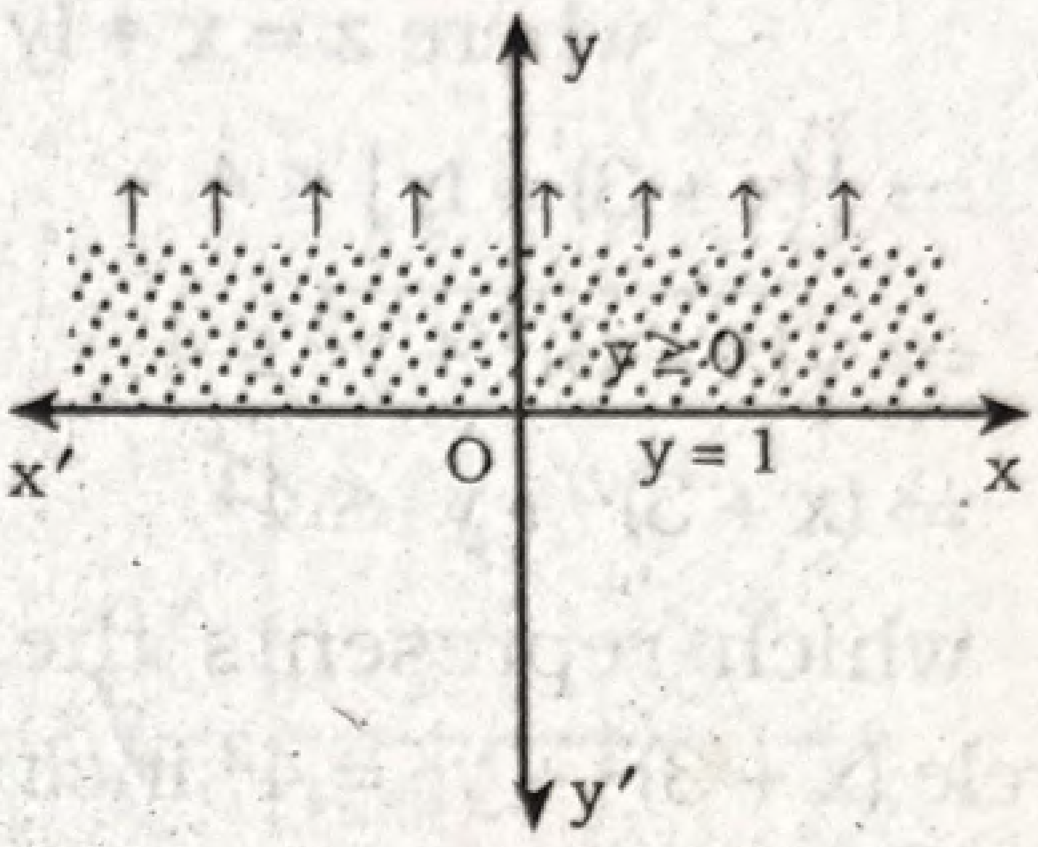
$\Rightarrow \sqrt{x^2 + (y - 1)^2} \leq \sqrt{x^2 + (y + 1)^2}$

$\Rightarrow x^2 + (y - 1)^2 \leq x^2 + (y + 1)^2$

$\Rightarrow (y + 1)^2 - (y - 1)^2 \geq 0$

$\Rightarrow 4y \geq 0$

$\Rightarrow y \geq 0$



which represents the region of the upper part of the x-axis including the x-axis.

(viii) The given expression is

$|z - 2| \leq |z + 2|$

$\Rightarrow |x + iy - 2| \leq |x + iy + 2|$, where $z = x + iy$

$\Rightarrow |(x - 2) + iy| \leq |(x + 2) + iy|$

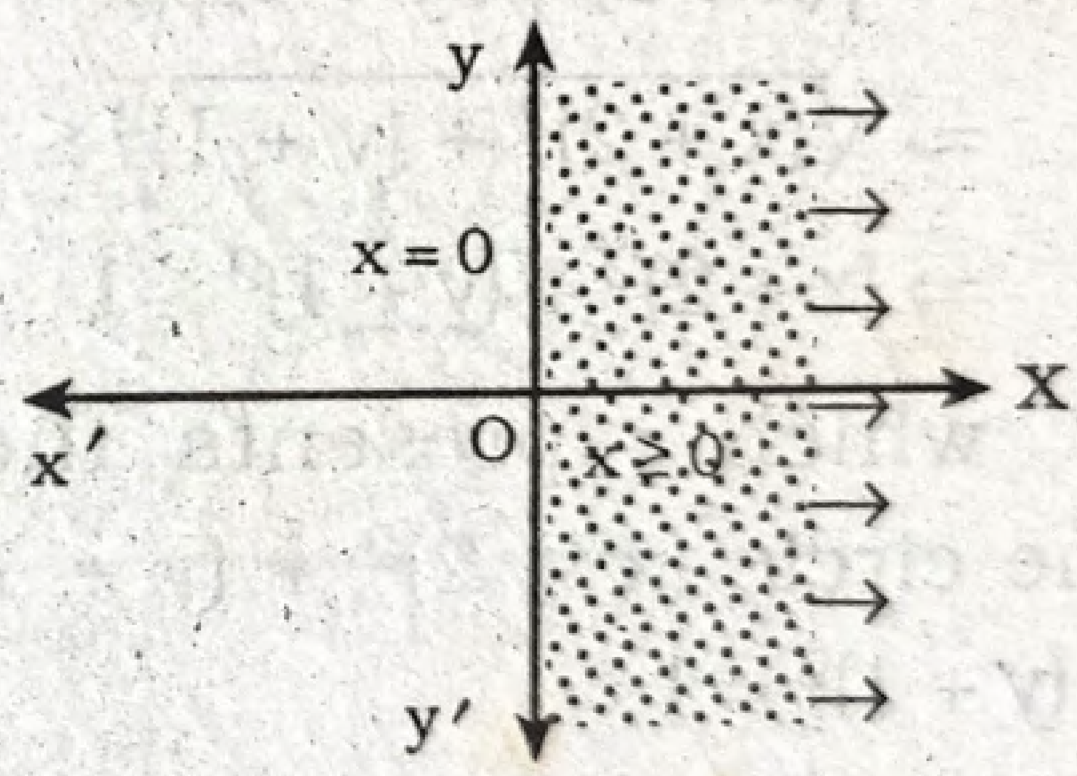
$\Rightarrow \sqrt{(x - 2)^2 + y^2} \leq \sqrt{(x + 2)^2 + y^2}$

$\Rightarrow (x - 2)^2 \leq (x + 2)^2$

$\Rightarrow (x + 2)^2 - (x - 2)^2 \geq 0$

$\Rightarrow 4.2.x \geq 0$

$\Rightarrow x \geq 0$



which represents the region of the right hand side of the y-axis including y-axis.

(ix) The given expression is

$|z + 3i| \geq 4$

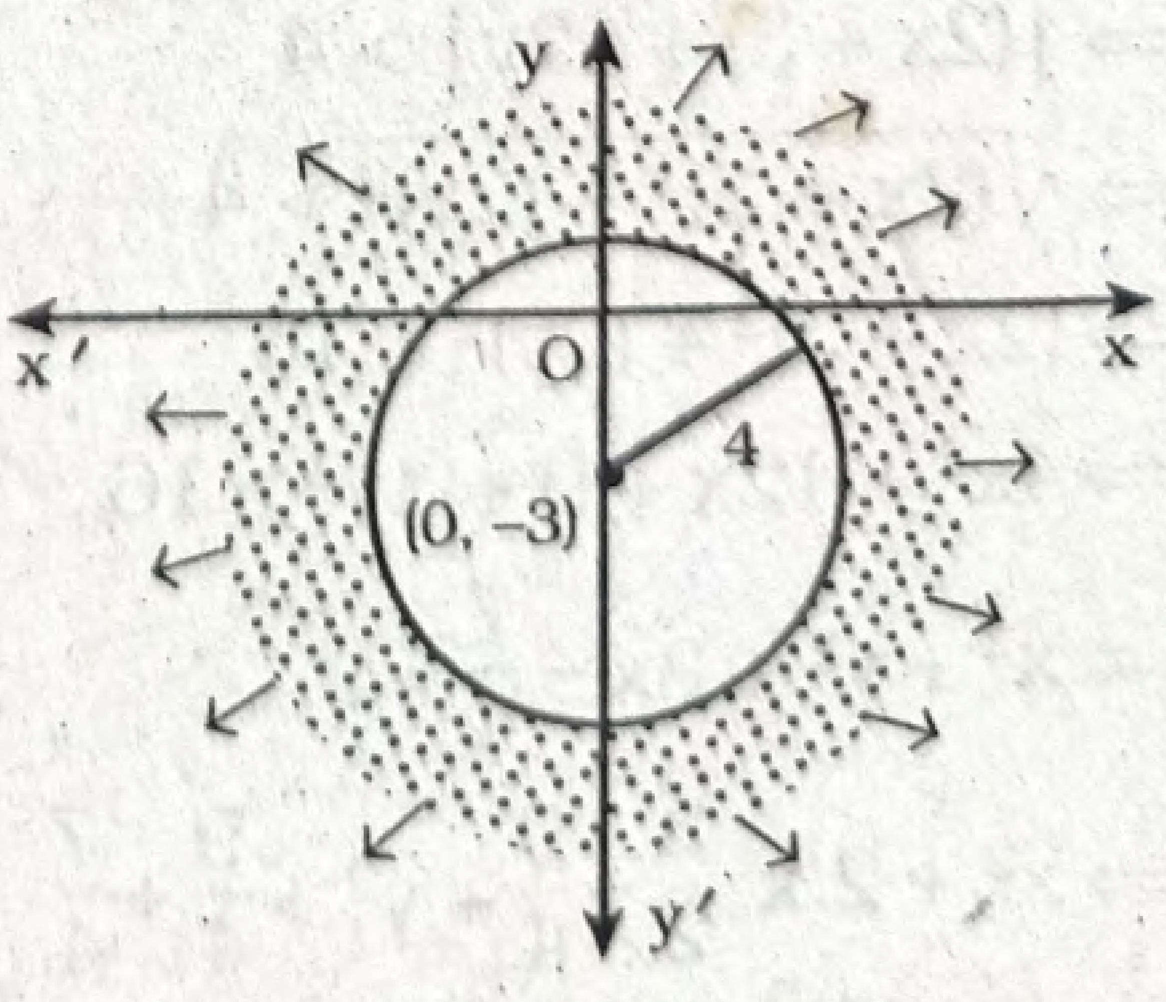
$\Rightarrow |x + iy + 3i| \geq 4$,

where $z = x + iy$

$\Rightarrow |x + i(y + 3)| \geq 4$

$\Rightarrow \sqrt{x^2 + (y + 3)^2} \geq 4$

$\Rightarrow x^2 + (y + 3)^2 \geq 4^2$



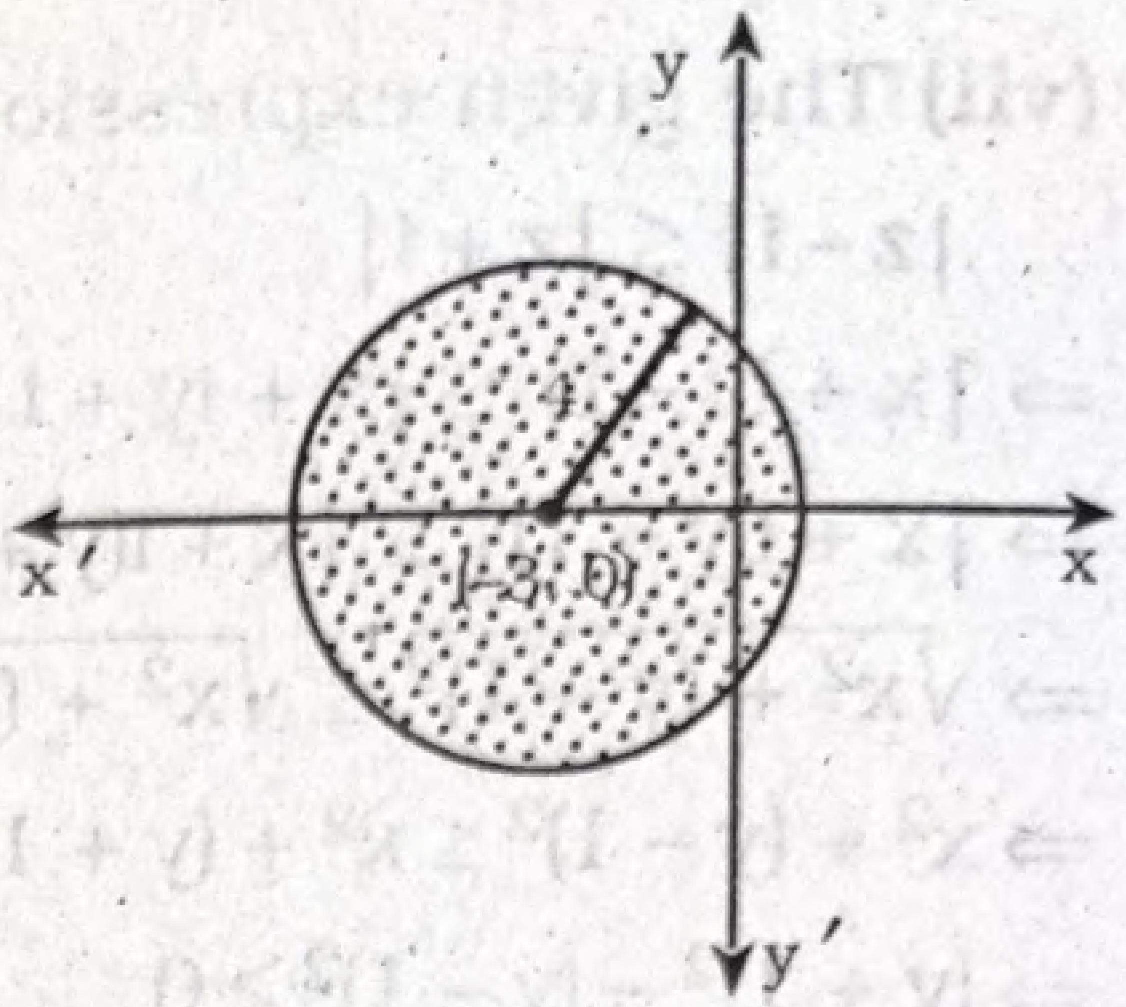
which represents the region of the whole exterior part of the circle $x^2 + (y + 3)^2 = 4^2$ including the circle $x^2 + (y + 3)^2 = 4^2$.

(x) The given expression is

$$|z+3| \le 4$$

$\Rightarrow |x + iy + 3| \le 4,$

where $z = x + iy$



$\Rightarrow |(x + 3) + iy| \le 4$

$\Rightarrow \sqrt{(x + 3)^2 + y^2} \le 4$

$\Rightarrow (x + 3)^2 + y^2 \le 4^2$

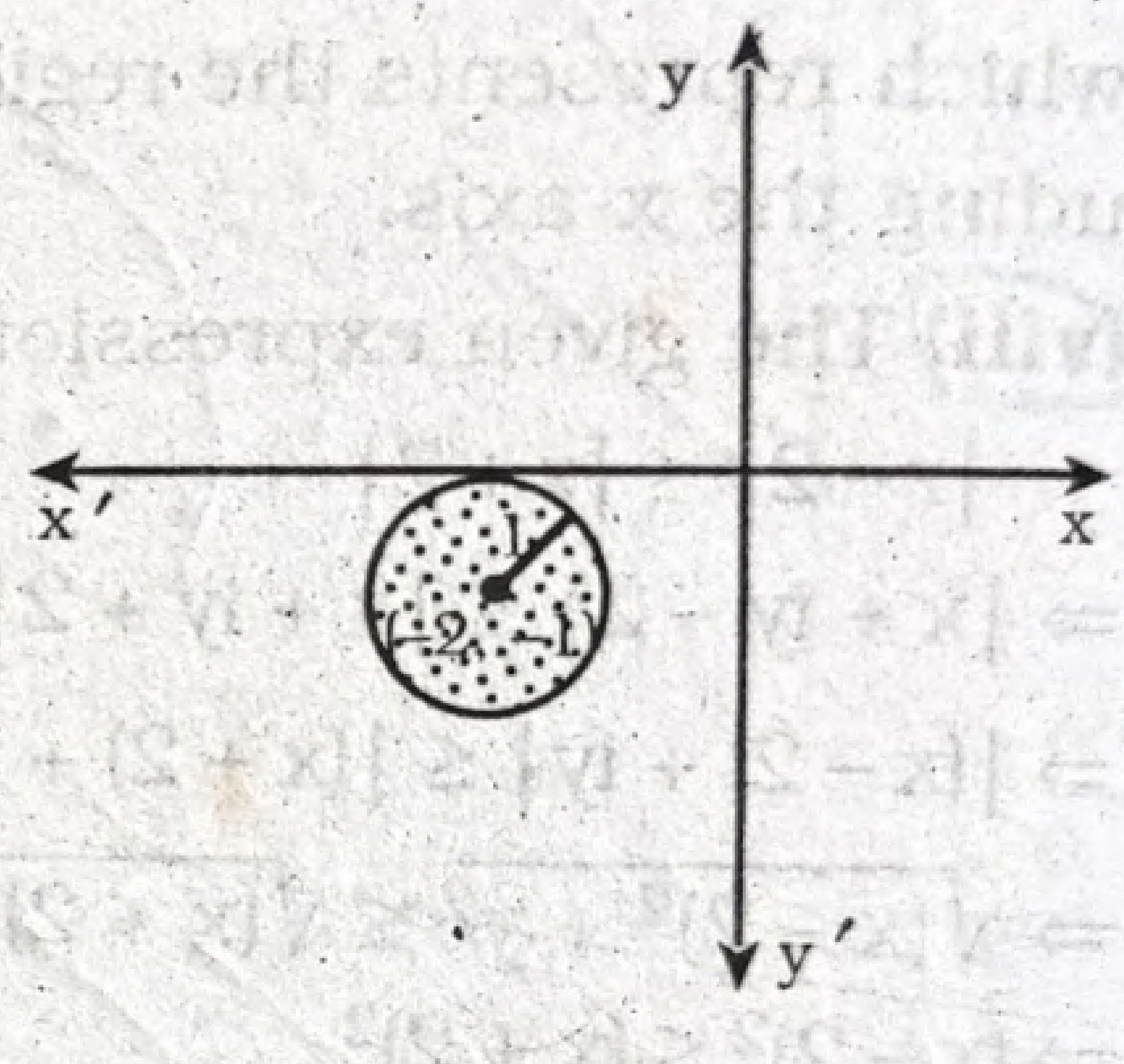
which represents the region of all interior points of the circle $(x + 3)^2 + y^2 = 4^2$ including the circle $(x + 3)^2 + y^2 = 4^2$.

(xi) The given expression is

$$|z+2+i| \le 1$$

$\Rightarrow |x + iy + 2 + i| \le 1,$

where $z = x + iy$



$\Rightarrow |(x + 2) + i(y + 1)| \le 1$

$\Rightarrow \sqrt{(x + 2)^2 + (y + 1)^2} \le 1$

$\Rightarrow (x + 2)^2 + (y + 1)^2 \le 1$

which represents the region of all interior points of the circle $(x + 2)^2 + (y + 1)^2 = 1$ including the circle $(x + 2)^2 + (y + 1)^2 = 1$.

(xii) The given expression is

$$|2z+3| > 4$$

$\Rightarrow |2(x + iy) + 3| > 4$

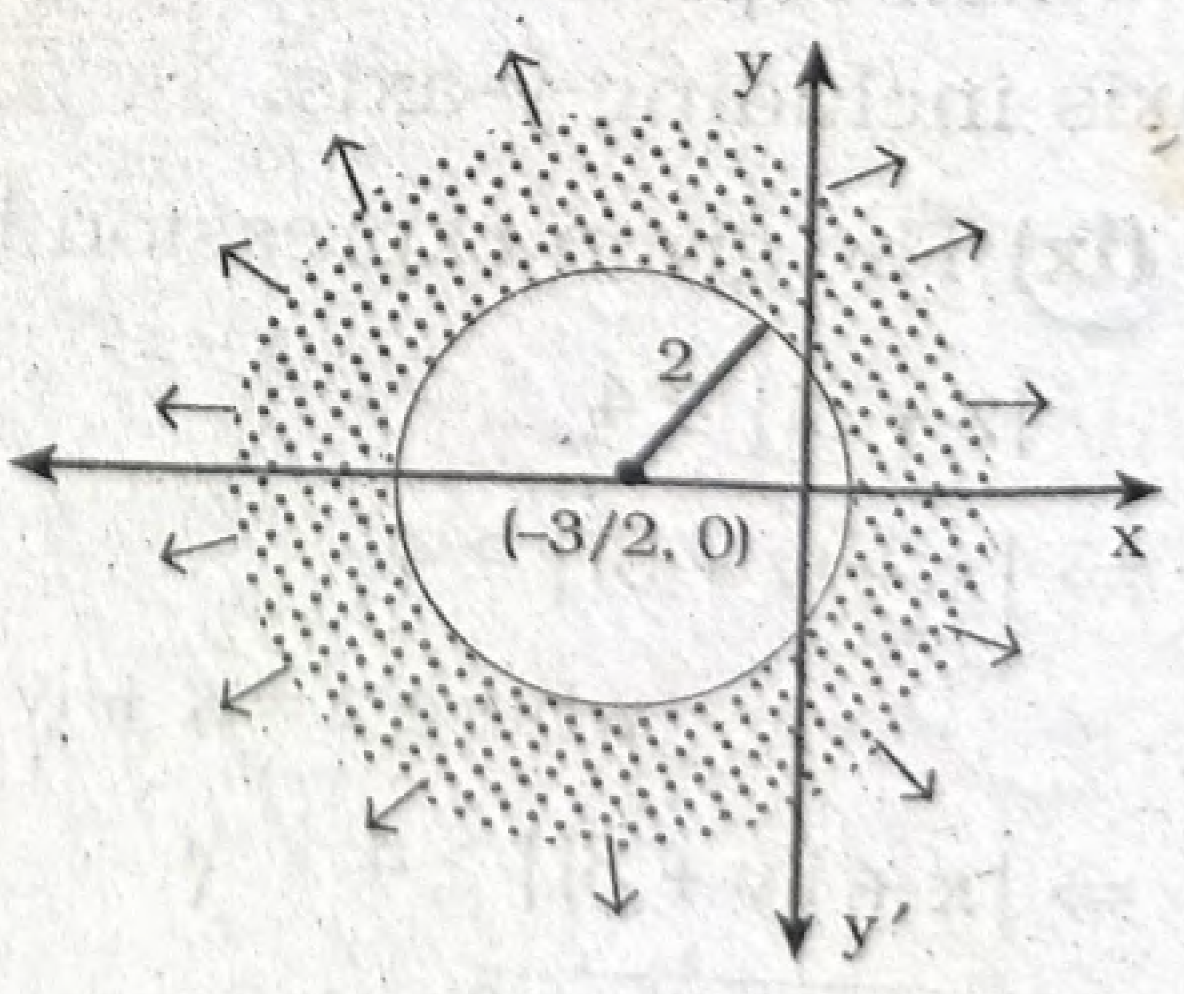
$\Rightarrow |(2x + 3) + i.2y| > 4$

$\Rightarrow \sqrt{(2x + 3)^2 + (2y)^2} > 4$

$\Rightarrow (2x + 3)^2 + (2y)^2 > 16$

$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$

$\Rightarrow x^2 + y^2 + 3x - \frac{7}{4} > 0$



$\Rightarrow x^2 + 2.x.\frac{3}{2} + \frac{9}{4} + y^2 > \frac{9}{4} + \frac{7}{4}$

$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > \frac{16}{4}$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2 + y^2 > 4 = 2^2$$

which represents the region of all exterior points of the circle $\left(x+\frac{3}{2}\right)^2 + y^2 = 2^2$.

(xiii) the given expression is

$$|z-1| \geq 3$$

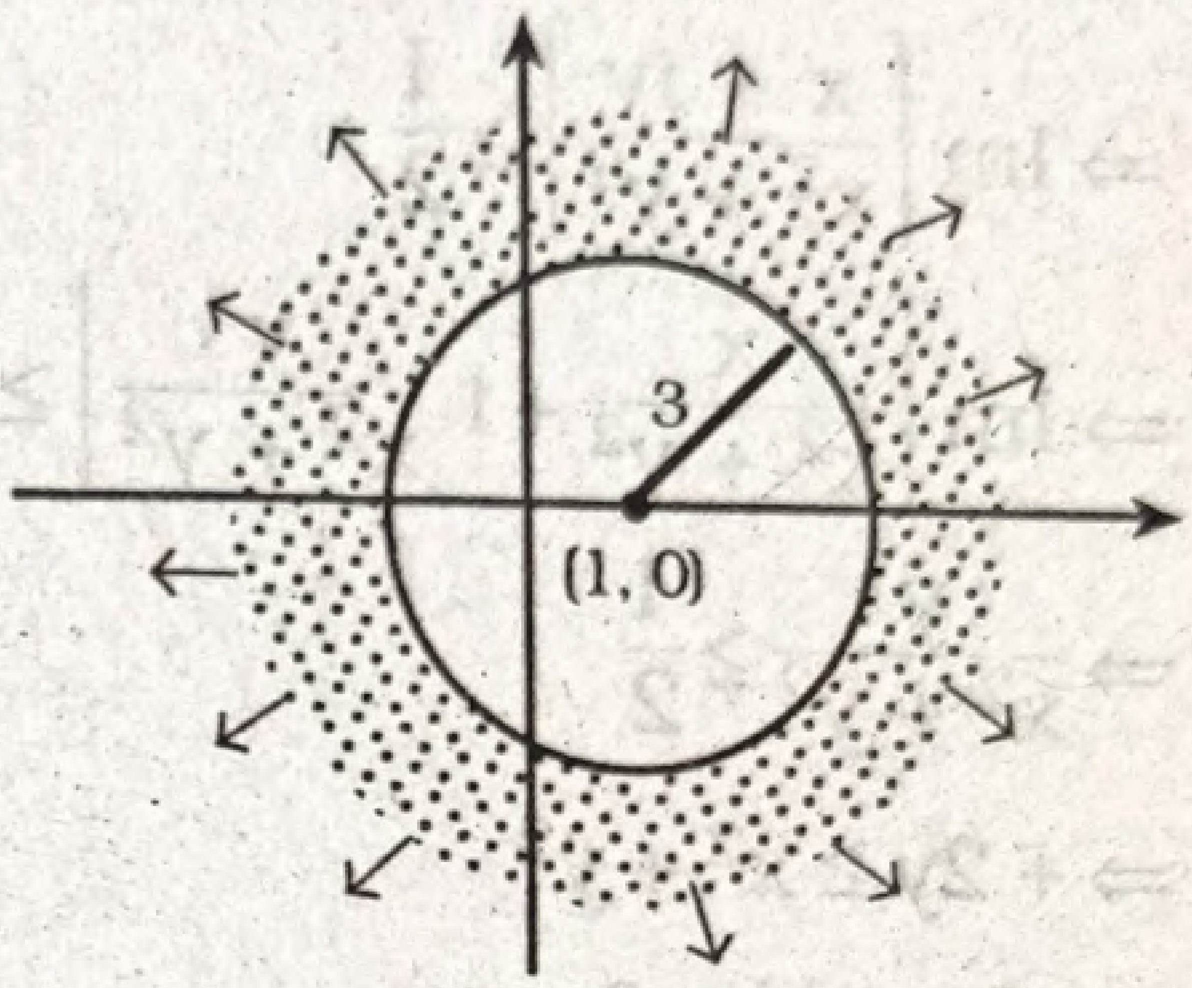
$$\Rightarrow |x+iy-1| \geq 3,$$

where $z = x + iy$

$$\Rightarrow |(x-1)+iy| \geq 3$$

$$\Rightarrow \sqrt{(x-1)^2+y^2} \geq 3$$

$$\Rightarrow (x-1)^2 + y^2 \geq 3^2$$



which represents the region of all exterior points of the circle $(x-1)^2 + y^2 = 3^2$ including the circle $(x-1)^2 + y^2 = 3^2$.

(xiv) The given expression is

$$\text{Re}\left(\frac{1}{z}\right) \leq \frac{1}{2}$$

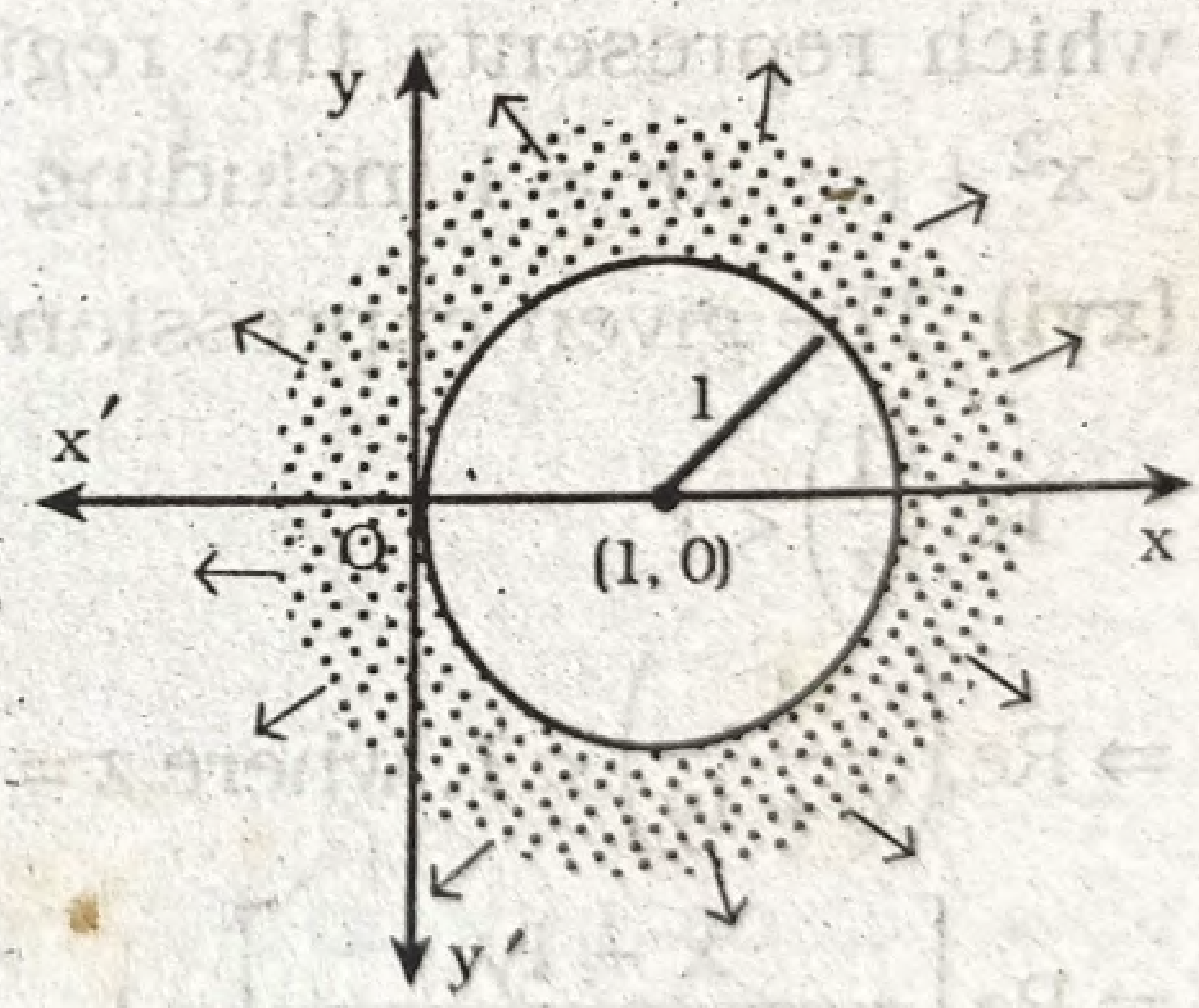
$$\Rightarrow \text{Re}\left(\frac{1}{x+iy}\right) \leq \frac{1}{2},$$

where $z = x + iy$

$$\Rightarrow \text{Re}\left[\frac{(x-iy)}{(x+iy)(x-iy)}\right] \leq \frac{1}{2}$$

$$\Rightarrow \text{Re}\left[\frac{x-iy}{x^2+y^2}\right] \leq \frac{1}{2}$$

$$\Rightarrow \text{Re}\left[\frac{x}{x^2+y^2} - i.\frac{y}{x^2+y^2}\right] \leq \frac{1}{2}$$



$$\Rightarrow \frac{x}{x^2+y^2} \leq \frac{1}{2}$$

$$\Rightarrow 2x \leq x^2 + y^2$$

$$\Rightarrow x^2 - 2x + y^2 \geq 0$$

$$\Rightarrow x^2 - 2x + 1 + y^2 \geq 1$$

$$\Rightarrow (x-1)^2 + y^2 \geq 1$$

which represents the region of all exterior points of the circle $(x-1)^2 + y^2 = 1$ including the circle $(x-1)^2 + y^2 = 1$.

(xv) The given expression is

$$\text{Im}\left(\frac{1}{\bar{z}}\right) \geq \frac{1}{2}$$

$$\Rightarrow \text{Im}\left[\frac{x+iy}{(x+iy)(x-iy)}\right] \geq \frac{1}{2}, \text{ where } z = x+iy \Rightarrow \bar{z} = x - iy$$

$$\Rightarrow \text{Im}\left[\frac{x+iy}{x^2+y^2}\right] \geq \frac{1}{2}$$

$$\Rightarrow \text{Im}\left[\frac{x}{x^2+y^2} + i\,\frac{y}{x^2+y^2}\right] \geq \frac{1}{2}$$

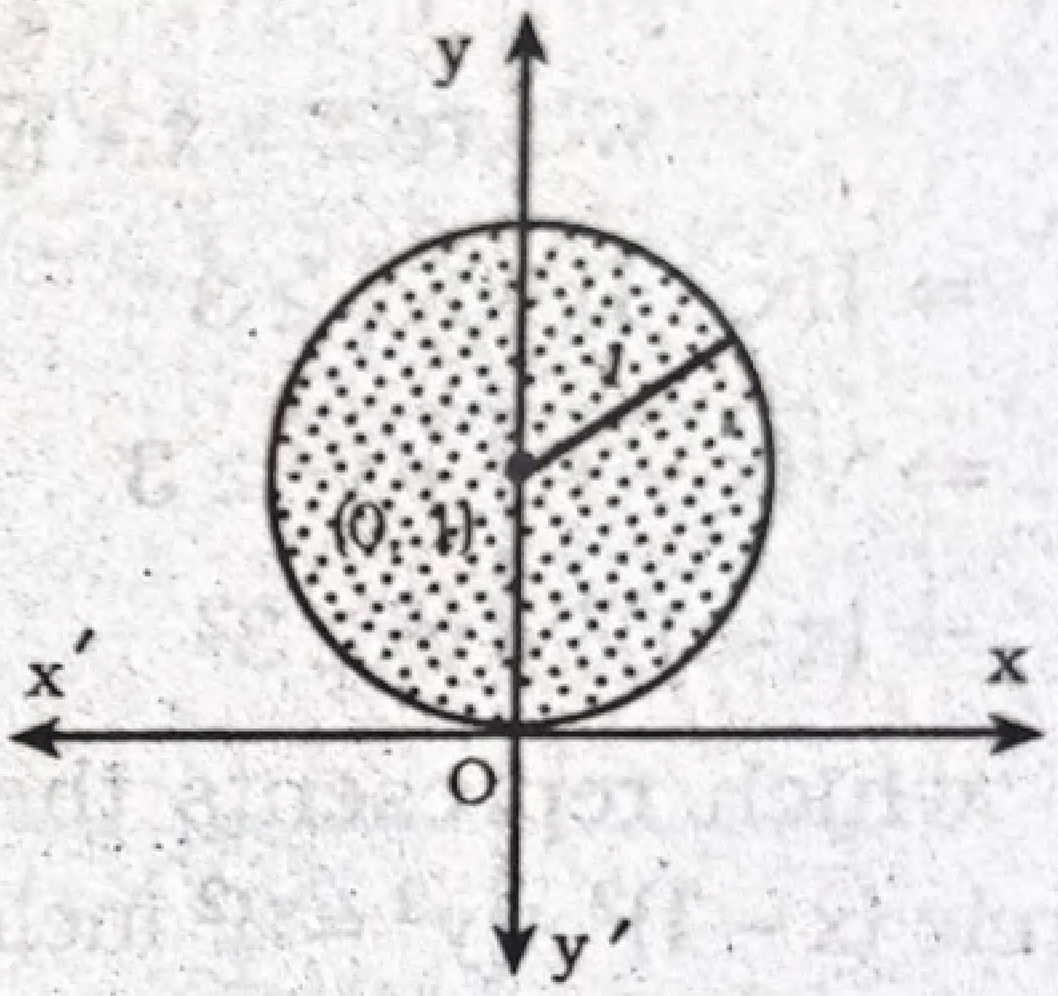
$$\Rightarrow \frac{y}{x^2+y^2} \geq \frac{1}{2}$$

$$\Rightarrow +2y \geq x^2 + y^2$$

$$\Rightarrow x^2 - 2y + y^2 \leq 0$$

$$\Rightarrow x^2 + y^2 - 2y + 1 \leq 1$$

$$\Rightarrow x^2 + (y-1)^2 \leq 1$$



which represents the region of all interior points of the circle $x^2 + (y-1)^2 = 1$ including the circle $x^2 + (y-1)^2 = 1$.

(xvi) The given expression is

$$\text{Re}\left(\frac{1}{z}\right) < 1$$

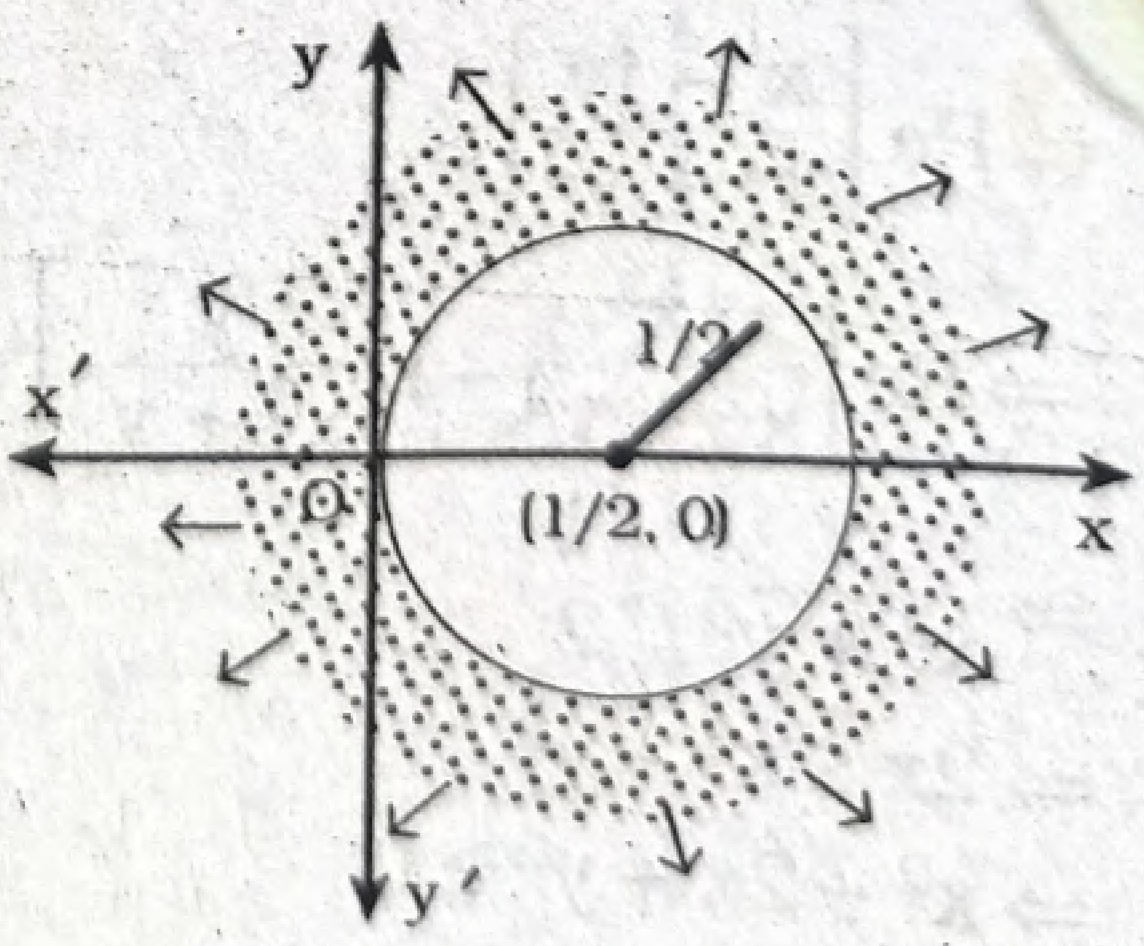
$$\Rightarrow \text{Re}\left(\frac{1}{x+iy}\right) < 1, \text{ where } z = x + iy$$

$$\Rightarrow \text{Re}\left[\frac{x-iy}{(x+iy)(x-iy)}\right] < 1$$

$$\Rightarrow \text{Re}\left[\frac{x-iy}{x^2+y^2}\right] < 1$$

$$\Rightarrow \text{Re}\left[\frac{x}{x^2+y^2} - i\,.\,\frac{y}{x^2+y^2}\right] < 1$$

$$\Rightarrow \frac{x}{x^2+y^2} < 1$$



$$\Rightarrow x^2 + y^2 > x$$

$$\Rightarrow x^2 - x + y^2 > 0$$

$$\Rightarrow x^2 - 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 + y^2 > 0 + \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(x - \frac{1}{2}\right)^2 + y^2 > \left(\frac{1}{2}\right)^2$$

which represents the region of the whole exterior part of the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$.

(xvii) The given expression expression is

$$|z + 1 - i| \le |z - 1 + i|$$

$$\Rightarrow |x + iy + 1 - i| \le |x + iy - 1 + i|$$

$$\Rightarrow |(x + 1) + i(y - 1)| \le |(x - 1) + i(y + 1)|$$

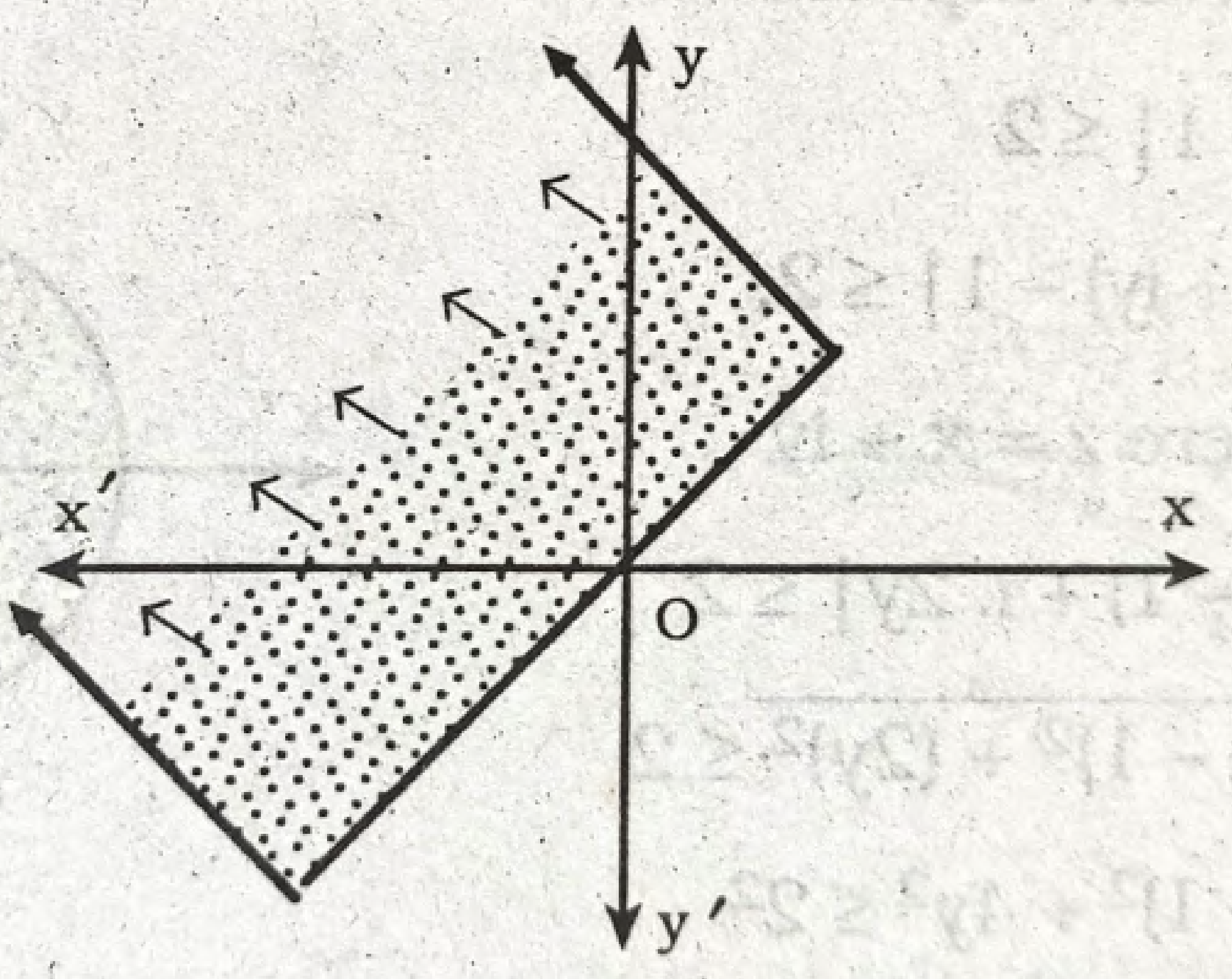
$$\Rightarrow \sqrt{(x + 1)^2 + (y - 1)^2} \le \sqrt{(x - 1)^2 + (y + 1)^2}$$

$$\Rightarrow (x + 1)^2 + (y - 1)^2 \le (x - 1)^2 + (y + 1)^2$$

$$\Rightarrow (x + 1)^2 - (x - 1)^2 \le (y + 1)^2 - (y - 1)^2$$

$$\Rightarrow 4x \le 4y$$

$$\Rightarrow y \ge x$$



which represents the region containing whole second quadrant, the upper half of first quadrant and the upper half of third quadrant including the line $y = x$.

(xviii) The given expression is

$$1 < |z - 2i| < 2$$

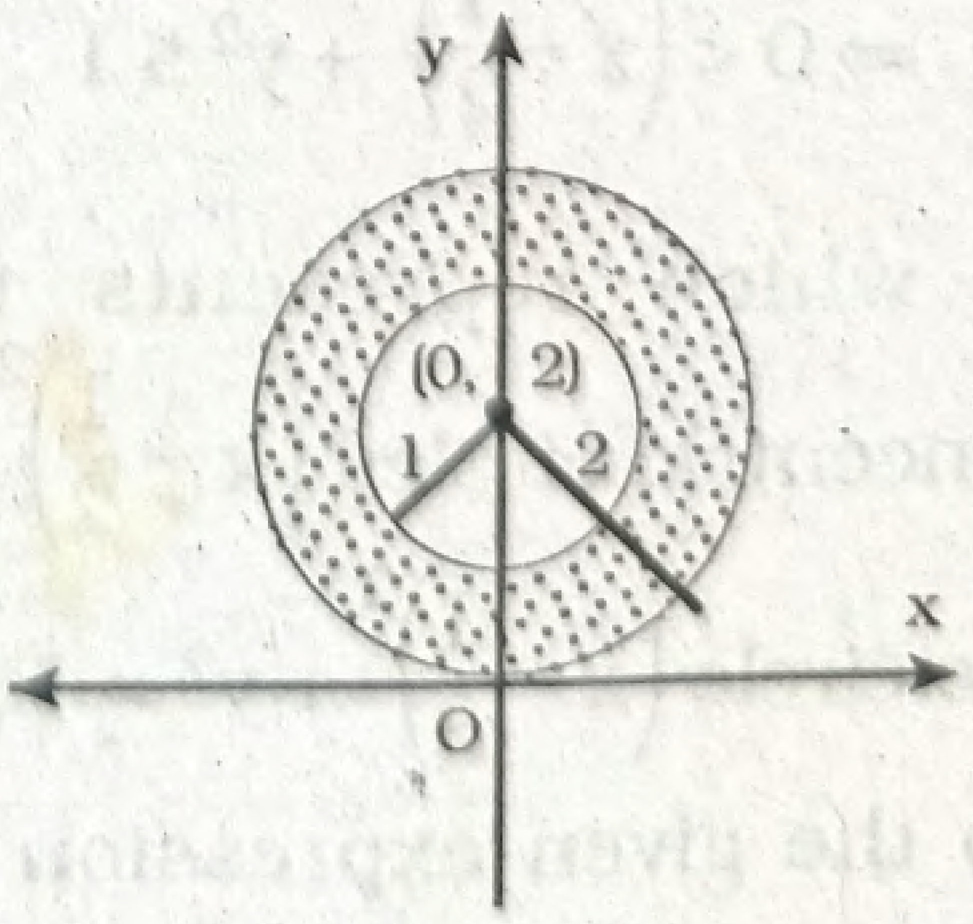
$$\Rightarrow 1 < |x + iy - 2i| < 2,$$

where $z = x + iy$

$$\Rightarrow 1 < |x + i(y - 2)| < 2$$

$$\Rightarrow 1 < \sqrt{x^2 + (y - 2)^2} < 2$$

$$\Rightarrow 1 < x^2 + (y - 2)^2 < 2^2$$



which represents the annular region between the circles $x^2 + (y - 2)^2 = 1$ and $x^2 + (y - 2)^2 = 2^2$.

(xix) The given expression is

$1 < |z + i| \leq 2$

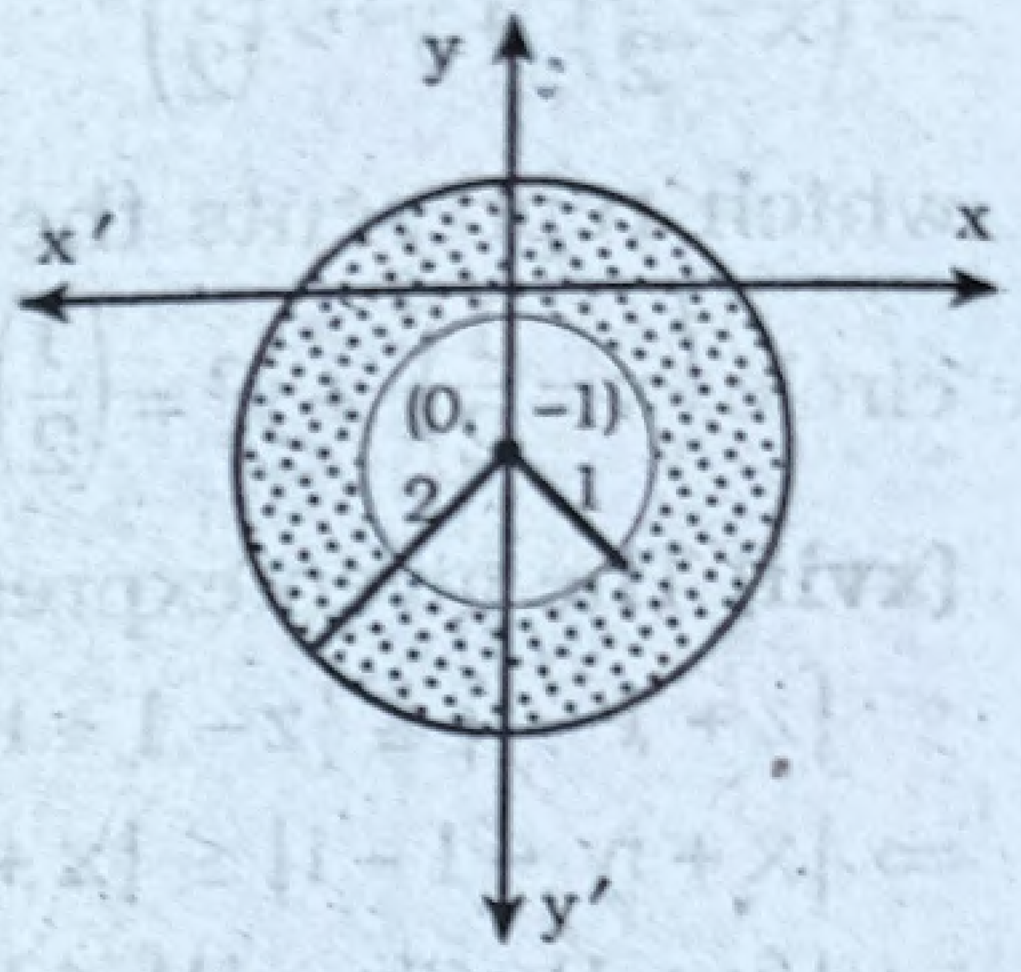
$\Rightarrow 1 < |x + iy + i| < 2,$

where $z = x + iy$

$\Rightarrow 1 < |x + i(y + 1)| \leq 2$

$\Rightarrow 1 < \sqrt{x^2 + (y + 1)^2} \leq 2$

$\Rightarrow 1 < x^2 + (y + 1)^2 \leq 2^2$



which represents the annular region between the concentric circles $x^2 + (y + 1)^2 = 1$ and $x^2 + (y + 1)^2 = 2^2$ including the circle $x^2 + (y + 1)^2 = 2^2$.

(xx) The given expression is

$0 < |2z - 1| \leq 2$

$\Rightarrow 0 < |2(x + iy) - 1| \leq 2,$

where $z = x + iy$

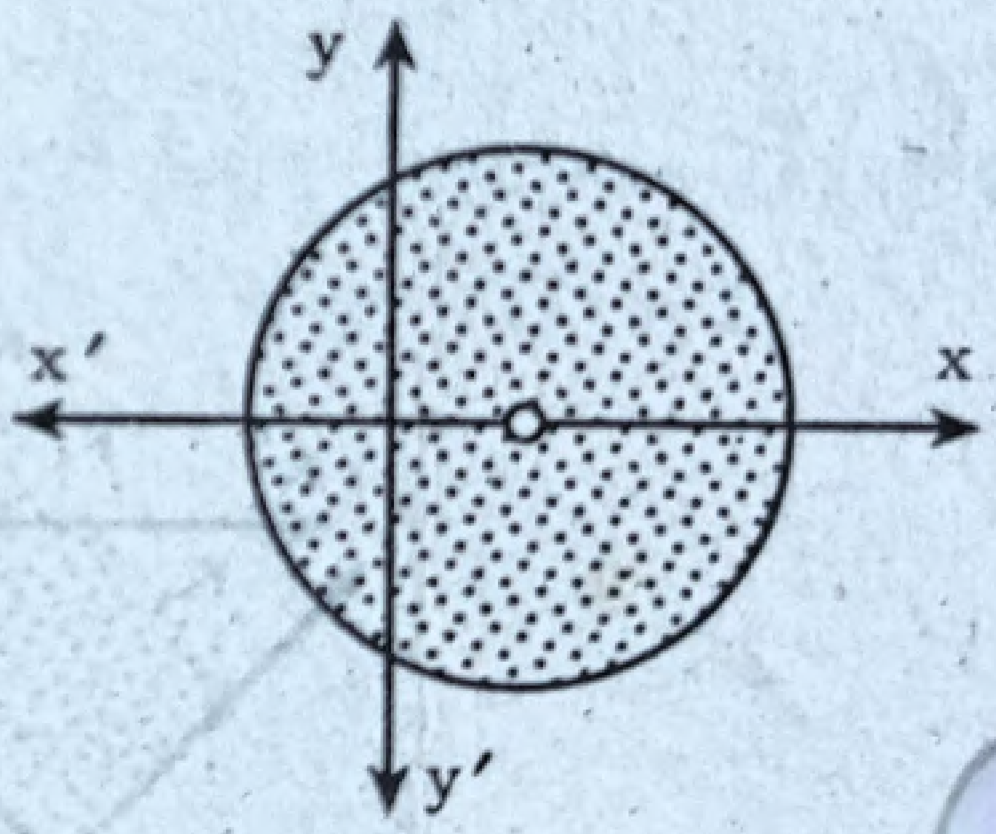
$\Rightarrow 0 < |(2x - 1) + i.\, 2y| \leq 2$

$\Rightarrow 0 < \sqrt{(2x - 1)^2 + (2y)^2} \leq 2$

$\Rightarrow 0 < (2x - 1)^2 + 4y^2 \leq 2^2$

$\Rightarrow 0 < 4\left(x - \frac{1}{2}\right)^2 + 4y^2 \leq 4$

$\Rightarrow 0 < \left(x - \frac{1}{2}\right)^2 + y^2 \leq 1$



which represents the annular region between the concentric circles $\left(x - \frac{1}{2}\right)^2 + y^2 = 1$ and $\left(x - \frac{1}{2}\right)^2 + y^2 = 0$ including the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = 1$. Since $\left(x - \frac{1}{2}\right)^2 + y^2 = 0$ is a point circle. So the given expression represents the region of all interior points of the circle $\left(x - \frac{1}{2}\right)^2 + y^2 = 1$ excluding the point $\left(\frac{1}{2}, 0\right)$ the centre of the circles.

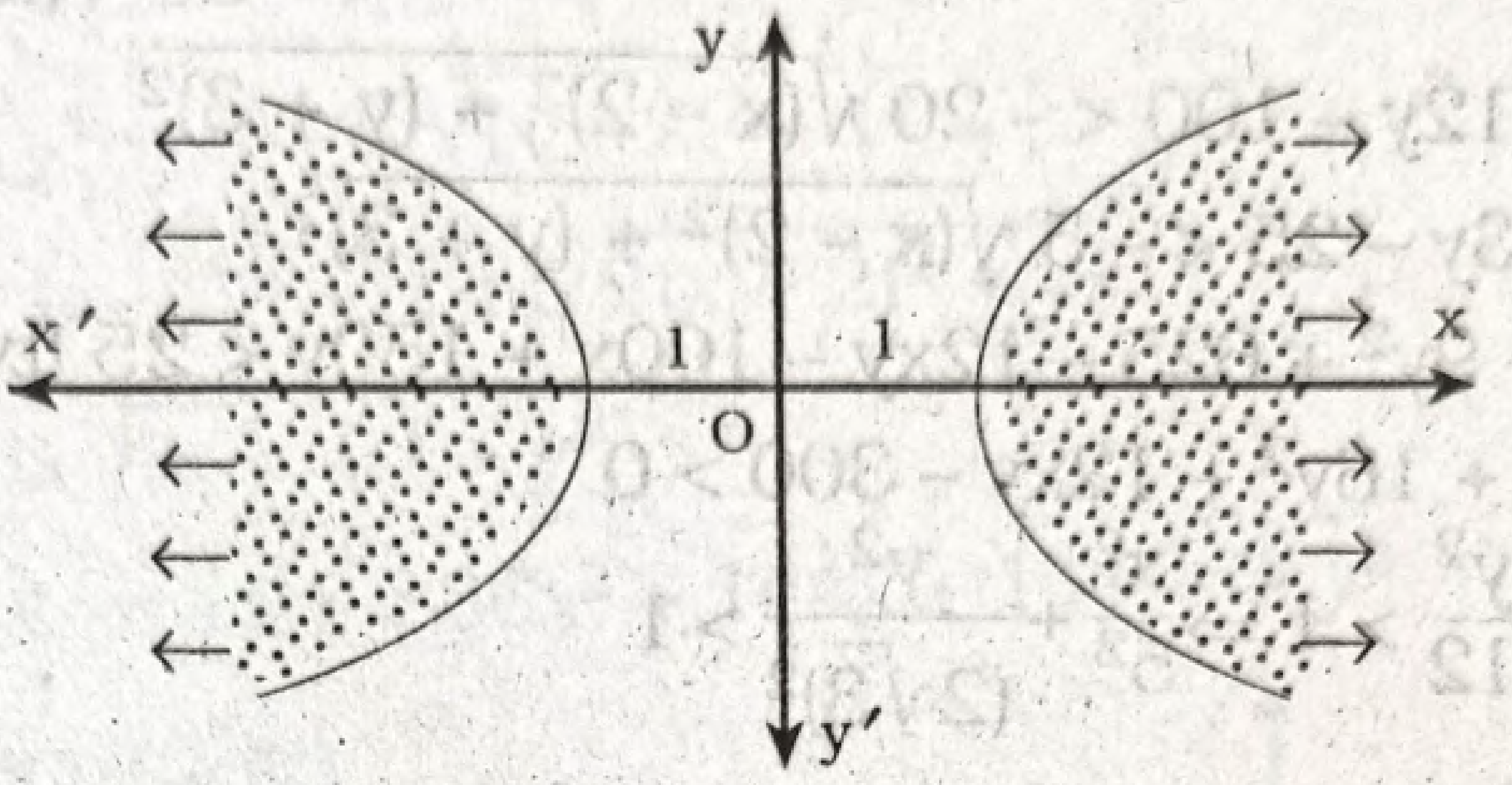
(xxi) The given expression is

$Re(z^2) > 1$

$\Rightarrow Re[(x + iy)^2] > 1$, where $z = x + iy$

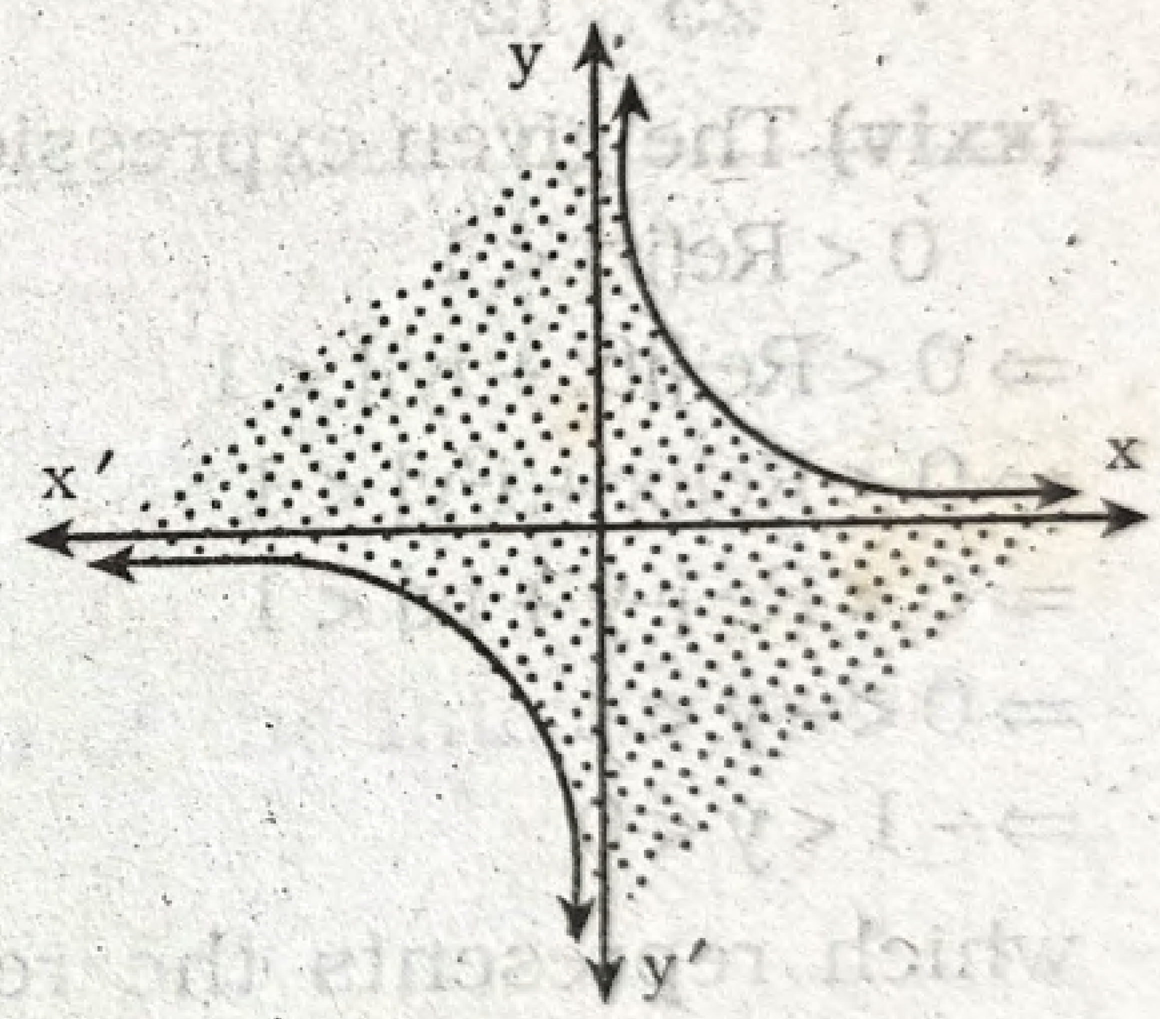
$\Rightarrow Re[x^2 - y^2 + i.2xy] > 1$

$\Rightarrow x^2 - y^2 > 1$



which represents the region of all exterior points of the hyperbola $x^2 - y^2 = 1$.

(xxii) The given expression is



$Im(z^2) \le 1$

$\Rightarrow Im(x + iy)^2 \le 1$,

where $z = x + iy$

$\Rightarrow Im(x^2 - y^2 + i.2xy) \le 1$

$\Rightarrow 2xy \le 1$

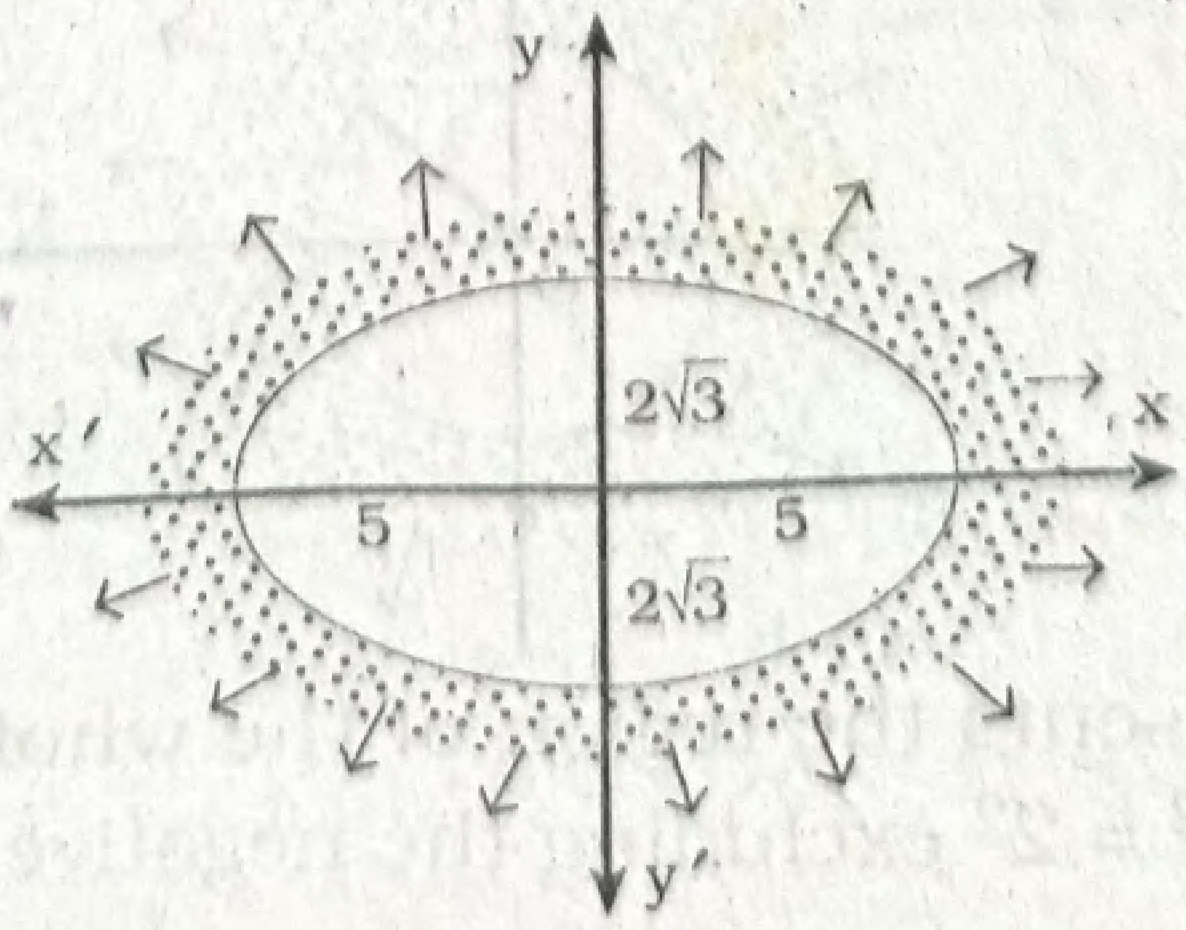
$\Rightarrow xy \le \frac{1}{2}$

which represents the region of all interior points of the hyperbola $xy = \frac{1}{2}$.

(xxiii) The given expression is

$|z + 2 - 3i| + |z - 2 + 3i| < 10$

$\Rightarrow |x + iy + 2 - 3i| + |x + iy - 2 + 3i| < 10$

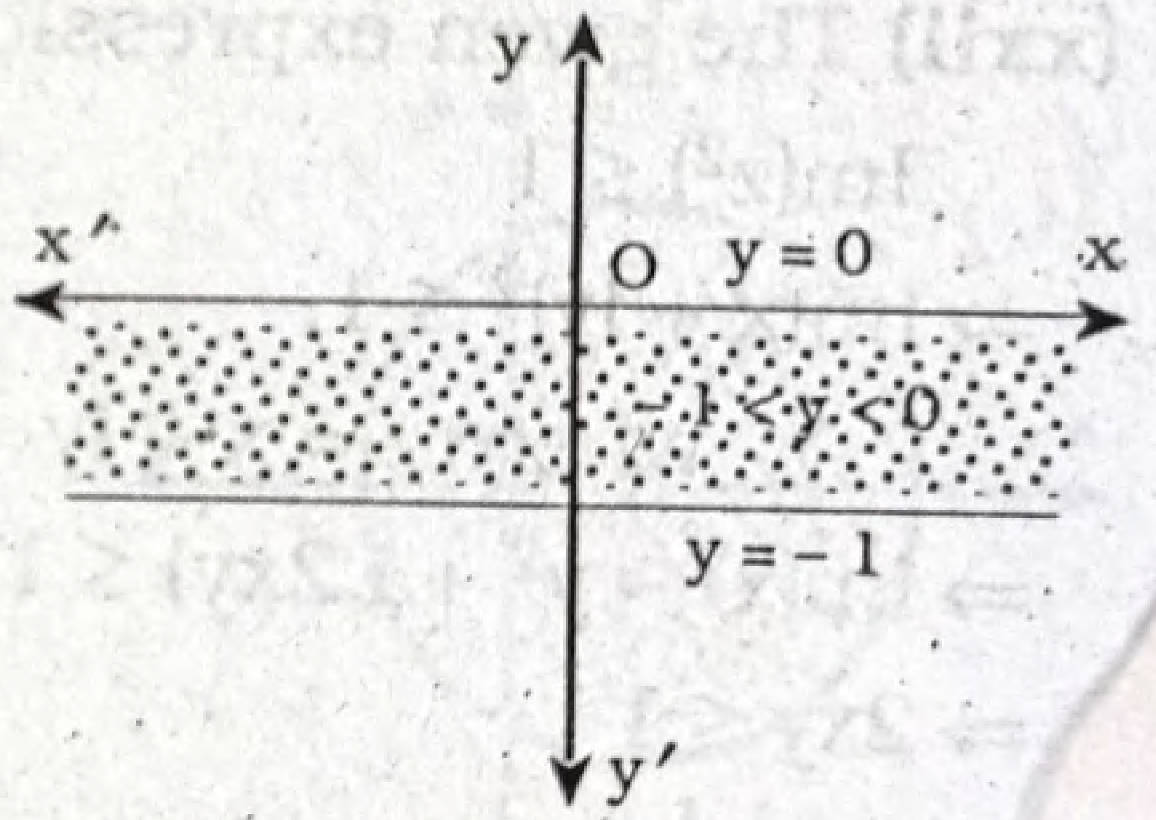


$$\Rightarrow |(x+2)+i(y-3)| + |(x-2)+i(y+3)| < 10$$
$$\Rightarrow \sqrt{(x+2)^2+(y-3)^2} + \sqrt{(x-2)^2+(y+3)^2} < 10$$
$$\Rightarrow (x+2)^2+(y-3)^2 < 100 - 20\sqrt{(x-2)^2+(y+3)^2} + (x-2)^2+(y+3)^2$$
$$\Rightarrow (x+2)^2-(x-2)^2+(y-3)^2-(y+3)^2-100 < -20\sqrt{(x-2)^2+(y+3)^2}$$
$$\Rightarrow 8x-12y-100 < -20\sqrt{(x-2)^2+(y+3)^2}$$
$$\Rightarrow 2x-3y-25 < -5\sqrt{(x-2)^2+(y+3)^2}$$
$$\Rightarrow 4x^2+9y^2+625-12xy-100x+150y < 25[(x-2)^2+(y+3)^2]$$
$$\Rightarrow 21x^2+16y^2+12xy-300 > 0$$
$$\Rightarrow \frac{x^2}{25}+\frac{y^2}{12} > 1 \Rightarrow \frac{x^2}{5^2}+\frac{y^2}{(2\sqrt{3})^2} > 1$$

which represents the region of the circle exterior points of the ellipse $\frac{x^2}{25}+\frac{y^2}{12}=1$.

(xxiv) The given expression is

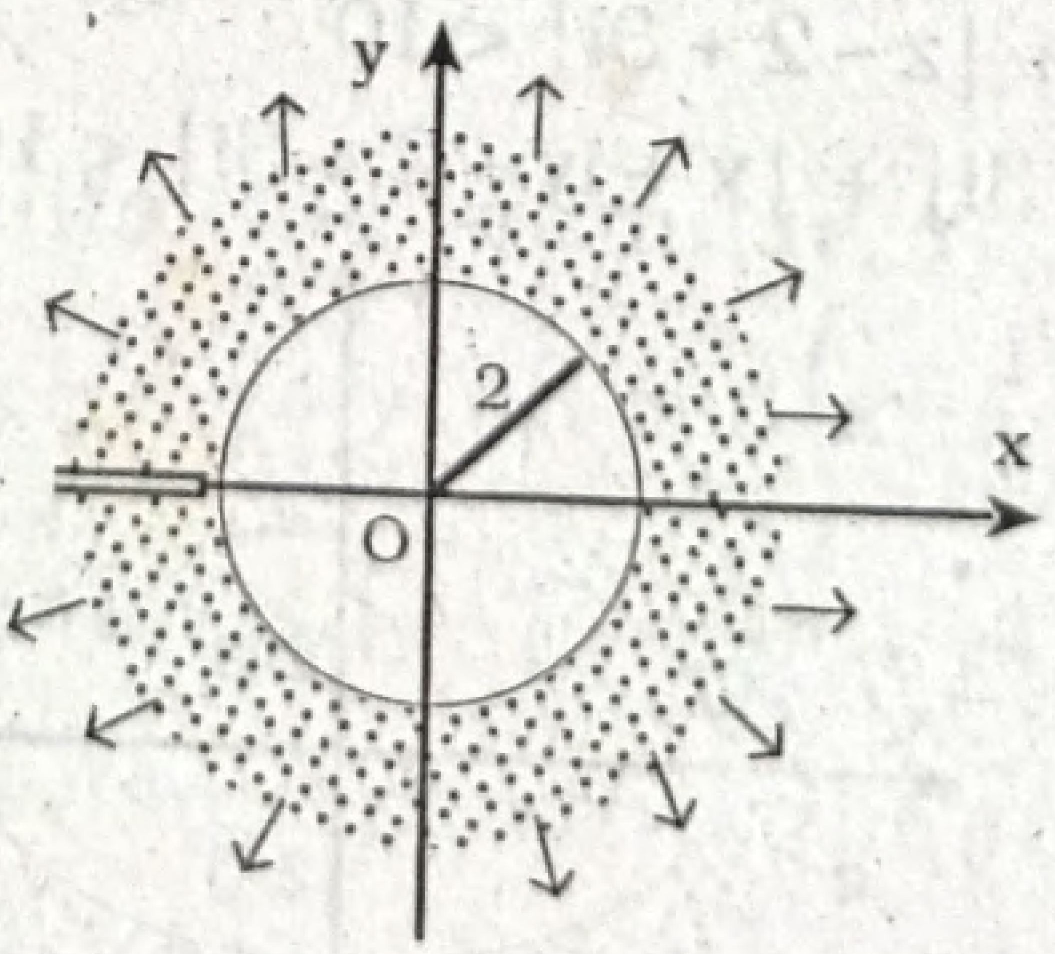
$$0 < \text{Re}(iz) < 1$$
$$\Rightarrow 0 < \text{Re}\,[i(x+iy)] < 1$$
$$\Rightarrow 0 < \text{Re}[ix+i^2y] < 1$$
$$\Rightarrow 0 < \text{Re}\,[-y+ix] < 1$$
$$\Rightarrow 0 < -y < 1$$
$$\Rightarrow -1 < y < 0$$



which represents the region between the lines $y = -1$ and $y = 0$.

(xxv) Then given expression is

$$-\pi < \arg(z) < \pi,\ |z| > 2$$
$$\Rightarrow -\pi < \theta < \pi,\ x^2+y^2 > 2^2, \text{ where } z = x+iy = re^{i\theta},\ \arg(z) = \theta.$$



which represents the region of the whole exterior points of the circle $x^2+y^2=2^2$ excluding the negative part of x-axis.

(xxvi) The given expression is

$$|z-2|-|z+2|>3$$

$\Rightarrow |x+iy-2|-|x+iy+2|>3$, where $z=x+iy$

$\Rightarrow |(x-2)+iy|-|(x+2)+iy|>3$

$\Rightarrow \sqrt{(x-2)^2+y^2}-\sqrt{(x+2)^2+y^2}>3$

$\Rightarrow (x-2)^2+y^2>9-6\sqrt{(x+2)^2+y^2}+(x+2)^2+y^2$

$\Rightarrow (x+2)^2-(x-2)^2+9<6\sqrt{(x+2)^2+y^2}$

$\Rightarrow 8x+9<6\sqrt{(x+2)^2+y^2}$

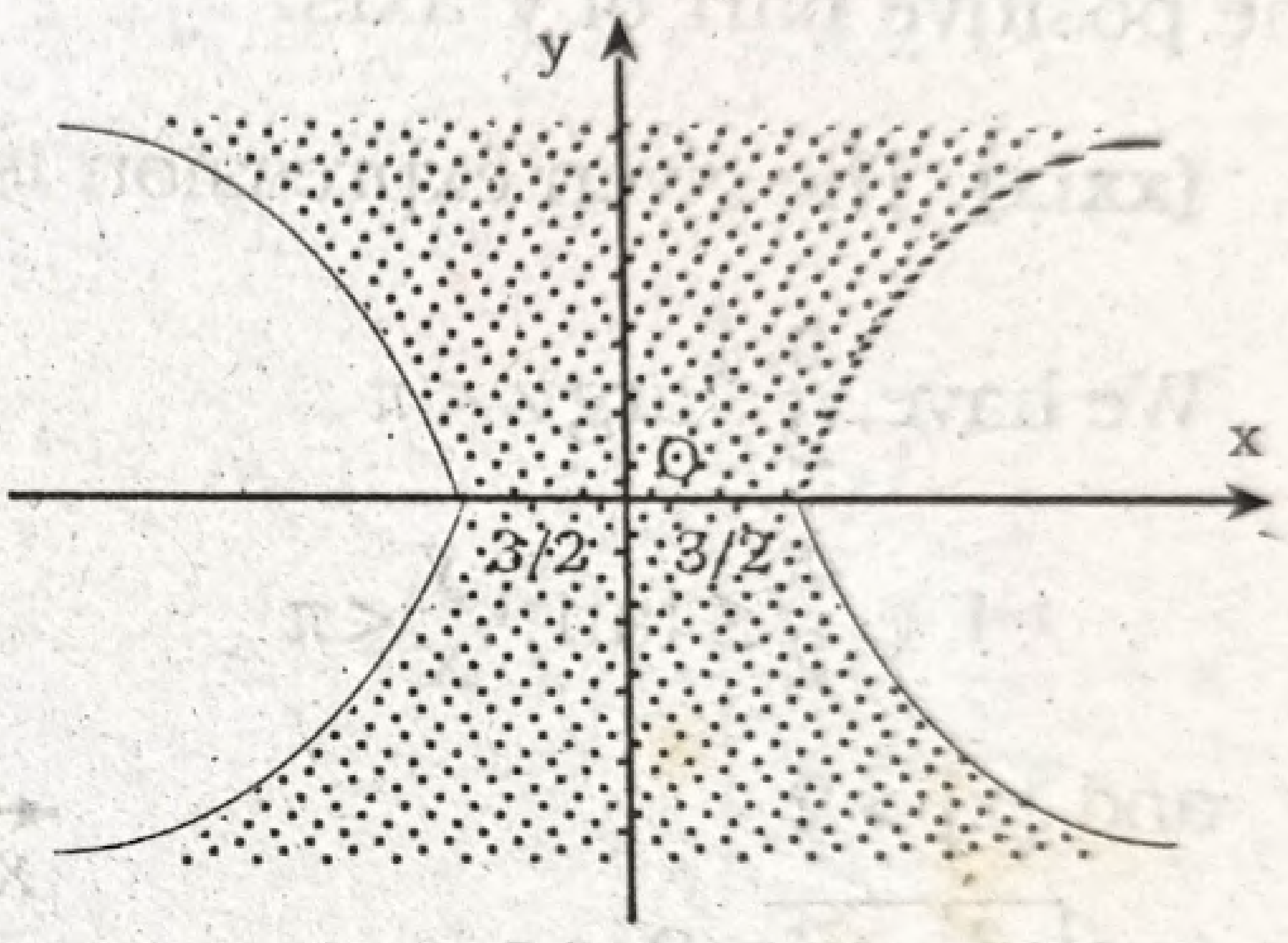
$\Rightarrow 64x^2+144x+81<36x^2+144x+144+36y^2$

$\Rightarrow 28x^2-36y^2<63$

$\Rightarrow \frac{28x^2}{63}-\frac{36y^2}{63}<1$

$\Rightarrow \frac{x^2}{9/4}-\frac{y^2}{7/4}<1$

$\Rightarrow \frac{x^2}{(3/2)^2}-\frac{y^2}{(\sqrt{7}/2)^2}<1$



which represents the region of the interior points of the hyperbola $\frac{x^2}{(3/2)^2}-\frac{y^2}{(\sqrt{7}/2)^2}=1$.

(xxvii) The given expression $0<\arg z\leq\frac{\pi}{2}$, $|z|>2$.

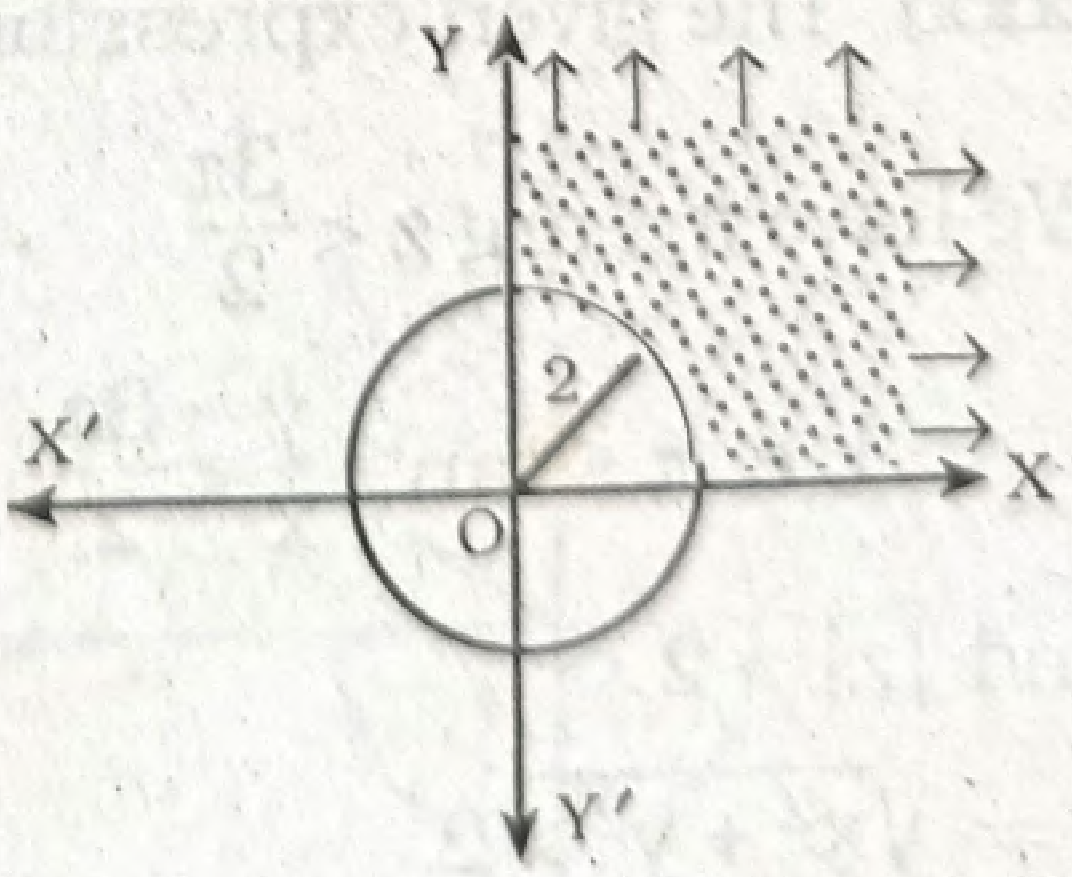
We have $0<\arg z\leq\frac{\pi}{2}$

i, e. $0<\tan^{-1}\frac{y}{x}\leq\frac{\pi}{2}$

and $|z|>2$

or, $\sqrt{x^2+y^2}>2$

or, $x^2+y^2>2^2$



The expression represents a region in the 1st quadrant in the exterior of the circle $x^2+y^2=4$ excluding the positive part of x-axis and including the positive part of y-axis.

(xxviii) The given expression is

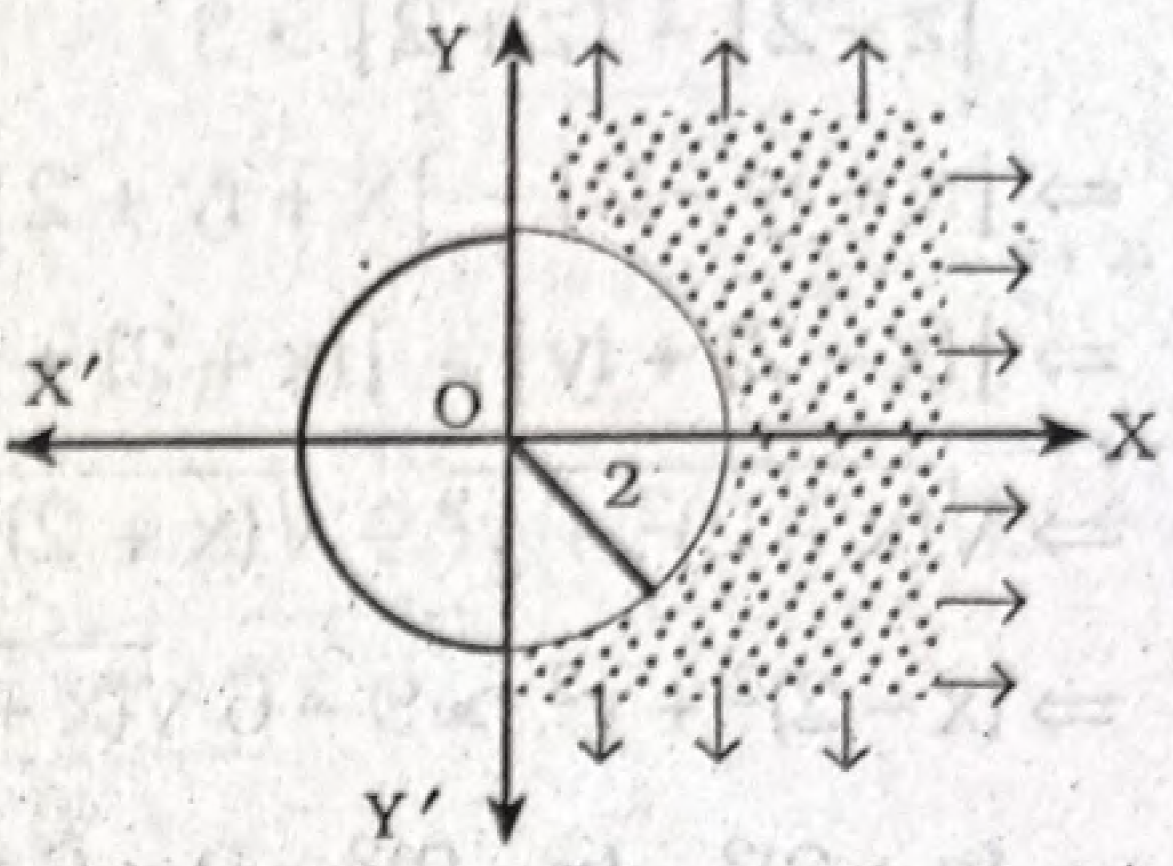
$-\frac{\pi}{2} \le \arg z < \frac{\pi}{2}, \ |z| > 2$

i, e. $-\frac{\pi}{2} \le \tan^{-1}\frac{y}{x} < \frac{\pi}{2}$

and, $|z| > 2$

or, $\sqrt{x^2 + y^2} > 2$

or, $x^2 + y^2 > 4$



The given expression represents a region in the first quadrant and fourth quadrant in the exterior of the circle $x^2 + y^2 = 4$ including the negative part of y-axis and excluding the positive part of y-axis.

(xxix) The given expression is $\frac{\pi}{2} \le \arg z < \pi, \ |z| > 2$

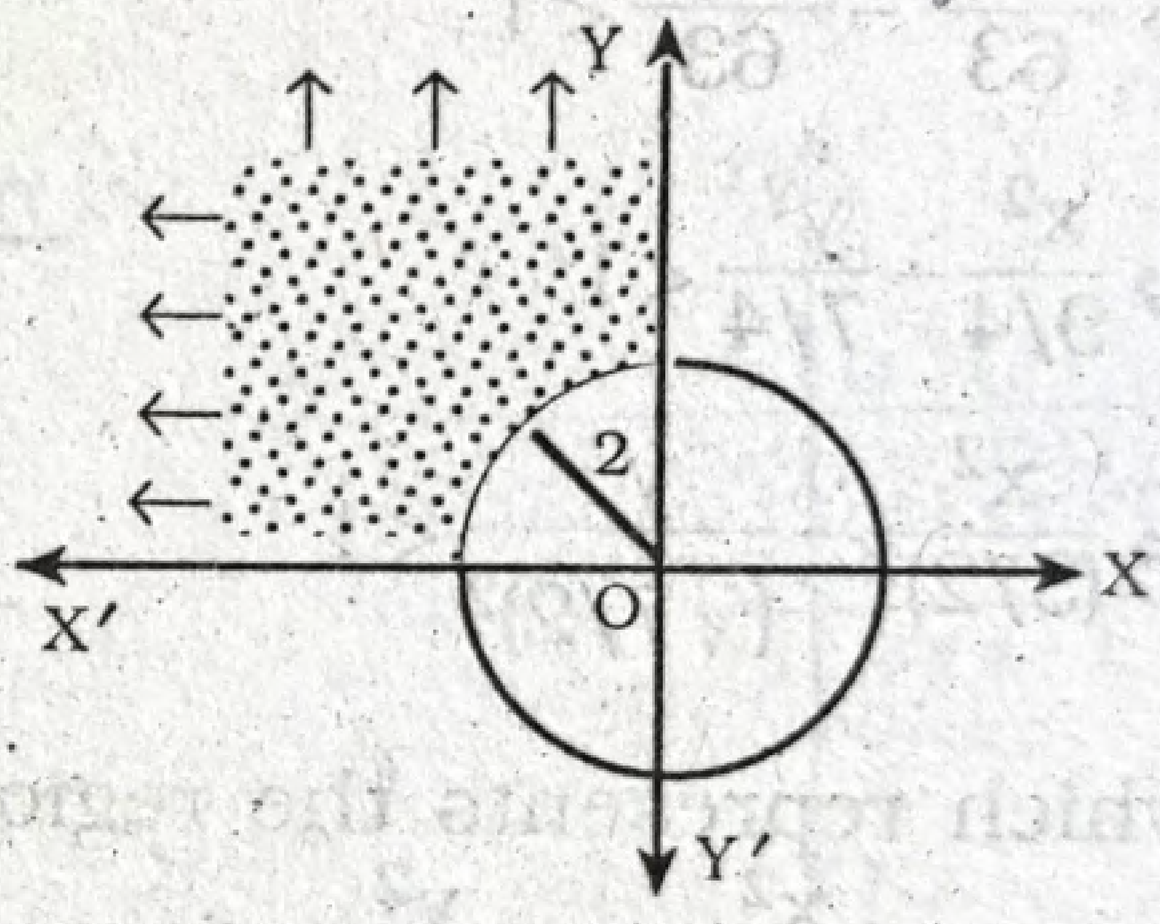
We have, $\frac{\pi}{2} \le \arg z < \pi$

i, e. $\frac{\pi}{2} \le \tan^{-1}\frac{y}{x} < \pi$

and $|z| > 2$

$\Rightarrow \sqrt{x^2 + y^2} > 2$

$\Rightarrow x^2 + y^2 > 4$



The expression represents a region in the 2nd quadrant in the exterior of the circle $x^2 + y^2 = 4$ including the positive part of y-axis and excluding the negative part of x-axis.

(xxx) The given expression $\pi \le \arg z < \frac{3\pi}{2}, \ |z| > 2.$

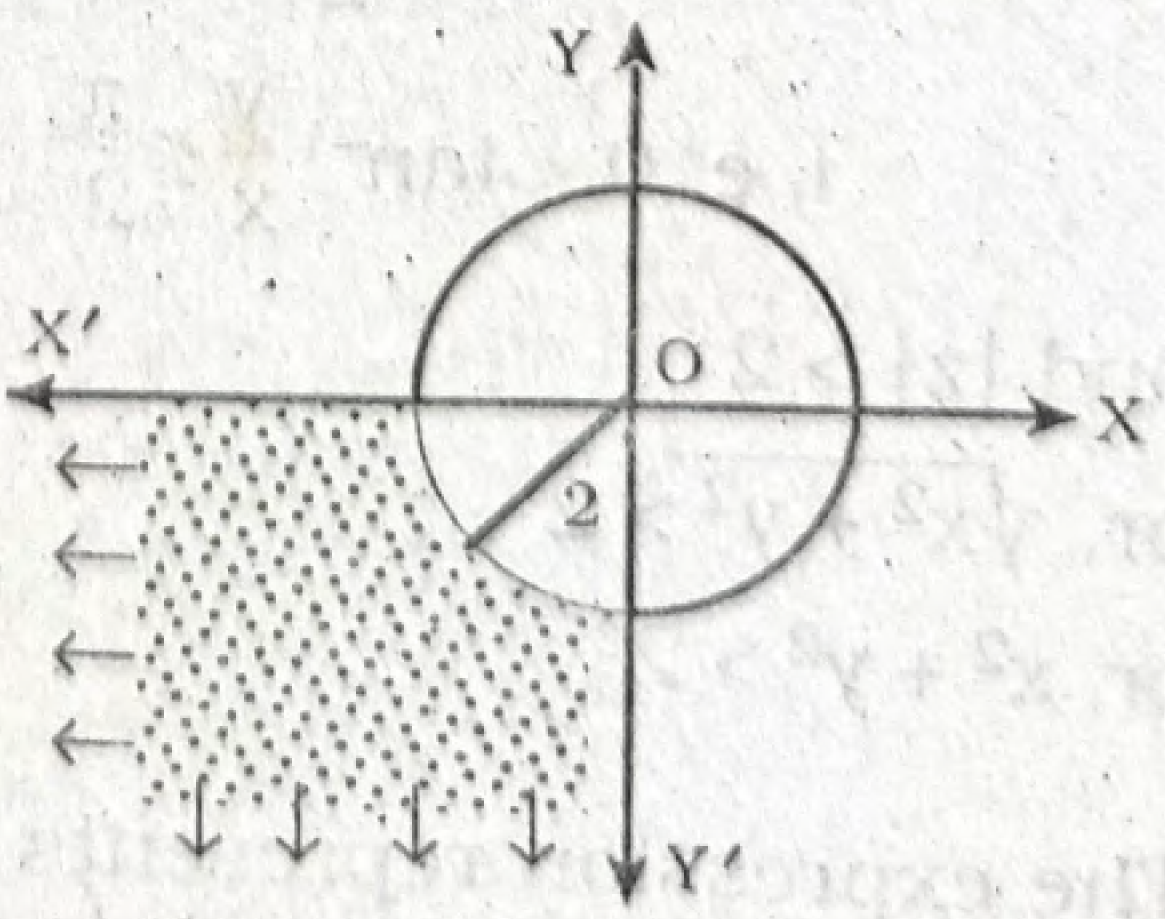
We have $\pi \le \arg z < \frac{3\pi}{2}$

i, e. $\pi \le \tan^{-1}\frac{y}{x} < \frac{3\pi}{2}$

and $|z| > 2$

$\Rightarrow \sqrt{x^2 + y^2} > 2$

$\Rightarrow x^2 + y^2 > 4$



The expression represents a region in the 3rd quadrant in the exterior of the circle $x^2 + y^2 = 4$ including the negative part of x-axis and excluding the negative part of y-axis.

(xxxi) The given expression is $\frac{3\pi}{2} \leq \arg z < 2\pi;\ |z| > 2.$

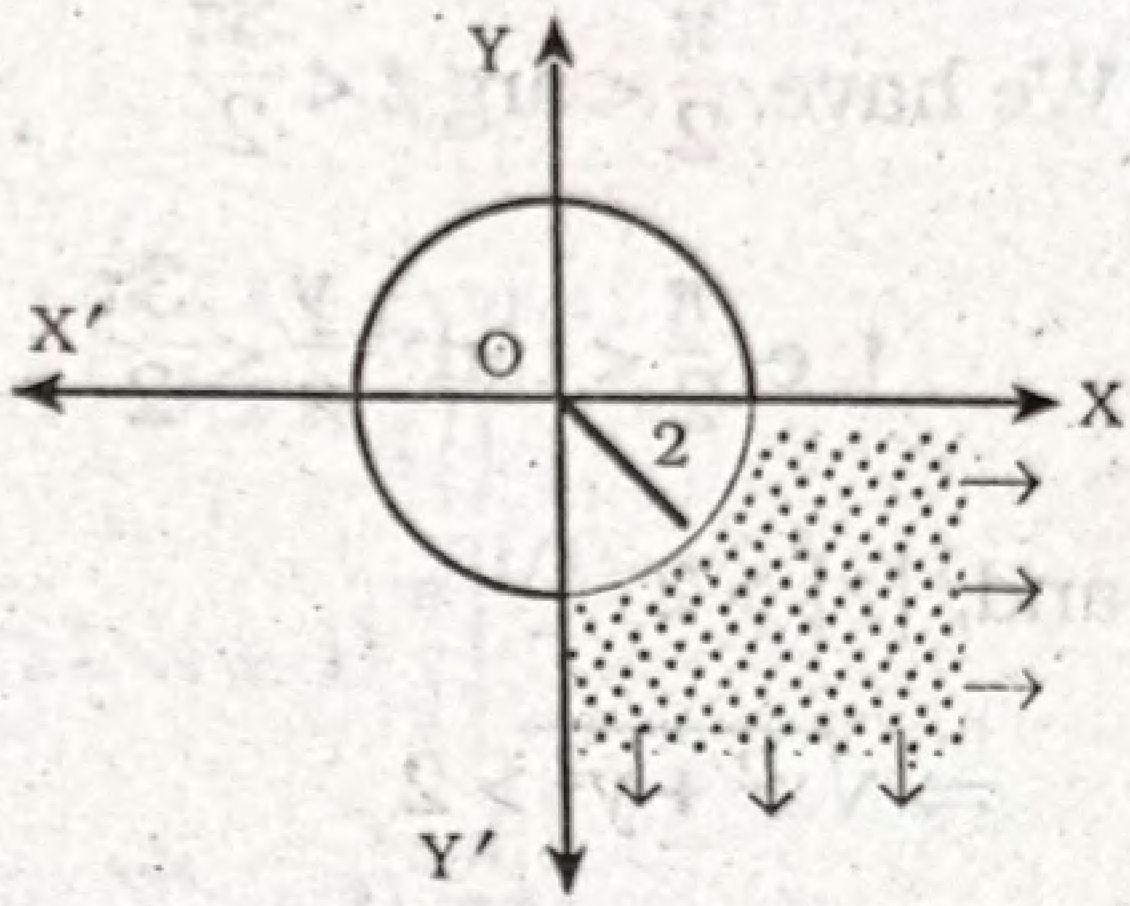
We have $\frac{3\pi}{2} \leq \arg z < 2\pi$

i, e. $\frac{3\pi}{2} \leq \tan^{-1}\frac{y}{x} < 2\pi$

And, $|z| > 2$

or $\sqrt{x^2 + y^2} > 2$

or $x^2 + y^2 > 4$



The expression represents a region in the 4th quadrant in the exterior of the circle $x^2 + y^2 = 4$ including the negative part of y-axis and excluding the positive part of x-axis.

(xxxii) The given expression is

$0 < \arg z < 2\pi;\ |z| > 2.$

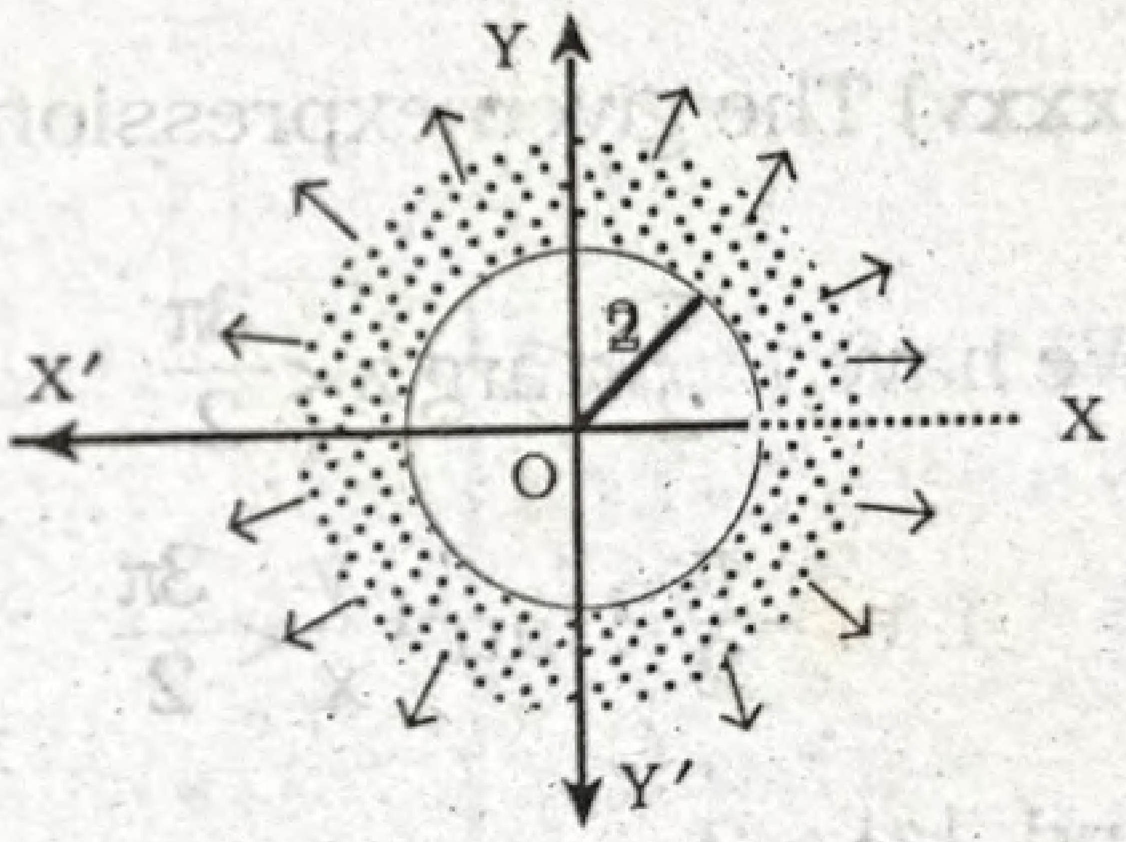
We have $0 < \arg z < 2\pi$

i, e. $\therefore 0 < \tan^{-1}\frac{y}{x} < 2\pi$

and, $|z| > 2$

or, $\sqrt{x^2 + y^2} > 2$

or, $x^2 + y^2 > 4$



The expression represents a region in the whole exterior part of the circle $x^2 + y^2 = 4$ excluding the positive part of x-axis.

(xxxiii) The given expression is

$0 < \arg z < \pi;\ |z| > 2$

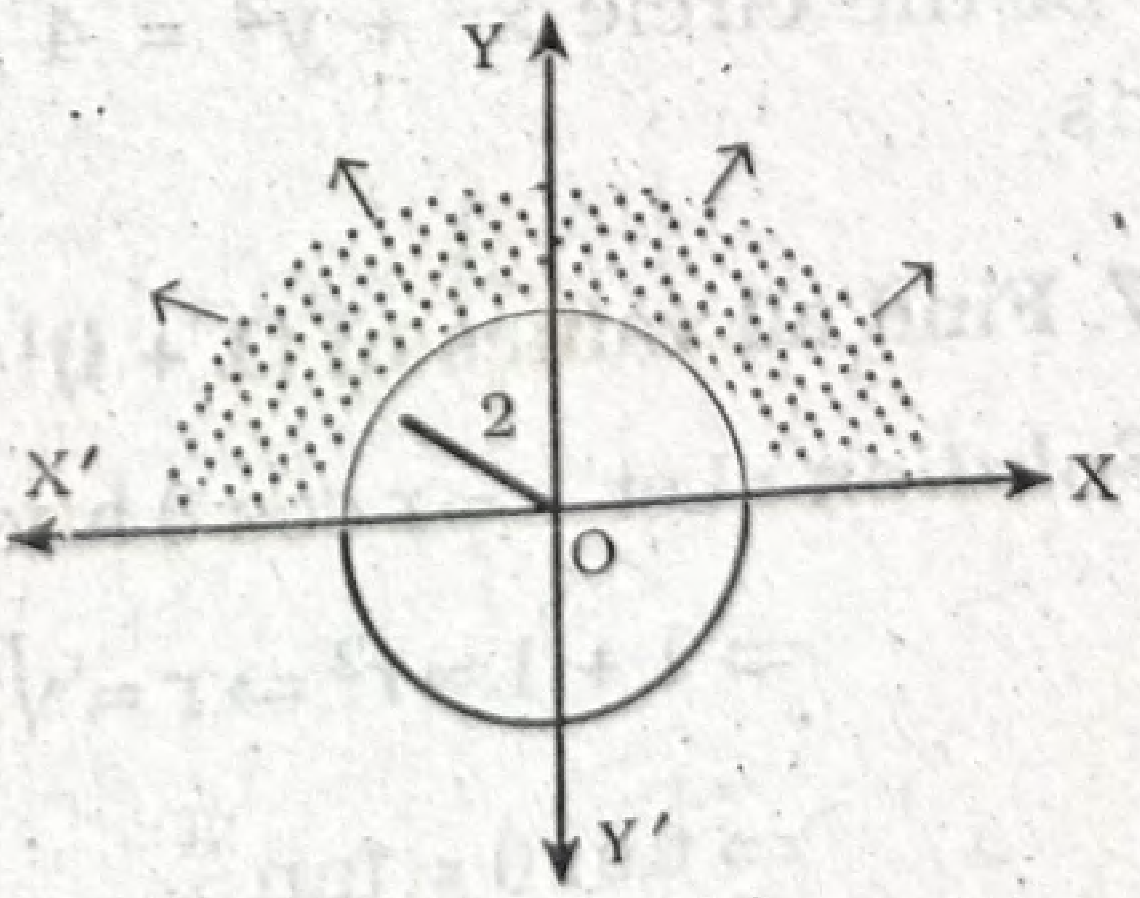
We have, $0 < \arg z < \pi$

i. e. $0 < \tan^{-1}\frac{y}{x} < \pi$

and, $|z| > 2$

$\Rightarrow \sqrt{x^2 + y^2 -} > 2$

$\Rightarrow x^2 + y^2 > 4$



The expression represents a region in the 1st and 2nd quadrant in the exterior of the circle $x^2 + y^2 = 4$ excluding x-axis.

(xxxiv) The given expression is $\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$; $|z| > 2$

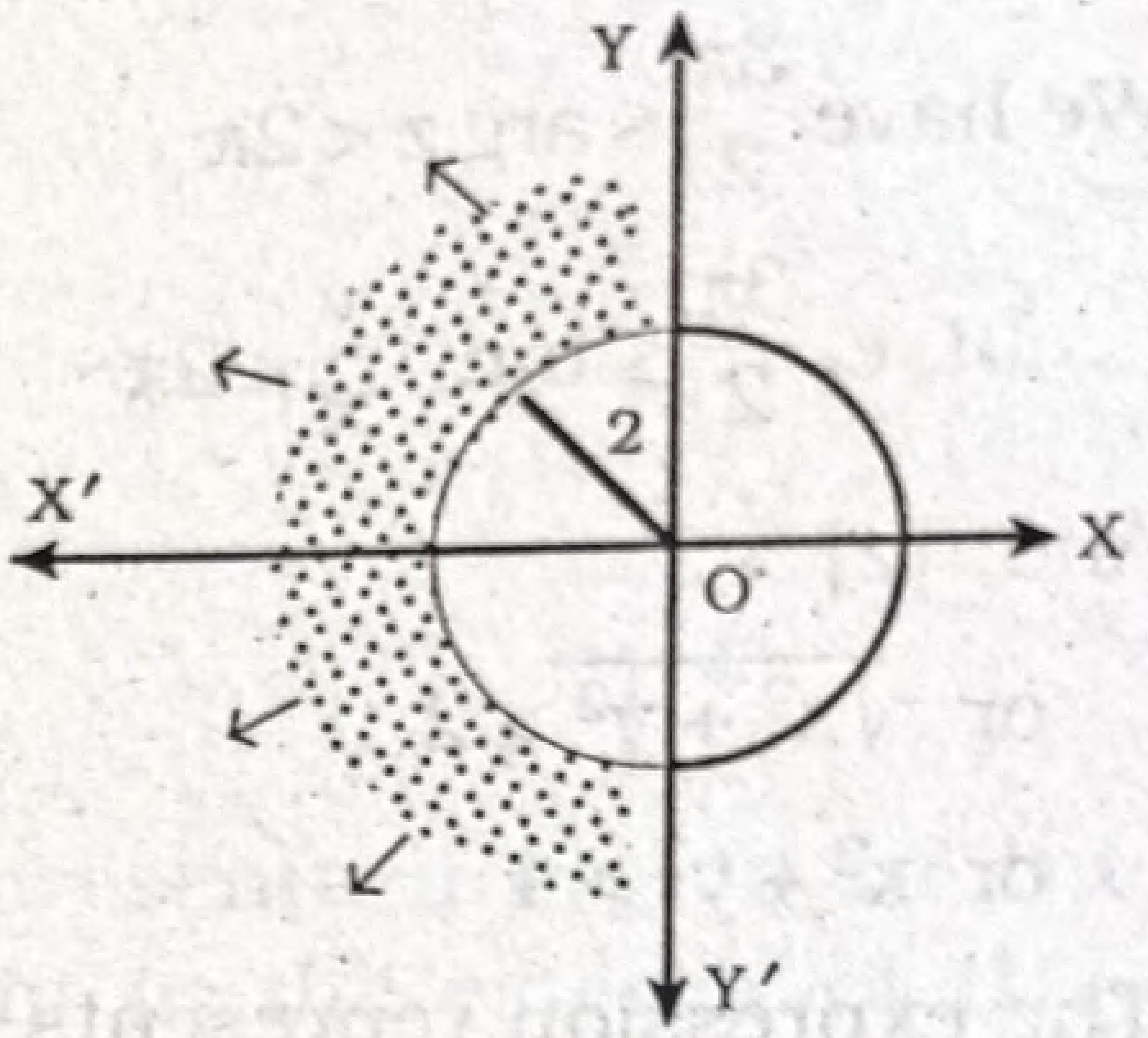
We have, $\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

i, e. $\frac{\pi}{2} < \tan^{-1}\frac{y}{x} < \frac{3\pi}{2}$

and, $|z| > 2$

$\Rightarrow \sqrt{x^2 + y^2} > 2$

$\Rightarrow x^2 + y^2 > 2^2$



The expression represents a region in the 2nd and 3rd quadrant in the exterior of the circle $x^2 + y^2 = 4$ excluding y-axis.

(xxxv) The given expression is $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$; $|z| > 2$.

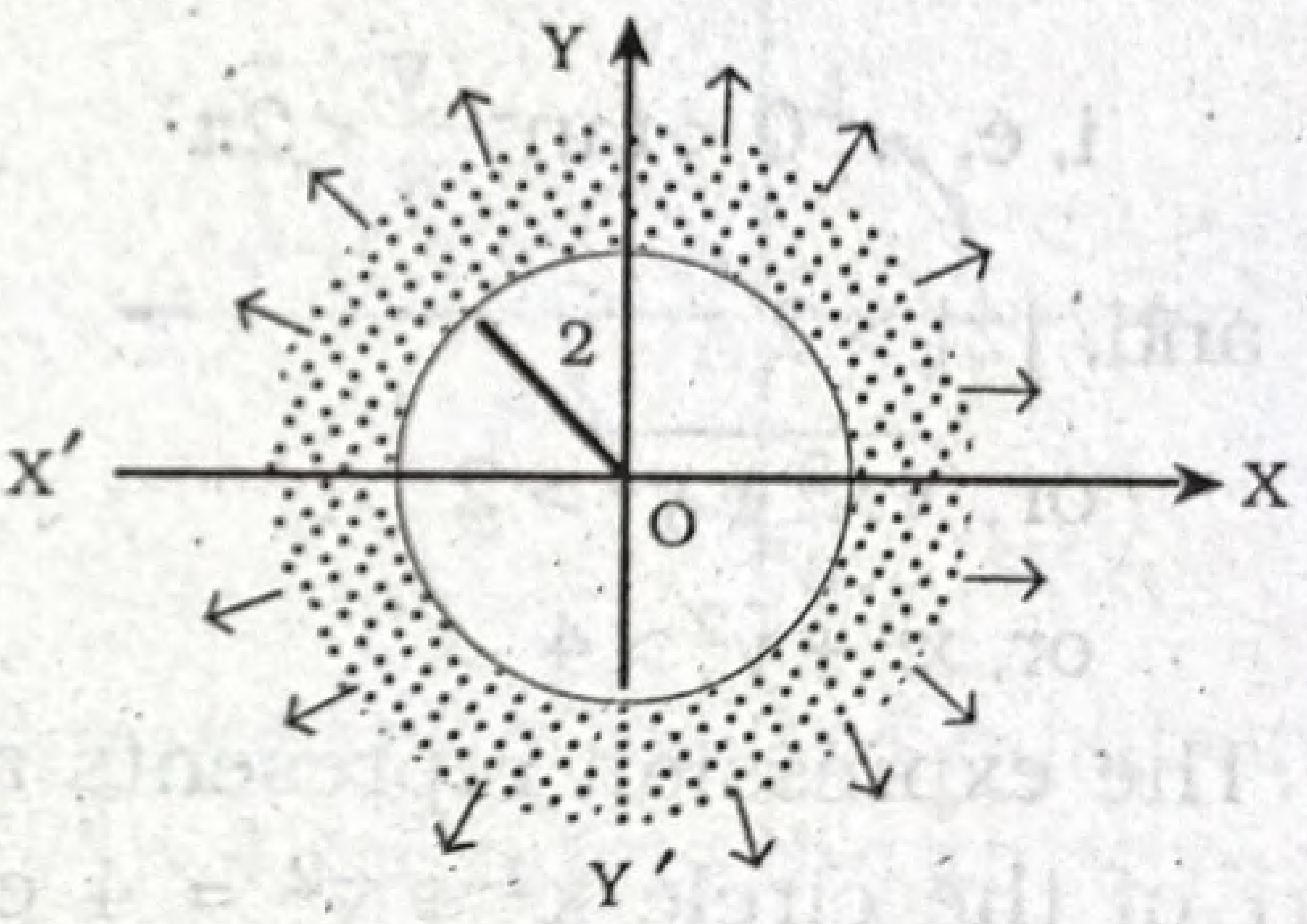
We have, $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

i, e. $-\frac{\pi}{2} < \tan^{-1}\frac{y}{x} < \frac{3\pi}{2}$

and, $|z| > 2$

$\Rightarrow \sqrt{x^2 + y^2} > 2$

$\Rightarrow x^2 + y^2 > 4$



The expression represents a region in the whole exterior part of the circle $x^2 + y^2 = 4$ excluding the negative part of y-axis.

7. Find all values of $(1 + i)^{1/4}$. **[N. U. H. 2001, 2005]**

Solution : Let $1 = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow 1 + 1 = r^2 \Rightarrow r = \sqrt{2}$ and $\tan\theta = 1$

$\Rightarrow \tan\theta = \tan\frac{\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{4}$

Let $z = (1 + i)^{1/4}$

$$= (r\cos\theta + i\,.\,r\sin\theta)^{1/4}$$

$$= 2^{1/8}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{1/4}$$

$$= 2^{1/8}\left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/4}$$

$$= 2^{1/8}\left[\cos(8k+1)\frac{\pi}{4} + i\sin(8k+1)\frac{\pi}{4}\right]^{1/4}$$

$$= 2^{1/8}\left[\cos(8k+1)\frac{\pi}{16} + i\sin(8k+1)\frac{\pi}{16}\right]$$

[by De-Moiver's th^m]

where k = 0, 1, 2, 3.

8. Find all values of $(-8i)^{1/3}$ or the cube roots of– 8i.

Solution : Let $z = (-8i)^{1/3}$

$$= \{0 + i(-8)\}^{1/3}$$

$$\therefore z = \{r\cos\theta + i.r\sin\theta\}^{1/3}$$

$$= \left[8\cos\left(-\frac{\pi}{2}\right) + i8\sin\left(-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= 8^{1/3}\left[\cos\left(2k\pi - \frac{\pi}{2}\right) + i\sin\left(2k\pi - \frac{\pi}{2}\right)\right]^{1/3}$$

$$= 2\left[\cos(4k-1)\frac{\pi}{2} + i\sin(4k-1)\frac{\pi}{2}\right]^{1/3}$$

$$= 2\left[\cos(4k-1)\frac{\pi}{6} + i\sin(4k-1)\frac{\pi}{6}\right]$$

[By De-Moiven's th^m]

Putting $0 = r\cos\theta$
and $-8 = r\sin\theta$
$\therefore 8^2 = r^2 \Rightarrow r = 8$
and $\tan\theta = \frac{-8}{0}$
$= \tan\left(-\frac{\pi}{2}\right)$
$\Rightarrow \theta = -\pi/2$

where k = 0, 1, 2.

when k = 0, then $z_0 = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$$= 2\left[\frac{\sqrt{3}}{2} - i.\frac{1}{2}\right]$$

$$= \sqrt{3} - i$$

when k = 1, then $z_1 = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$$= 2i$$

CHAPTER-2

ANALYTIC FUNCTION

2.1 : *Differentiability of a Complex function :*

Let S be an open set in the complex plane. Let us suppose that f(z) is a complex function. Then the function f(z) is said to be differentiable at a point z_0 of S if $\underset{z \to z_0}{\text{Lt}} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists,

which is denoted by $f'(z_0)$, here the limit is independent of the path along $z \to z_0$ in the complex plane. So we can write

$$f'(z_0) = \underset{z \to z_0}{\text{Lt}} \frac{f(z) - f(z_0)}{z - z_0}$$

If we put $z - z_0 = \Delta z_0$

$$\Rightarrow z = z_0 + \Delta z_0$$

Limit : if $z \to z_0 \Rightarrow \Delta z_0 \to 0$

then we have $f'(z_0) = \underset{\Delta z_0 \to 0}{\text{Lt}} \dfrac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0}$

If we replace z_0 by z, then we have

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

2.2, (a) : *Definition of single-Valued function :*

If for each value of z, the complex function f(z) corresponds only one value, then f(z) is said to be a single-valued function.

As for example, if $f(z) = z^4$, then to each value of z there is only one value of f(z), hence $f(z) = z^4$ is a single-valued function.

2.2. (b) : *Definition of Multiple–Valued function :*

If for each value of z, the complex function f(z) corresponds mor han one value, then f(z) is called Multiple–valued function.

As for example, if $f(z) = z^{1/3}$, then to each value of z there are three values of f(z), hence $f(z) = z^{1/3}$ is a multiple-valued function of z [In this case f(z) is three-valued function]

2.3 : Derivative of a Complex function :

Let $w = f(z)$ be a complex function. If $f(z)$ is single-valued in some region S of the complex plane [z-plane], then the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

2.3.(a) : $\in-\delta$ definition of the derivative at a point :

If the single-valued function $f(z)$ has a derivative $f'(z_0)$ at the point z_0, $\forall \in > 0 \; \exists \; \delta > 0$ such that

$$\left| \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} - f'(z_0) \right| < \in$$

whenever $|\Delta z_0| < \delta$

2.4(a) : Definition of analytic function at a point :

[N. U. H. 2005, 2009]

A single-valued complex function $f(z)$ is said to be analytic at a point z_0 if it is differentiable in some neighbourhood $|z - z_0| < \delta$ of z_0.

2.4. (b) : Definition of Analytic function :

[N. U. H. 2006]

A singled-valued complex function $f(z)$ is said to be analytic in a domain D if it is analytic at each point of the domain D.

Note : The terms Holomorphic functions and Regular functions are the synonym of Analytic function.

2.4. (c) : Definition of Entire function :

A complex function $f(z)$ is said to be entire if it is analytic in the whole complex plane.

2.5 : Singular point or Singularity : **[N. U. H. 2003]**

If a function $f(z)$ fails to be analytic at a point z_0 but in every neighbourhood of z_0 there exist at least one point where the function is analytic, then z_0 is said to be a **singular point** or **singularity** of $f(z)$.

The function $f(z) = \frac{1}{z}$ is analytic except at $z = 0$, and in each deleted neighbourhood $f(z)$ is analytic, so that by definition $z = 0$ is a singular point.

Again for the function $f(z) = \frac{z^3 + 1}{(z^2 + 4)(z - 1)}$, the points $z = 1$, $z = 2i$ and $z = -2i$ are singular points or singularities of f(z), since the function f(z) is analytic except the points $z = 1, z = \pm 2i$.

2.6(a) : *Laplacian* :

The second order partial differential operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ is called the Laplacian.

2.6. (b) : *Laplace's equation in two dimension* :

The second order partial differential equation $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ $\Rightarrow \nabla^2 \psi = 0$ is called the Laplace's equation in two dimensions, where $\psi(x, y)$ is the real valued function.

2.6.(C) : *Definition of Harmonic function* :

Any real function of two variables [x and y] is said to be **harmonic** in a domain D, if throughout D it has second order continuous partial derivatives and satisfies the Laplace's equation.

2.6. (D) : *Conjugate harmonic function of a harmonic function* : [N. U. H. 2004]

If u(x, y) is a harmonic function, then u(x, y) satisfies the Laplace equation, that is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. In this case there exists an another function v(x, y), such that u + iv is analytic in a given region. Then we have $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$, so v(x, y) is a harmonic function. Again, v(x, y) is a conjugate function of u(x, y)

Then v(x, y) is called **Conjugate harmonic function** of u(x, y).

Note : If the complex function f(z) = u(x, y) + iv(x, y) is analytic. Then v is the conjugate function of u and u is also the conjugate function of v.

2.7 : Cauchy-Riemann equations :

The partial differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called **Cauchy-Riemann** equations of the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic.

2.8 : Cauchy-Riemann equations :

***Theorem-1* : Necessary conditions for f(z) to be analytic.**

[N. 2004(Old) , 2005, 2007, 2009]

Statement : The complex function $w = f(z) = u(x, y) + iv(x, y)$ to be analytic (differentiable) at any point $z = x + iy$ of its domain D is that the four partial derivatives u_x, u_y, v_x and v_y should exist and satisfy the Cauchy-Riemann partial differential equations $u_x = v_y$ and $u_y = -v_x$.

Proof : Let $f(z) = u(x, y) + iv(x, y)$ be analytic at any point $z = x + iy$ of its domain D, therefore,

$$f'(z) = \underset{\partial z \to 0}{Lt} \frac{f(z + \partial z) - f(z)}{\partial z}$$ exists and which is independent of the path along $\partial z \to 0$.

$\because f(z) = u(x, y) + iv(x, y)$

$\Rightarrow f(z + \partial z) = u(x + \partial x, y + \partial y) + iv(x + \partial x, y + \partial y)$

Also $z = x + iy \Rightarrow \partial z = \partial x + i\partial y$

As Limit $\partial z \to 0 \Rightarrow \partial x \to 0$ and $\partial y \to 0$

$$\text{Then } f'(z) = \underset{\substack{\partial x \to 0 \\ \partial y \to 0}}{Lt} \frac{[u(x + \partial x, y + \partial y) + iv(x + \partial x, y + \partial y)] - [u(x, y) + iv(x, y)]}{\partial x + i\partial y}$$

$$= \underset{\substack{\partial x \to 0 \\ \partial y \to 0}}{Lt} \frac{u(x + \partial x, y + \partial y) - u(x, y)}{\partial x + i\partial y} + i\underset{\substack{\partial x \to 0 \\ \partial y \to 0}}{Lt} \frac{v(x + \partial x, y + \partial y) - v(x, y)}{\partial x + i\partial y}$$

If ∂z is purely real, then $\partial z = \partial x$ as $\partial z \to 0 \Rightarrow \partial x \to 0$ and $\partial y = 0$

$$\text{Then } f'(z) = \underset{\partial x \to 0}{Lt} \frac{u(x + \partial x, y) - u(x, y)}{\partial x} + i\underset{\partial x \to 0}{Lt} \frac{v(x + \partial x, y) - v(x, y)}{\partial x}$$

$f'(z) = u_x(x, y) + iv_x(x, y)$ ... ... (i) [by definition]

Again, if ∂z is purely imaginary, then $\partial z = i\partial y$ also as $\partial z \to 0$ $\Rightarrow \partial y \to 0$ and $\partial x = 0$

$$\text{Then } f'(z) = \underset{\partial y \to 0}{\text{Lt}} \frac{u(x, y + \partial y) - u(x, y)}{i\partial y} + i\underset{\partial y \to 0}{\text{Lt}} \frac{v(x, y + \partial y) - v(x, y)}{i\partial y}$$

$$= \frac{1}{i}\underset{\partial y \to 0}{\text{Lt}} \frac{u(x, y + \partial y) - u(x, y)}{\partial y} + \underset{\partial y \to 0}{\text{Lt}} \frac{v(x, y + \partial y) - v(x, y)}{\partial y}$$

$$= \frac{-i^2}{i} u_y(x, y) + v_y(x, y)$$

$$\therefore f'(z) = v_y(x, y) - iu_y(x, y) \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

From the equations (i) and (ii), we get

$$u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y)$$

$$\Rightarrow u_x = v_y \text{ and } u_y = -v_x.$$

Which are known as Cauchy-Riemann partial differential equations.

***Theorem–2* : Sufficient condition for f(z) to be analytic.**

[N. U. H. 2006]

Statement : The function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, if

(i) u, v are differentiable in D and $u_x = v_y$; $u_y = -v_x$

(ii) The partial derivatives u_x, u_y, v_x and v_y all are continuous in D.

***Proof* :** Let $w = u + iv$ ... (i)

$$\Rightarrow \partial w = \partial u + i\partial v \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Also, we have $u = u(x, y)$ ... ... (iii)

$$\Rightarrow u + \partial u = u(x + \partial x, y + \partial y)$$

$$\Rightarrow u + \partial u - u = u(x + \partial x, y + \partial y) - u(x, y)$$

$$\Rightarrow \partial u = [u(x + \partial x, y + \partial y) - u(x, y + \partial y)] + [u(x, y + \partial y) - u(x, y)]$$

$$\Rightarrow \partial u = \partial x\, u_x(x + \theta_1 \partial x, y + \partial y) + \partial y\, u_y(x, y + \theta_2 \partial y) \quad \ldots \quad \ldots \quad \text{(iv)}$$

Whe[illegible] $< \theta_1 < 1$ and $0 < \theta_2 < 1$ [By Mean Value theorem]

Again, u_x and u_y are given to be continuous so that

$$|u_x(x + \theta_1 \partial x, y + \partial y) - u_x(x, y)| < \epsilon$$

and $|u_y(x, y + \theta_2 \partial y) - u_y(x, y)| < \eta$

If $\epsilon_1 < \epsilon$ and $\eta_1 < \eta$, then we can write

$$\left.\begin{aligned} u_x(x + \theta_1 \partial x, y + \partial y) - u_x(x, y) &= \epsilon_1 \\ \text{and } u_y(x, y + \theta_2 \partial y) - u_y(x, y) &= \eta_1 \end{aligned}\right\} \quad \cdots \quad \cdots \quad \text{(v)}$$

Using (v), we get from (iv)

$$\partial u = \partial x\{u_x(x, y) + \epsilon_1\} + \partial y\{u_y(x, y) + \eta_1\}$$

$$\Rightarrow \partial u = u_x \partial x + u_y \partial y + \epsilon_1 \partial x + \eta_1 \partial y \quad \cdots \quad \cdots \quad \text{(vi)}$$

Similarly, we get

$$\partial v = v_x \partial x + v_y \partial y + \epsilon_2 \partial x + \eta_2 \partial y \cdots \quad \text{(vii)}$$

Using (vi) and (vii), we get from (i)

$$\partial w = (u_x \partial x + u_y \partial y + \epsilon_1 \partial x + \eta_1 \partial y)$$

$$+ i(v_x \partial x + v_y \partial y + \epsilon_2 \partial x + \eta_2 \partial y)$$

$$= (u_x + iv_x) \partial x + (u_y + iv_y) \partial y + (\epsilon_1 + i\epsilon_2) \partial x + (\eta_1 + i\eta_2) \partial y$$

$$= (u_x + iv_x) \partial x + (-v_x + iu_x) \partial y + \epsilon_3 \partial x + \eta_3 \partial y$$

[$\because u_x = v_y$ and $u_y = -v_x$] and where $\epsilon_1 + i\epsilon_2 = \epsilon_3$

$\eta_1 + i\eta_2 = \eta_3$

$$\Rightarrow \partial w = (u_x + iv_x) \partial x + (i^2 v_x + iu_x) \partial y + \epsilon_3 \partial x + \eta_3 \partial y$$

$$= (u_x + iv_x) \partial x + i(u_x + iv_x) \partial y + \epsilon_3 \partial x + \eta_3 \partial y$$

$$\partial w = (u_x + iv_x)(\partial x + i\partial y) + \epsilon_3 \partial x + \eta_3 \partial y$$

Dividing throughout by $\partial z = \partial x + i\partial y$

$$\therefore \frac{\partial w}{\partial z} = \frac{(u_x + iv_x)(\partial x + i\partial y)}{(\partial x + i\partial y)} + \frac{\epsilon_3 \partial x + \eta_3 \partial y}{\partial x + i\partial y}$$

Taking limit $\partial z \to 0 \Rightarrow \partial x \to 0$ and $\partial y \to 0$, $\epsilon_3 \to 0$, $\eta_3 \to 0$

Then $\displaystyle \mathop{Lt}_{\partial z \to 0} \frac{\partial w}{\partial z} = u_x + iv_x$

$$\Rightarrow \frac{dw}{dz} = u_x + iv_x$$

$$\Rightarrow \frac{d}{dz}\{f(z) = u_x + iv_x$$

$$\Rightarrow f'(z) = u_x + iv_x \quad \cdots \quad \cdots \quad \cdots \quad (✪)$$

Since u_x and v_x exist and are unique, so we conclude that $f'(z)$ exists, that is f(z) is analytic at an arbitrary point z of D.

Hence f(z) is analytic in the domain D.

Cor–I : From (✪) we have $\frac{dw}{dz} = u_x + iv_x$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial x}(u + iv)$$

$$\boxed{\frac{dw}{dz} = \frac{\partial w}{\partial x}} \quad \ldots \quad \ldots \quad (A)$$

Cor–II : Again from (✪) we have

$$\frac{dw}{dz} = u_x + iv_x$$

$$= v_y + i(-u_y) \quad \text{[By Cauchy-Riemann equations]}$$

$$= - iu_y + v_y$$

$$= - iu_y - i^2v_y$$

$$= - i(u_y + iv_y)$$

$$= - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= - i\frac{\partial}{\partial y}(u + iv)$$

$$\boxed{\frac{dw}{dz} = - i\frac{\partial w}{\partial y} = \frac{1}{i}\frac{\partial w}{\partial y}} \quad \ldots \quad \ldots \quad \ldots \quad (B)$$

Note : Equations (A) and (B) should be kept in mind.

2.9 : *Cauchy-Riemann equations in Polar form :*

[N. U. H–2003, 2004, 2008]

Statement **:** If w = f(z) is an analytic function, then the Cauchy-Riemann equations in Polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof **:** We know that the Cauchy-Riemann equations in Cartesian Co-ordinates are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Again, we know that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial x}{\partial r} &= \cos\theta \\ \text{and } \frac{\partial x}{\partial \theta} &= -r\sin\theta \end{aligned}\right\} \text{...... (iii)} \qquad \left.\begin{aligned} \frac{\partial y}{\partial r} &= \sin\theta \\ \text{and } \frac{\partial y}{\partial \theta} &= r\cos\theta \end{aligned}\right\} \text{...... (iv)}$$

But we have, $u = u(x, y)$.

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \text{[By chain rule]}$$

$$= \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta \quad \text{[by (iii) and (iv)]}$$

$$\therefore \frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \quad \dots \quad \dots \quad \dots \quad \text{(v)}$$

$$\text{And } \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} \quad \text{[By chain rule]}$$

$$= \frac{\partial u}{\partial x}\cdot(-r\sin\theta) + \frac{\partial u}{\partial y}\cdot r\cos\theta \quad \text{[by (iii) and (iv)]}$$

$$\therefore \frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} \quad \dots \quad \dots \quad \dots \quad \text{(vi)}$$

Again, we have

$$v = v(x, y)$$

$$\Rightarrow \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \text{[By chain rule]}$$

$$\Rightarrow \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\cos\theta + \frac{\partial v}{\partial y}\cdot\sin\theta \quad \text{[by (iii) and (iv)]}$$

$$\Rightarrow \frac{\partial v}{\partial r} = \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial r} = \cos\theta\left(-\frac{\partial u}{\partial y}\right) + \sin\theta\frac{\partial u}{\partial x} \quad \text{[by (i) and (ii)]}$$

$$\Rightarrow \frac{\partial v}{\partial r} = -\left[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow -\frac{\partial v}{\partial r} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y} \quad \dots \quad \dots \quad \dots \quad \text{(vii)}$$

$$\text{and } \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} \quad \text{[By chain rule]}$$

$$\Rightarrow \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot(-r\sin\theta) + \frac{\partial v}{\partial y}\cdot r\cos\theta \quad \text{[by (iii) and (iv)]}$$

$$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\sin\theta \frac{\partial v}{\partial x} + \cos\theta . \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = -\sin\theta . \left(-\frac{\partial u}{\partial y}\right) + \cos\theta \left(\frac{\partial u}{\partial x}\right) \quad \text{[by (i) and (ii)]}$$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \cos\theta . \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

From (v) and (viii), we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

and from (vi) and (vii), we get

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \ldots \quad \ldots \quad \ldots \quad \text{(x)}$$

Thus equations (ix) and (x) are the Cauchy-Riemann equations in Polar form.

2.10 : *Orthogonal system :*

Definition : Two families of curves $u(x, y) = c_1$ and $u(x, y) = c_2$ are said to be orthogonal to each other if they intersect at right angle at each their points of intersection.

Theorem–3 : If $w = f(z) = u(x, y) + iv(x, y)$ be analytic at $z = x + iy$ then families of curve given by $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system.

Proof : We have

$$u(x, y) = c_1$$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{[by theorem on total derivatives]}$$

$$\Rightarrow u_x\, dx + u_y\, dy = 0$$

$$\Rightarrow u_y\, dy = -u_x\, dx$$

$$\therefore \left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{u_x}{u_y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

Again, $v(x, y) = c_2$

$$\Rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \quad \text{[by theorem on total derivatives]}$$

$$\Rightarrow -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0 \quad \text{[By Cauchy-Riemann equations]}$$

$$\Rightarrow u^2\left(\frac{\partial u}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

$$\Rightarrow (u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0$$

$$\Rightarrow k^2 \; |f'(z)|^2 = 0 \quad \text{[by (i) and (ii)]}$$

$$\Rightarrow |f'(z)|^2 = 0 \quad [\because k^2 \neq 0]$$

$$\Rightarrow f'(z) = 0$$

Integrating, we get f(z) = constant

Hence proved the theorem.

2.12 : *Theorem* : If the complex function f(z) = u(x, y) + iv(x, y) is analytic in a region R, then u(x, y) and v(x, y) are harmonic in R. If they have continuous second partial derivatives in R.

Proof : Let f(z) = u(x, y) + iv(x, y) be an analytic function in R, then the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ...... (i) and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ...... (ii) are satisfied in R.

Let us assume that u and v have the continuous second partial derivatives.

Now differentiating equations (i) and (ii) w. r. to x and y respectively. Then we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \, \partial y} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

$$\text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \, \partial y} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \text{(iv)}$$

Adding (iii) and (iv), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which shows that u(x, y) is harmonic in R.

Again, differentiating partially equation (i) and (ii) w. r. to y and x respectively. Then we have

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial y^2} \quad \cdots \quad \cdots \quad \cdots \quad (v)$$

and $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$

$$\Rightarrow -\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 v}{\partial x^2} \quad \cdots \quad \cdots \quad \cdots \quad (vi)$$

Adding (v) and (vi) we get

$$0 = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Which shows that v(x, y) is harmonic in R.

2.13 : *Determination of the conjugate function :*

If f(z) = u(x, y) + iv(x, y) be an analytic function where both u(x, y) and v(x, y) are conjugate functions. If one of these say u(x, y) be given then we have to determine the other.

We have

$$v = v(x, y)$$

$$\Rightarrow dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \qquad \text{[by the theorem on total derivative]}$$

$$\Rightarrow dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \cdots \quad \cdots \quad \cdots \quad (i)$$

$$\left[\text{by Cauchy-Riemann equations as } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

R. H. S. of (i) is of the form

Mdx + Ndy, where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

$$\therefore \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2} \quad \cdots \quad \cdots \quad (ii)$$

As u(x, y) satisfies Laplaces equation, so that, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

WORKOUT EXAMPLES

Ex–1 : Using the definition, find the derivative of the following functions :

(i) $f(z) = 3z^2 - 2z + 4$

(ii) $f(z) = z^3 + 2z^2 + i$

(iii) $f(z) = z^3 - 2z$

Solution : We know that by definition, the derivative of f(z) at any point z is

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

(i) Given that $f(z) = 3z^2 - 2z + 4$

$$\Rightarrow f(z + \Delta z) = 3(z + \Delta z)^2 - 2(z + \Delta z) + 4$$

$$\Rightarrow f(z + \Delta z) = 3[z^2 + 2z\Delta z + (\Delta z)^2] - 2z - 2\Delta z + 4$$

$$\therefore \ f(z + \Delta z) - f(z) = 3z^2 + 6z\Delta z + 3(\Delta z)^2 - 2z - 2\Delta z + 4 - 3z^2 + 2z - 4$$

$$\Rightarrow f(z + \Delta z) - f(z) = 6z\Delta z + 3(\Delta z)^2 - 2\Delta z \quad \ldots \quad \ldots \quad (2)$$

Using (2), we get from (1)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{6z\Delta z + 3(\Delta z)^2 - 2\Delta z}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{(6z + 3\Delta z - 2)\,\Delta z}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \ 6z + 3\Delta z - 2 = 6z - 2.$$

(ii) Given that, $f(z) = z^3 + 2z^2 + i$

$$\Rightarrow f(z + \Delta z) = (z + \Delta z)^3 + 2(z + \Delta z)^2 + i$$

$$\therefore \ f(z + \Delta z) - f(z) = z^3 + 3z^2\,\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 + 2z^2 + 4z\Delta z + 2(\Delta z)^2 + i - z^3 - 2z^2 - i$$

$$= 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 + 4z\Delta z + 2(\Delta z)^2$$

$$\Rightarrow f(z + \Delta z) - f(z) = [3z^2 + 3z(\Delta z) + (\Delta z)^2 + 4z + 2\Delta z]\,\Delta z \quad \ldots \quad (3)$$

Using (3), we get from (1)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{[3z^2 + 3z(\Delta z) + (\Delta z)^2 + 4z + 2\Delta z](\Delta z)}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \; 3z^2 + 3z\Delta z + (\Delta z)^2 + 4z + 2\Delta z$$

$\therefore f'(z) = 3z^2 + 4z.$

(iii) Given that

$f(z) = z^3 - 2z$

$\Rightarrow f(z + \Delta z) = (z + \Delta z)^3 - 2(z + \Delta z)$

$= z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2z - 2\Delta z$

$\therefore f(z + \Delta z) - f(z) = z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2z - 2\Delta z - z^3 + 2z$

$= 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2\Delta z$

$\Rightarrow f(z + \Delta z) - f(z) = [3z^2 + 3z(\Delta z) + (\Delta z)^2 - 2]\Delta z$ ... ... (4).

Using (4), we get from (1)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{[3z^2 + 3z\Delta z + (\Delta z)^2 - 2]\Delta z}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \; 3z^2 + 3z\Delta z + (\Delta z)^2 - 2$$

$= 3z^2 - 2.$

Ex-2 **:** Using the definition, find the derivative of the following functions at the indicated points.

(a) $f(z) = 3z^2 + 3iz - 5 + i$, at $z = 2$

(b) $f(z) = \dfrac{2z - i}{z + 2i}$, at $z = -i$

(c) $f(z) = 3z^{-2}$, at $z = 1 + i$.

Solution **:** We know that by definition, the derivative of f(z) at any point z is

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad \dots \quad \dots \quad (i)$$

(a) Given that

$f(z) = 3z^2 + 4iz - 5 + i$ ... ... (ii)

$$\Rightarrow f(z + \Delta z) = 3(z + \Delta z)^2 + 4i(z + \Delta z) - 5 + i$$

$$= 3z^2 + 6z\Delta z + 3(\Delta z)^2 + 4iz + 4i\Delta z - 5 + i$$

$$= (3z^2 + 4iz - 5 + i) + 6z\Delta z + 3(\Delta z)^2 + 4i\Delta z$$

$$\Rightarrow f(z + \Delta z) = f(z) + [6z + 3\Delta z + 4i]\, \Delta z \quad \text{[by (ii)]}$$

$$\therefore f(z + \Delta z) - f(z) = [6z + 3\Delta z + 4i]\, \Delta z \quad \cdots \quad \cdots \quad \text{(iii)}$$

Using (iii), we get from (i)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{[6z + 3\Delta z + 4i]\Delta z}{\Delta z}$$

$$= \underset{\Delta z \to 0}{\text{Lt}}\ 6z + 3\Delta z + 4i$$

$$\therefore f'(z) = 6z + 4i.$$

And at $z = 2$, $f'(z) = 6.2 + 4i$

$$= 12 + 4i.$$

(b) Given that

$$f(z) = \frac{2z - i}{z + 2i}$$

$$\Rightarrow f(z + \Delta z) = \frac{2(z + \Delta z) - i}{z + \Delta z + 2i}$$

$$\therefore f(z + \Delta z) - f(z) = \frac{2(z + \Delta z) - i}{z + \Delta z + 2i} - \frac{2z - i}{z + 2i}$$

$$= \frac{2z^2 + 2z\Delta z - iz + 4iz + 4i\Delta z - 2i^2 - 2z^2 - 2z\Delta z - 4iz + iz + i\Delta z + 2i^2}{(z + \Delta z + 2i)(z + 2i)}$$

$$f(z + \Delta z) - f(z) = \frac{5i\Delta z}{(z + \Delta z + 2i)(z + 2i)} \quad \cdots \quad \cdots \quad \text{(iv)}$$

Using (iv), we get from (i)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \left[\frac{5i(\Delta z)}{(z + \Delta z + 2i)(z + 2i)}\right] \frac{1}{(\Delta z)}$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{5i}{(z + \Delta z + 2i)(z + 2i)}$$

$$f'(z) = \frac{5i}{(z + 2i)(z + 2i)}$$

$$\therefore f'(z) = \frac{5i}{(z + 2i)^2}$$

And at $z = -i$, $f'(z) = \frac{5i}{(-i + 2i)^2}$

$$= \frac{5i}{i^2}$$

$$= -5i.$$

(c) Given that

$$f(z) = 3z^{-2}$$

$$= \frac{3}{z^2}$$

$$\Rightarrow f(z + \Delta z) = \frac{3}{(z + \Delta z)^2}$$

$$\therefore f(z + \Delta z) - f(z) = \frac{3}{(z + \Delta z)^2} - \frac{3}{z^2}$$

$$= 3\left[\frac{z^2 - (z + \Delta z)^2}{(z + \Delta z)^2 z^2}\right]$$

$$= 3\left[\frac{(z + z + \Delta z)\ (z - z - \Delta z)}{(z + \Delta z)^2 z^2}\right]$$

$$f(z + \Delta z) - f(z) = \frac{-3(2z + \Delta z)\Delta z}{(z + \Delta z)^2 z^2} \quad \ldots \ldots \quad (v)$$

Using (v), we get from (i)

$$f'(z) = \underset{\Delta z \to 0}{\text{Lt}} \frac{1}{(\Delta z)}\left[\frac{-3(2z + \Delta z)\ (\Delta z)}{(z + \Delta z)^2 z^2}\right]$$

$$= \underset{\Delta z \to 0}{\text{Lt}} \frac{-3(2z + \Delta z)}{(z + \Delta z)^2 z^2}$$

$$= \frac{-3(2z + 0)}{z^2 \,.\, z^2}$$

$$f'(z) = \frac{-6z}{z^4}$$

at $z = 1 + i$, $f'(z) = \frac{-6(1 + i)}{(1 + i)^4}$

$$= \frac{-6(1 + i)}{1 + 4i + 6i^2 + 4i^3 + i^4}$$

$$= \frac{-6(1 + i)}{1 + 4i - 6 - 4i + 1}$$

$$= \frac{3}{2}(1 + i).$$

Ex-3 : Show that $\frac{d}{dz}(\bar{z})$ does not exist any where.

Proof : Let $f(z) = \bar{z}$ ... ... (i)

$\Rightarrow f(z + \Delta z) = \overline{z + \Delta z}$ ... ... (ii)

By definition, we know that

$$\frac{d}{dz}\{f(z)\} = \underset{\Delta z \to 0}{Lt} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow \frac{d}{dz}(\bar{z}) = \underset{\Delta z \to 0}{Lt} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} \quad \ldots \quad \ldots \quad \text{(iii)}$$

we have $z = x + iy$

$$\left.\begin{aligned} \Rightarrow \Delta z &= \Delta x + i\Delta y \\ \Rightarrow \Delta \bar{z} &= \Delta x - i\Delta y \end{aligned}\right\} \quad \ldots \quad \ldots \quad \text{(iv)}$$

when $\Delta z \to 0$, then $\Delta x \to 0$ and $\Delta y \to 0$ ... (v)

$$\therefore \frac{d}{dz}(\bar{z}) = \underset{\Delta z \to 0}{Lt} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} \quad [\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2]$$

$$= \underset{\Delta z \to 0}{Lt} \frac{\overline{\Delta z}}{\Delta z}$$

$$\frac{d}{dz}(\bar{z}) = \underset{\substack{\Delta x \to 0 \\ \Delta y \to 0}}{Lt} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \quad \ldots \quad \ldots \quad \text{(vi)[by (iv) and (v)]}$$

If Δz is purely real, then $\Delta z = \Delta x$ and $\Delta y = 0$, then

$$\frac{d}{dz}(\bar{z}) = \underset{\Delta x \to 0}{Lt} \frac{\Delta x}{\Delta x} = 1$$

Again, if Δz is purely imaginary, then $\Delta z = i\Delta y$ and $\Delta x = 0$, then

$$\frac{d}{dz}(\bar{z}) = \underset{\Delta y \to 0}{Lt} \frac{-i\Delta y}{i\Delta y} = -1$$

Since in two manner in which $\Delta z \to 0$, $\frac{d}{dz}(\bar{z})$ is not same, so that $\frac{d}{dz}(\bar{z})$ does not exist anywhere that is $f(z)$ is not analytic.

Now taking limit through the point (x_0, y_0) and parallel to the x-axis, then we have $y = y_0$, $x \to x_0$ and equation (i), becomes

$$f'(z_0) = \underset{x \to x_0}{\text{Lt}} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\underset{x \to x_0}{\text{Lt}} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}$$

$$\Rightarrow f'(z_0) = \frac{\partial u(x_0, y_0)}{\partial x} + i\frac{\partial v(x_0, y_0)}{\partial x}$$

$$\Rightarrow f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \quad \textbf{(Proved)}$$

Again, taking the limit through the point (x_0, y_0) and parallel to the y-axis, then we have $x = x_0$, $y \to y_0$ and equation (i) becomes

$$f'(z_0) = \underset{y \to y_0}{\text{Lt}} \frac{u(x_0, y) - u(x_0, y_0)}{i(y - y_0)} + i\underset{y \to y_0}{\text{Lt}} \frac{v(x_0, y) - v(x_0, y_0)}{i(y - y_0)}$$

$$= \frac{1}{i}\frac{\partial u(x_0, y_0)}{\partial y} + \frac{\partial v(x_0, y_0)}{\partial y}$$

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \quad \textbf{(Proved)}$$

Ex–7 **:** Prove that the function $f(z) = u(x, y) + iv(x, y)$, where

$$f(z) = \begin{cases} \dfrac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous and the Cauchy-Riemann equations are satisfied at origin. Yet $f'(z)$ does not exist there.

Proof **: 1st part :** Given that

$$f(z) = \frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} \quad \ldots \ldots \quad \text{(A)}$$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2}$$

$$\therefore u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \text{ and } v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

Here both $u(x, y)$ and $v(x, y)$ are rational and finite for all values of z except $z = 0$. So $u(x, y)$ and $v(x, y)$ are continuous at all those points for which $z \neq 0$. That is $f(z)$ is continuous where $z \neq 0$.

Again, Given that, at z = 0, f(z) = 0

$$\Rightarrow f(0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(B)}$$

$$\Rightarrow f(0 + i0) = 0 + i0$$

$$\Rightarrow u(0, 0) = 0 \text{ and } v(0, 0) = 0$$

which are finite and unique.

So, u(x, y) and v(x, y) are continuous at origin. Hence f(z) is continuous at everywhere.

2nd part : We know that by definition at any point (x, y)

$$\frac{\partial}{\partial x}\{u(x, y)\} = \underset{\partial x \to 0}{\text{Lt}} \frac{u(x + \partial x, y) - u(x, y)}{\partial x}$$

$$\therefore \text{ at origin, } \frac{\partial u}{\partial x} = \underset{\partial x \to 0}{\text{Lt}} \frac{u(\partial x, 0) - u(0, 0)}{\partial x}$$

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} \quad \ldots \quad \ldots \quad \text{(i)}$$

[Replacing ∂x by x]

But, we have $u(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$

$$\Rightarrow u(x, 0) = \frac{x^3}{x^2} = x \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

$$\Rightarrow u(0, y) = \frac{-y^3}{y^2} = -y \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

Also given that u(0, 0) = 0 ... ... (iv)

Using (ii) and (iv), we get from (i)

$$\frac{\partial u}{\partial x} = \frac{x - 0}{x} = 1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

Again, we have at origin

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

Using (iii) and (iv), we get from (vi)

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{-y - 0}{y} = -1 \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii)}$$

Similarly, we have at origin

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

and $$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(ix)}$$

But, we have

$$v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\Rightarrow v(x, 0) = \frac{x^3}{x^2} = x \quad \cdots \quad \cdots \quad \cdots \quad \text{(x)}$$

$$\Rightarrow v(0, y) = \frac{y^3}{y^2} = y \quad \cdots \quad \cdots \quad \cdots \quad \text{(xi)}$$

also, given that, $v(0, 0) = 0$ ... ... (xii)

Using (x), (xi) and (xii), we get from (viii) and (ix) respectively

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{x - 0}{x} = 1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xiii)}$$

and $$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{y - 0}{y} = 1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xiv)}$$

From (v) and (xiv), we get $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and from (vii) and (xiii), we get $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

The above equations show that the Cauchy-Rieman equations are satisfied at origin.

3rd part : We know that the derivative of f(z) at $z = z_0$ is

$$f'(z_0) = \underset{z \to z_0}{\text{Lt}} \frac{f(z) - f(z_0)}{z - z_0}$$

Thus the derivative of f(z) at $z = 0$ is

$$f'(z) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} \quad \cdots \quad \cdots \quad \cdots \quad \text{(xv)}$$

$$= \underset{z \to 0}{\text{Lt}} \frac{1}{x + iy}\left[\frac{x^3(1 + i) - y^3(1 - i)}{x^2 + y^2} - 0\right] \quad \text{[by (A) and (B)]}$$

Along $y = x,\ z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = \underset{x\to 0}{\text{Lt}} \frac{1}{(x + ix)}\left[\frac{(x^3 - x^3) + i(x^3 + x^3)}{x^2 + x^2}\right]$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1}{(1 + i)x} \cdot \frac{2x^3 i}{2x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{i}{1 + i} = \frac{i}{1 + i}$$

$$= \frac{(1 - i)i}{1 - i^2}$$

$$\therefore f'(0) = \frac{1}{2}(1 + i) \quad \cdots \quad \cdots \quad \cdots \quad \text{(xvi)}$$

Again, along $y = 0,\ z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = \underset{x\to 0}{\text{Lt}} \frac{(x^3 - 0) + i(x^3 + 0)}{(x^2 + 0)(x + 0)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x^3(1 + i)}{x^3}$$

$\because z = x + iy$ for $y = 0,\ z = x$
so, $z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = 1 + i \quad \cdots \quad \cdots \quad \cdots \quad \text{(xvii)}$$

From equation (xvi) and (xvii), we conclude that $f'(0)$ is not unique, that is the values of $f'(0)$ are not same as $z \to 0$ along different paths.

Hence $f'(z)$ does not exist at the origin.

Ex–8 : Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied at the point (0, 0).

Proof : Let $f(z) = u(x, y) + iv(x, y)$ be a function.

Given that,

$$f(z) = \sqrt{|xy|} \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\Rightarrow u(x, y) + iv(x, y) = \sqrt{|xy|}$$

$$\Rightarrow u(x, y) = \sqrt{|xy|} \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

$$\text{and } v(x, y) = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

From (xi) and (xiv), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad (xv)$$

Also from (xii) and (xiii), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad (xvi)$$

The above equations (xv) and (xvi) show that the Cauchy-Riemann equations are satisfied at the origin.

Again, We know that the derivative of f(z) at origin is

$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z}$$

$$= \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|}}{x + iy} \quad \text{[by (i) and } \because z = x + iy]$$

along $y = mx,\ z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|x.mx|}}{x + imx}$$

$$f'(0) = \underset{x \to 0}{\text{Lt}} \frac{\sqrt{m}}{1 + im}$$

which depends on m, so $f'(0)$ is not unique.

Hence f(z) is not regular at the origin although Cauchy-Riemann equations are satisfied there.

Ex-9 : Let $f(z) = \dfrac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{x^2 + y^2}$, when $z \neq 0$

and $f(0) = 0$, when $z = 0$

Show that f(z) is continuous and the Cauchy-Riemann equations are satisfied but f(z) is not differentiable at z = 0.

[N. U. H. 2000, 2003, 2005]

***Proof* : 1st Part :** Let f(z) = u(x, y) + iv(x, y) be a function.

Given that

$$f(z) = \frac{(x^3 - 3xy^2) + i(y^3 - 3x^2y)}{x^2 + y^2},\ z \neq 0 \quad \cdots \quad \cdots \quad (i)$$

and $f(0) = 0$, when $z = 0$ $\quad \cdots \quad \cdots \quad \cdots \quad$ (ii)

From (i), we get

$$u(x, y) + iv(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} + i\frac{y^3 - 3x^2y}{x^2 + y^2}$$

$$\Rightarrow u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\text{and } v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2} \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

From (ii), we get

$$u(0, 0) + iv(0, 0) = 0 + i.0$$

$$\Rightarrow u(0, 0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

$$\text{and } v(0, 0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

Here $u(x, y)$ and $v(x, y)$ both are rational and finite for all values of z except $z = 0$. So that $u(x, y)$ and $v(x, y)$ are continuous at all those points in which $z \neq 0$.

Hence $f(z)$ is continuous anywhere except the point $z = 0$.

Again, we have $u(0, 0) = 0$ and $v(0, 0) = 0$ that is at the origin u and v are finite and unique. So u and v are continuous at the origin, that is $f(z)$ is continuous at the origin.

Hence, we conclude that $f(z)$ is continuous everywhere.

2nd Part : We know that the partial derivatives of $u(x, y)$ and $v(x, y)$ at the origin respectively

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii)}$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} \quad \ldots \quad \ldots \quad \ldots \quad \text{(ix)}$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(x)}$$

But, we have $u(x, y) = \dfrac{x^3 - 3xy^2}{x^2 + y^2}$

$$\Rightarrow u(x, 0) = \frac{x^3}{x^2} = x \quad \ldots \quad \ldots \quad \ldots \quad \text{(xi)}$$

$$\text{and } u(0, y) = \frac{0}{y^2} = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xii)}$$

$$\text{and } v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$$

$$\Rightarrow v(x, 0) = \frac{0}{x^2} = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xiii)}$$

$$\text{and } v(0, y) = \frac{y^3}{y^2} = y \quad \cdots \quad \cdots \quad \cdots \quad \text{(xiv)}$$

Using (xi) and (v), we get from (vii)

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{x - 0}{x} = 1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xv)}$$

Using (xii) and (v), we get from (viii)

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xvi)}$$

Using (xiii) and (vi), we get from (ix)

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xvii)}$$

Using (xiv) and (vi), we get from (x)

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{y - 0}{y} = 1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(xviii)}$$

From (xv) and (xviii), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(xix)}$$

Again, from (xvi) and (xvii), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(xx)}$$

The above equations (xix) and (xx) show that the Cauchy-Riemann equations are satisfied at the origin.

3rd part : We know that the derivative of f(z) at the point $z = 0$ is

$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z}$$

$$= \underset{z \to 0}{\text{Lt}} \frac{1}{x + iy}\left[\frac{x^3 - 3xy^2}{x^2 + y^2} + i\,\frac{y^3 - 3x^2y}{x^2 + y^2} - 0\right]$$

[by (i) and (ii)]

Along $y = x$, $z \to 0 \Rightarrow x \to 0$

Then, $f'(0) = \underset{x \to 0}{\text{Lt}} \dfrac{1}{x + ix}\left[\dfrac{x^3 - 3x^3}{x^2 + x^2} + i.\dfrac{x^3 - 3x^3}{x^2 + x^2}\right]$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1}{(1 + i)x}\left[\dfrac{-2x^3}{2x^2} + i\,.\dfrac{-2x^3}{2x^2}\right]$

$= \underset{x \to 0}{\text{Lt}} \dfrac{1}{(1 + i)x}[-(1 + i)x]$

$\therefore f'(0) = -1$

Again, along $y = 0$, $z \to 0 \Rightarrow x \to 0$

Then $f'(0) = \underset{x \to 0}{\text{Lt}} \dfrac{1}{x}\left[\dfrac{x^3}{x^2} + i\dfrac{0}{x^2}\right]$

$= \underset{x \to 0}{\text{Lt}} \dfrac{x^3}{x^3} = 1$

$\therefore f'(0) = 1$

Here, we see that $f'(0)$ is not unique, that is the values of $f'(0)$ are not same as $z \to 0$ along different paths.

Hence $f'(z)$ does not exists at $z = 0$ that is $f(z)$ is not differentiable at $z = 0$.

Ex–10 **:** Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}$, $z \neq 0$ and $f(0) = 0$ in the region including the origin.

Solution **:** Let $f(z) = u(x, y) + iv(x, y)$ be a function.

Given that

$$f(z) = \frac{x^2y^5(x + iy)}{x^4 + y^{10}} \quad \cdots \quad \cdots \quad \cdots \quad (A)$$

$$\Rightarrow u(x, y) + iv(x, y) = \frac{x^3y^5}{x^4 + y^{10}} + i\frac{x^2y^6}{x^4 + y^{10}}$$

$$\Rightarrow u(x, y) = \frac{x^3y^5}{x^4 + y^{10}} \quad \cdots \quad \cdots \quad \cdots \quad (i)$$

$$\text{and } v(x, y) = \frac{x^2y^6}{x^4 + y^{10}} \quad \cdots \quad \cdots \quad \cdots \quad (ii)$$

$$\text{Then } f'(0) = \underset{x\to 0}{\text{Lt}} \frac{1}{(x+iy)} \times \frac{x^2.x^2(x+iy)}{x^4+(x^2)^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x^4}{2x^4}$$

$$\therefore f'(0) = \frac{1}{2}$$

Here, we see that $f'(0)$ is different in different paths, so that $f'(0)$ does not exist.

Hence f(z) is not differentiable at the origin although it is continuous at the origin and Cauchy-Riemann equations are satisfied at the origin.

Ex-11 **:** Show that the function f(z) = xy + iy is everywhere continuous but is not analytic.

Proof : Given that

$$f(z) = xy + iy \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

But, we have f(z) = u(x, y) + iv(x, y) ... ... (ii)

From (i) and (ii), we have

$$u(x, y) + iv(x, y) = xy + iy$$

$$\Rightarrow u(x, y) = xy \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

$$\text{and } v(x, y) = y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$$

It is clear that u(x, y) and v(x, y) both exist at all points of the complex plane. So that they are continuous everywhere, that is f(z) is continuous at everywhere.

Partially differentiating equations (iii) and (iv) w. r. to x and y respectively.

$$\therefore \frac{\partial u}{\partial x} = y \quad \cdots \quad \cdots \quad \cdots \quad \text{(v)}$$

$$\text{and } \frac{\partial u}{\partial y} = x \quad \cdots \quad \cdots \quad \cdots \quad \text{(vi)}$$

$$\text{And } \frac{\partial v}{\partial x} = 0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(vii)}$$

$$\text{and } \frac{\partial v}{\partial y} = 1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

From (v) and (viii), we get $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ ... ... (ix)

from (vi) and (vii), we get $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$ ... ... (x)

The above equations (ix) and (x) show that the Cauchy-Riemann equations are not satisfied and hence f(z) is not an analytic function.

Ex-12 **:** Prove that $f(z) = z\bar{z}$ is nowhere analytic.

Proof **:** Given that $f(z) = z\bar{z}$

Let $z = x + iy \Rightarrow \bar{z} = x - iy$

$\therefore f(z) = (x + iy)(x - iy)$

$\Rightarrow u(x, y) + iv(x, y) = (x^2 + y^2) + i.0$

$\Rightarrow u(x, y) = x^2 + y^2$ ... ... (i)

and $v(x, y) = 0$ ... ... (ii)

Partially differentiating equation (i) and (ii), w. r. to x and y respectively.

$\therefore \frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = 2y$

and $\frac{\partial v}{\partial x} = 0; \quad \frac{\partial v}{\partial y} = 0$

The above equations show that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous every where. But the Cauchy-Riemann equations are satisfied only at the origin. Hence z = 0 is only the point at which $f'(z)$ exists. Thus $f(z) = z\bar{z}$ is nowhere analytic.

Ex-13 **:** Show that the function $f(z) = e^{-z^{-4}}$, $[z \neq 0]$ and $f(0) = 0$ is not analytic at z = 0 although Cauchy-Riemann equations are satisfied at the point. How would you explain this?

Proof **:** Given that

$f(z) = e^{-z^{-4}}, z \neq 0$ ... ... (i)

and $f(0) = 0$ ... ... (ii)

Let $f(z) = u + iv$ ... ... (iii)

$\therefore u + iv = e^{-z^{-4}}$ ... ... (iv)

Ex-15 : Show that the function $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, $[z \neq 0]$ and $f(0) = 0$ is not analytic at the point $z = 0$.

Proof : Given that

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4}, z \neq 0 \quad \ldots \quad \ldots \quad \text{(i)}$$

and $f(0) = 0 \quad \ldots \quad \ldots \quad \ldots$ (ii)

We know that the derivative of f(z) at z = 0 is

$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z}$$

$$= \underset{z \to 0}{\text{Lt}} \frac{1}{(x+iy)}\left[\frac{xy^2(x+iy)}{x^2+y^4} - 0\right] \quad \text{[by (i) and (ii)]}$$

$$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{xy^2}{x^2+y^4}$$

Along $y = mx$, $z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = \underset{x \to 0}{\text{Lt}} \frac{x.m^2x^2}{x^2+m^4x^4}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{m^2x^3}{x^2(1+m^4x^2)}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{m^2x}{1+m^4x^2}$$

$$\Rightarrow f'(0) = 0$$

Again, along the curve $y^2 = x$, $z \to 0 \Rightarrow x \to 0$

$$\therefore f'(0) = \underset{x \to 0}{\text{Lt}} \frac{x.x}{x^2+(x)^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^2}{2x^2}$$

$$\Rightarrow f'(0) = \frac{1}{2}$$

Here, we see that the values of $f'(0)$ is different in different paths. So that $f'(0)$ does not exist. Hence the given function f(z) is not analytic at the point z = 0.

By the theorem on total differential, we have from (iii)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \quad \text{[by (i) and (ii)]}$$ **(Proved)**

Again, by the theorem on total differential, we have from (iv)

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\therefore dv = \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx.$$ **(Proved)**

Ex-19 **:** For what values of z do the function w defined by the equation. $z = e^{-v}(\cos u + i \sin u)$, $w = u + iv$ cease to be analytic.

Solution **:** Given that

$$z = e^{-v}(\cos u + i \sin u)$$

$$= e^{i^2 v} . e^{iu}$$

$$= e^{iu + i^2 v} = e^{i(u+iv)}$$

$$z = e^{iw} \quad [\because w = u + iv]$$

$$\Rightarrow iw = \ln z$$

$$\Rightarrow w = \frac{1}{i} . \ln z$$

$$\Rightarrow w = -i \ln z$$

It is clear that $\frac{dw}{dz}$ becomes infinite at $z = 0$, so that w is not analytic at $z = 0$.

Ex-20 **:** Verify that the Cauchy-Riemann equations are satisfied for the functions :

(i) $f(z) = e^{z^2}$ (ii) $f(z) = \cos 2z$

(iii) $f(z) = \sin 2z$ (iv) $f(z) = \cosh 4z$

(v) $f(z) = \sinh 4z$ (vi) $f(z) = \frac{1}{x + iy}$

(vii) $f(z) = e^y (\cos x + i \sin x)$ (viii) $f(z) = e^x(\cos y + i \sin y)$

(ix) $f(z) = \sin x \cosh y + i \cos x \sinh y$

(x) $f(z) = z^2$ **(xi)** $f(z) = (z^2 - 2)e^{-z}$

(xii) $f(z) = e^{-y}(\sin x - i\cos x)$ **(xiii)** $f(z) = z^n$

(xiv) $f(z) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$

(xv) $f(z) = \bar{z} + 1$ **(xvi)** $f(z) = \ln z.$

Solution : **(i)** Given that

$$f(z) = e^{z^2}$$

$\Rightarrow u + iv = e^{(x+iy)^2}$, where $f(z) = u + iv$

$\Rightarrow u + iv = e^{x^2-y^2+2xyi}$ and $z = x + iy$

$$= e^{x^2-y^2} . e^{i2xy}$$

$$= e^{x^2-y^2} [\cos 2xy + i \sin 2xy]$$

$$\Rightarrow u + iv = e^{x^2-y^2} \cos 2xy + i.e^{x^2-y^2} \sin 2xy$$

Equating the real and imaginary parts, we get

$$u = e^{x^2-y^2} \cos 2xy \quad \dots \quad \dots \quad \dots \quad (i)$$

$$\text{and } v = e^{x^2-y^2} \sin 2xy \quad \dots \quad \dots \quad \dots \quad (ii)$$

Partially differentiating equation (i) and (ii) w. r. to x and y respectively.

$$\therefore \frac{\partial u}{\partial x} = e^{x^2-y^2}. (2x). \cos 2xy + e^{x^2-y^2} (-\sin 2xy). 2y$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2e^{x^2-y^2} [x \cos 2xy - y \sin 2xy] \quad \dots \quad \dots \quad (iii)$$

$$\text{and } \frac{\partial u}{\partial y} = e^{x^2-y^2} (-2y). \cos 2xy + e^{x^2-y^2} (-\sin 2xy). 2x$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2e^{x^2-y^2} [x \sin 2xy + y \cos 2xy] \quad \dots \quad \dots \quad (iv)$$

$$\text{Again, } \frac{\partial v}{\partial x} = e^{x^2-y^2}. (2x). \sin 2xy + e^{x^2-y^2}. \cos 2xy. (2y)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 2e^{x^2-y^2} [x \sin 2xy + y \cos 2xy] \quad \dots \quad \dots \quad (v)$$

$$\text{and } \frac{\partial v}{\partial y} = e^{x^2-y^2} .(-2y). \sin 2xy + e^{x^2-y^2}. \cos 2xy.(2x)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2e^{x^2-y^2} [x \cos 2xy - y \sin 2xy] \quad \dots \quad \dots \quad (vi)$$

From equations (iii) and (vi), and we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots \quad \dots \quad \dots \quad (vii)$$

From equations (iv) and (v), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function $f(z) = e^{z^2}$.

(ii). Given that

$$f(z) = \cos 2z$$

$$\Rightarrow u + iv = \cos 2(x + iy), \text{ where } f(z) = u + iv \text{ and } z = x + iy$$

$$\Rightarrow u + iv = \cos(2x + i2y)$$

$$\Rightarrow u + iv = \cos 2x \cos(i2y) - \sin 2x \sin(i2y)$$

$$\Rightarrow u + iv = \cos 2x \cosh 2y - \sin 2x . i \sinh 2y$$

$$[\because \cos ix = \cosh x \text{ and } \sin ix = i \sinh y]$$

$$\Rightarrow u + iv = \cos 2x \cosh 2y - i \sin 2x \sinh 2y$$

Equating the real and imaginary parts, we get

$$u = \cos 2x \cosh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

$$\text{and } v = -\sin 2x \sinh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(ii)}$$

Partially differentiating equations (i) and (ii) w. r. to x and y respectively, we get

$$\therefore \frac{\partial u}{\partial x} = -2 \sin 2x \cosh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$\text{and } \frac{\partial u}{\partial y} = 2 \cos 2x \sinh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

$$\text{Again, } \frac{\partial v}{\partial x} = -2 \cos 2x \sinh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

$$\text{and } \frac{\partial v}{\partial y} = -2 \sin 2x \cosh 2y \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots \quad \ldots \quad \text{(vii)}$$

From (iv) and (v), we get $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ... ... (viii)

The above equations (vii) and (viii) show that the Cuchy-Riemann equations are satisfied for the given function $f(z) = \cos 2z$.

(iii). Given that

$f(z) = \sin 2z$

$\Rightarrow u + iv = \sin 2(x + iy)$, where $f(z) = u + iv$ and $z = x + iy$

$\Rightarrow u + iv = \sin(2x + i.2y)$

$\Rightarrow u + iv = \sin 2x \cos(i2y) + \cos 2x \sin(i2y)$

$\Rightarrow u + iv = \sin 2x \cosh 2y + i \cos 2x \sinh 2y$

[$\because \cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$]

Equating the real and imaginary parts, we get

$u = \sin 2x \cosh 2y$ ... ... (i)

and $v = \cos 2x \sinh 2y$ ... ... (ii)

Partially differentiating equations (i) and (ii), w. r. to x and y respectively, we get

$\frac{\partial u}{\partial x} = 2 \cos 2x \cosh 2y$ ... ... (iii)

and $\frac{\partial u}{\partial y} = 2 \sin 2x \sinh 2y$ ... ... (iv)

Again, $\frac{\partial v}{\partial x} = -2 \sin 2x \sinh 2y$ ... ... (v)

and $\frac{\partial v}{\partial y} = 2 \cos 2x \cosh 2y$ ... ... (vi)

From (iii) and (vi), we get

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ... ... (vii)

From (vi) and (v) we get

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ... ... (viii)

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function $f(z) = \sin 2z$.

(iv). Given that

$f(z) = \cosh 4z$, where $f(z) = u + iv$ and $z = x + iy$

$\Rightarrow f(z) = \cos(i4z)$ [$\because \cos ix = \cosh x$]

$\Rightarrow u + iv = \cos\{4i(x + iy)\}$

$\Rightarrow u + iv = \cos(4xi - 4y)$

$\Rightarrow u + iv = \cos(4xi)\cos 4y + \sin(4xi)\sin 4y$

$\Rightarrow u + iv = \cosh 4x \cos 4y + i \sinh 4x \sin 4y$

[$\because \cos ix = \cosh x$ and $\sin ix = i \sinh x$]

Equating the real and imaginary parts, we get

$u = \cosh 4x \cos 4y$ ... ... (i)

and $v = \sinh 4x \sin 4y$ ... ... (ii)

Partially-differentiating equations (i) and (ii) w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x} = 4 \sinh 4x \cos 4y \quad \ldots \quad \ldots \quad \ldots \quad \text{(iii)}$$

and $$\frac{\partial u}{\partial y} = -4 \cosh 4x \sin 4y \quad \ldots \quad \ldots \quad \ldots \quad \text{(iv)}$$

Again, $$\frac{\partial v}{\partial x} = 4 \cosh 4x \sin 4y \quad \ldots \quad \ldots \quad \ldots \quad \text{(v)}$$

and $$\frac{\partial v}{\partial y} = 4 \sinh 4x \cos 4y \quad \ldots \quad \ldots \quad \ldots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots \quad \ldots \quad \ldots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function $f(z) = \cosh 4z$.

(v). Given that

$f(z) = \sinh 4z$

$\Rightarrow f(z) = -i \sin(i4z)$

$\Rightarrow u + iv = -i \sin\{4i(x + iy)\}$, where $f(z) = u + iv$ and $z = x + iy$

$\Rightarrow u + iv = -i \sin(4xi - 4y)$

$\Rightarrow u + iv = -i[\sin(i4x)\cos 4y - \cos(i4x)\sin 4y]$

$\Rightarrow u + iv = -i[i \sinh 4x \cos 4y - \cosh 4x \sin 4y]$

[$\because \sin ix = i \sinh x$ and $\cos ix = \cosh x$]

$\Rightarrow u + iv = \sinh 4x \cos 4y + i \cosh 4x \sin 4y$

Equating the real and imaginary parts, we get

$$u = \sinh 4x \cos 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\text{and } v = \cosh 4x \sin 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

Partially differentiating equations (i) and (ii), w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x} = -4 \cosh 4x \cos 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

$$\text{and } \frac{\partial u}{\partial y} = 4 \sinh 4x \sin 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$$

$$\text{Again, } \frac{\partial v}{\partial x} = -4 \sinh 4x \sin 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(v)}$$

$$\text{and } \frac{\partial v}{\partial y} = -4 \cosh 4x \cos 4y \quad \cdots \quad \cdots \quad \cdots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function f(z) = sinh4z.

(vi). Given that

$$f(z) = \frac{1}{x + iy}$$

$$\Rightarrow u + iv = \frac{x - iy}{(x + iy)(x - iy)}, \quad \text{where } f(z) = u + iv$$

$$\Rightarrow u + iv = \frac{x - iy}{x^2 + y^2}$$

$$\Rightarrow u + iv = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}.$$

Equating the real and imaginary parts, we get

$$u = \frac{x}{x^2 + y^2} \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

$$\text{and } v = \frac{-y}{x^2 + y^2}$$

Partially differentiating equations (i) and (ii) w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2} \quad \dots \quad \dots \quad \dots \quad \text{(iii)}$$

and $\frac{\partial u}{\partial y}=x.\frac{-1}{(x^2+y^2)^2}\times 2y$

$$\Rightarrow \frac{\partial u}{\partial y}=\frac{-2xy}{(x^2+y^2)^2} \quad \dots \quad \dots \quad \dots \quad \text{(iv)}$$

Again, $\frac{\partial v}{\partial x}=(-y)\frac{-1}{(x^2+y^2)^2}\times 2x$

$$\Rightarrow \frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2} \quad \dots \quad \dots \quad \dots \quad \text{(v)}$$

and $\frac{\partial v}{\partial y}=-\left[\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right]$

$$=-\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$\frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2} \quad \dots \quad \dots \quad \dots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \dots \quad \dots \quad \dots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \quad \dots \quad \dots \quad \dots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the functions $f(z)=\frac{1}{x+iy}$.

(vii). Given that

$$f(z)=e^y(\cos x+i\sin x)$$

$\Rightarrow u+iv=e^y\cos x+ie^y\sin x$, where $f(z)=u+iv$

Equating the real and imaginary parts, we get

$$u=e^y\cos x \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

and $v=e^y\sin x \quad \dots \quad \dots \quad \dots \quad \text{(ii)}$

Partially differentiating equations (i) and (ii) w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x} = -e^y \sin x \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

and $$\frac{\partial u}{\partial y} = e^y \cos x \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$$

Again, $$\frac{\partial v}{\partial x} = e^y \cos x \quad \cdots \quad \cdots \quad \cdots \quad \text{(v)}$$

and $$\frac{\partial v}{\partial y} = e^y \sin x \quad \cdots \quad \cdots \quad \cdots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

The above equations (vii) and (viii), show that the Cauchy-Riemann equations are not satisfied for the function $f(z) = e^y (\cos x + i \sin x)$.

(viii). Given that

$$f(z) = e^x(\cos y + i \sin y)$$

$\Rightarrow u + iv = e^x \cos y + ie^x \sin y$, wher, $f(z) = u + iv$

Equating the real and imaginary parts, we get

$$u = e^x \cos y \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

and $$v = e^x \sin y \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

Partially differentiating equation (i) and (ii) w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

and $$\frac{\partial u}{\partial y} = -e^x \sin y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$$

Again, $$\frac{\partial v}{\partial x} = e^x \sin y \quad \cdots \quad \cdots \quad \cdots \quad \text{(v)}$$

and $$\frac{\partial v}{\partial y} = e^x \cos y \quad \cdots \quad \cdots \quad \cdots \quad \text{(vi)}$$

From (iii) and (vi), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function $f(z) = e^x (\cos y + i \sin y)$

(ix). Given that

$f(z) = \sin x \cosh y + i \cos x \sinh y$

$\Rightarrow u + iv = \sin x \cosh y + i \cos x \sin hy$, where $f(z) = u + iv$

Equating the real and imaginary parts, we get,

$$u = \sin x \cosh y \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

and $v = \cos x \sinh y \quad \cdots \quad \cdots \quad \cdots \quad$ (ii)

Partially differentiating equations (i) and (ii) w. r. to x and y respectively, we get

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad \cdots \quad \cdots \quad \cdots \quad \text{(iii)}$$

and $\frac{\partial u}{\partial y} = \sin x \sinh y \quad \cdots \quad \cdots \quad \cdots \quad$ (iv)

Again, $\frac{\partial v}{\partial x} = -\sin x \sinh y \quad \cdots \quad \cdots \quad \cdots \quad$ (v)

and $\frac{\partial v}{\partial y} = \cos x \cosh y \quad \cdots \quad \cdots \quad \cdots \quad$ (vi)

From (iii) and (vi) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \cdots \quad \cdots \quad \cdots \quad \text{(vii)}$$

From (iv) and (v), we get

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots \quad \cdots \quad \cdots \quad \text{(viii)}$$

The above equations (vii) and (viii) show that the Cauchy-Riemann equations are satisfied for the given function $f(z) = \sin x \cosh y + i \cos x \sinh y$.